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ON SOME PROPERTIES OF MULTIPLE MODULI OF CONTINUITY

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ABSTRACT. For functions of several variables a property is established similar to the well-known result of S. B. Stechkin about the metric property of almost increasing functions. In the case of one variable the proof is easier than the known one.

The following result by S. B. Stechkin is true.

Lemma (see [1], p. 92). *Let $\eta(\delta)$ $(0 < \delta \leq 1)$ be a positive, nondecreasing function, and $\zeta^{-1}\eta(\zeta) \leq c\delta^{-1}\eta(\delta)$ for $0 < \delta < \eta \leq 1$. If for some number $\alpha \in]0,1]$ the series $\sum\limits_{n=1}^{\infty} n^{-\alpha}\eta(n^{-1}) = \infty$, then there exists a sequence (B_n) such that $B_n \downarrow 0$, $\sum\limits_{n=1}^{\infty} n^{-\alpha}B_n = \infty$ and $\sum\limits_{k=1}^{N} B_k \leq N\eta(N^{-1})$ $(N = 1, 2, \dots)$.*

The proof of this lemma (see [1], p. 94) implies that $B_n \leq \eta(n^{-1})$ for all $n = 1, 2, \dots$.

One knows many applications of this result, for instance, in the theory of classes of conjugate functions, in the theory of embedding of certain classes of functions, etc.

In this paper a multidimensional analogue of S. B. Stechkin's result is obtained for $\alpha = 1$.

To present further results we need a definition. Let $[0,1]^n = [0,1] \times [0,1] \times \cdots \times [0,1]$. A function

$$\omega : [0,1]^n \to [0, +\infty[$$

is called a mixed modulus of continuity (see, e.g., [2], p. 179) if it satisfies the following conditions:

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(1) $\omega(0,\delta_2,\dots,\delta_n)=\omega(\delta_1,0,\dots,\delta_n)=\cdots=\omega(\delta_1,\delta_1,\dots,0)=0$ for all $0\le\delta_k\le 1$, $k=1,2,\dots,n$;
(2) ω is continuous on $[0,1]^n$;
(3) ω does not decrease on $[0,1]^n$ in the sense of Hardy [3];
(4) ω is a semi-additive function with respect to each of the variables.

Note that in what follows c will always denote absolute but, generally speaking, different positive constants.

Theorem 1. *Let the function $u:[0,1]^n\to[0,+\infty[$ not decrease with respect to each of the variables,*

$$(t_k')^{-1}u(t_1,\dots,t_k',\dots,t_n)\ge c(t_k'')^{-1}u(t_1,\dots,t_k'',\dots,t_n)$$
$$\text{for all}\quad 0<t_k'<t_k''\le 1,\quad 0\le t_i\le 1 \qquad (*)$$
$$(k=1,2,\dots,n,\quad i=1,2,\dots,n,\quad i\ne k)$$

and

$$\int_0^1\int_0^1\cdots\int_0^1(t_1t_2\cdots t_n)^{-1}u(t_1,t_2,\dots,t_n)\,dt_1\,dt_2\cdots dt_n=\infty.$$

Then there exists a non-negative function B which does not decrease either with respect to each of the variables and satisfies the conditions:

(1) $B(t_1,t_2,\dots,t_n)\le u(t_1,t_2,\dots,t_n)$ *for all* $(t_1,t_2,\dots,t_n)\in[0,1]^n$*;*

(2) $\displaystyle\int_0^1\int_0^1\cdots\int_0^1(t_1t_2\cdots t_n)^{-1}B(t_1,t_2,\dots,t_n)\,dt_1\,dt_2\cdots dt_n=\infty;$

(3) $\displaystyle\int_{x_1}^1\int_{x_2}^1\cdots\int_{x_n}^1(t_1t_2\cdots t_n)^{-2}B(t_1,t_2,\dots,t_n)\,dt_1\,dt_2\cdots dt_n\le$

$$\le c(x_1x_2\cdots x_n)^{-1}u(x_1,x_2,\dots,x_n)$$
$$\text{for all }\ 0<x_k\le 1,\ \ k=1,2,\dots,n.$$

Proof. We set

$$B(t_1,t_2,\dots,t_n)=u(t_1^\beta,t_2^\beta,\dots t_n^\beta)$$

for $(t_1,t_2,\dots,t_n)\in[0,1]^n$, where $\beta>1$ is some fixed number.

We have

$$\int_0^1\int_0^1\cdots\int_0^1(t_1t_2\cdots t_n)^{-1}B(t_1,t_2,\dots,t_n)\,dt_1\,dt_2\cdots dt_n=$$

$$= \int_0^1 \int_0^1 \cdots \int_0^1 (t_1 t_2 \cdots t_n)^{-1} u(t_1^\beta, t_2^\beta, \dots, t_n^\beta)\, dt_1\, dt_2 \cdots dt_n =$$

$$= \beta^{-n} \int_0^1 \int_0^1 \cdots \int_0^1 (t_1 t_2 \cdots t_n)^{-1/\beta} u(t_1, t_2, \dots, t_n)\, \frac{dt_1\, dt_2 \cdots dt_n}{t_1^{1-\frac{1}{\beta}} t_2^{1-\frac{1}{\beta}} \cdots t_n^{1-\frac{1}{\beta}}} =$$

$$= \beta^{-n} \int_0^1 \int_0^1 \cdots \int_0^1 (t_1 t_2 \cdots t_n)^{-1} u(t_1, t_2, \dots, t_n)\, dt_1\, dt_2 \cdots dt_n = \infty,$$

which means that condition (2) is fulfilled.

Since $(t_1, t_2, \dots, t_n) \in [0,1]^n$, $\beta > 1$, and u does not decrease with respect to each of the variables, we conclude that B does not decrease either with respect to each of the variables, and also

$$B(t_1, t_2, \dots, t_n) \le u(t_1, t_2, \dots, t_n)$$

for all $(t_1, t_2, \dots, t_n) \in [0,1]^n$. Therefore condition (1) is fulfilled.

Now we will show that relation (3) is true. Let x_k be some numbers, $0 < x_k < 1$ $(k = 1, 2, \dots, n)$.

In that case, if

$$\mathcal{I} = \int_{x_1}^1 \int_{x_2}^1 \cdots \int_{x_n}^1 (t_1 t_2 \cdots t_n)^{-2} B(t_1, t_2, \dots, t_n) dt_1\, dt_2 \cdots dt_n =$$

$$= \int_{x_1}^1 \int_{x_2}^1 \cdots \int_{x_n}^1 (t_1 t_2 \cdots t_n)^{-2} u(t_1^\beta, t_2^\beta, \dots, t_n^\beta) dt_1\, dt_2 \cdots dt_n =$$

$$= \int_{x_1}^{x_1^{1/\beta}} \int_{x_2}^1 \cdots \int_{x_n}^1 (t_1 t_2 \cdots t_n)^{-2} u(t_1^\beta, t_2^\beta, \dots, t_n^\beta) dt_1\, dt_2 \cdots dt_n +$$

$$+ \int_{x_1^{1/\beta}}^1 \int_{x_2}^1 \cdots \int_{x_n}^1 (t_1 t_2 \cdots t_n)^{-2} u(t_1^\beta, t_2^\beta, \dots, t_n^\beta) dt_1\, dt_2 \cdots dt_n,$$

then by Fubini's theorem

$$\mathcal{I} = \int_{x_1}^{x_1^{1/\beta}} t_1^{-2} \left(\int_{x_2}^1 \cdots \int_{x_n}^1 (t_2 \cdots t_n)^{-2} u(t_1^\beta, t_2^\beta, \dots, t_n^\beta)\, dt_2 \cdots dt_n \right) dt_1 +$$

$$+\int\limits_{x_1^{1/\beta}}^{1} t_1^{-2}\Bigg(\int\limits_{x_2}^{1}\cdots\int\limits_{x_n}^{1}(t_2\cdots t_n)^{-2}u(t_1^{\beta},t_2^{\beta},\dots,t_n^{\beta})\,dt_2\cdots dt_n\Bigg)dt_1\le$$

$$\le\int\limits_{x_1}^{x_1^{1/\beta}} t_1^{-2}\Bigg(\int\limits_{x_2}^{1}\cdots\int\limits_{x_n}^{1}(t_2\cdots t_n)^{-2}u(x_1,t_2,\dots,t_n)\,dt_2\cdots dt_n\Bigg)dt_1+$$

$$+\beta^{-1}\int\limits_{x_1}^{1}\Bigg(\tau_1^{-\frac{2}{\beta}}\int\limits_{x_2}^{1}\cdots\int\limits_{x_n}^{1}(t_2\cdots t_n)^{-2}u(\tau_1,t_2^{\beta},\dots,t_n^{\beta})dt_2\cdots dt_n\Bigg)\tau_1^{\frac{1}{\beta}-1}d\tau_1=$$

$$=\int\limits_{x_1}^{x_1^{1/\beta}} t_1^{-2}dt_1\int\limits_{x_2}^{1}\cdots\int\limits_{x_n}^{1}(t_2\cdots t_n)^{-2}u(x_1,t_2^{\beta},\dots,t_n^{\beta})\,dt_2\cdots dt_n+$$

$$+\beta^{-1}\int\limits_{x_1}^{1}\Bigg(\int\limits_{x_2}^{1}\cdots\int\limits_{x_n}^{1}\tau_1^{-1}(t_2\cdots t_n)^{-2}u(\tau_1,t_2^{\beta},\dots,t_n^{\beta})\,dt_2\cdots dt_n\Bigg)\tau_1^{-\frac{1}{\beta}}d\tau_1.$$

Using the property $(*)$ of the function u, we obtain

$$\mathcal{I}\le[t_1^{-1}]_{x_1^{\frac{1}{\beta}}}^{x_1}\int\limits_{x_2}^{1}\cdots\int\limits_{x_n}^{1}(t_2\cdots t_n)^{-2}u(x_1,t_2^{\beta},\dots,t_n^{\beta})\,dt_2\cdots dt_n+$$

$$+c\int\limits_{x_1}^{1}\Bigg(\int\limits_{x_2}^{1}\cdots\int\limits_{x_n}^{1}x_1^{-1}(t_2\cdots t_n)^{-2}u(x_1,t_2^{\beta},\dots,t_n^{\beta})\,dt_2\cdots dt_n\Bigg)\tau_1^{-\frac{1}{\beta}}d\tau_1=$$

$$=(x_1^{-1}-x_1^{-\frac{1}{\beta}})\int\limits_{x_2}^{1}\cdots\int\limits_{x_n}^{1}(t_2\cdots t_n)^{-2}u(x_1,t_2^{\beta},\dots,t_n^{\beta})\,dt_2\cdots dt_n+$$

$$+c\int\limits_{x_1}^{1}\tau_1^{-\frac{1}{\beta}}d\tau_1\cdot x_1^{-1}\int\limits_{x_2}^{1}\cdots\int\limits_{x_n}^{1}(t_2\cdots t_n)^{-2}u(x_1,t_2,\dots,t_n)\,dt_2\cdots dt_n\le$$

$$\le x_1^{-1}\int\limits_{x_2}^{1}\cdots\int\limits_{x_n}^{1}(t_2\cdots t_n)^{-2}u(x_1,t_2^{\beta},\dots,t_n^{\beta})\,dt_2\cdots dt_n+$$

$$+c[\tau_1^{1-\frac{1}{\beta}}]_{x_1}^{1}\cdot x_1^{-1}\int\limits_{x_2}^{1}\cdots\int\limits_{x_n}^{1}(t_2\cdots t_n)^{-2}u(x_1,t_2^{\beta},\dots,t_n^{\beta})\,dt_2\cdots dt_n\le$$

$$\leq cx_1^{-1}\int\limits_{x_2}^{1}\cdots\int\limits_{x_n}^{1}(t_2\cdots t_n)^{-2}u(x_1,t_2^{\beta},\dots,t_n^{\beta})\,dt_2\cdots dt_n\equiv c\mathcal{I}_1,$$

where

$$\mathcal{I}_1=x_1^{-1}\int\limits_{x_2}^{1}\cdots\int\limits_{x_n}^{1}(t_2\cdots t_n)^{-2}u(x_1,t_2^{\beta},\dots,t_n^{\beta})\,dt_2\cdots dt_n.$$

Next,

$$\mathcal{I}_1=x_1^{-1}\int\limits_{x_2}^{x_2^{1/\beta}}\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1}(t_2\cdots t_n)^{-2}u(x_1,t_2^{\beta},\dots,t_n^{\beta})\,dt_2\cdots dt_n+$$

$$+\,x_1^{-1}\int\limits_{x_2^{1/\beta}}^{1}\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1}(t_2\cdots t_n)^{-2}u(x_1,t_2^{\beta},\dots,t_n^{\beta})\,dt_2\cdots dt_n\leq$$

$$\leq x_1^{-1}\int\limits_{x_2}^{x_2^{1/\beta}}t_2^{-2}\biggl(\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1}(t_3\cdots t_n)^{-2}u(x_1,x_2,t_3^{\beta},\dots,t_n^{\beta})\,dt_3\cdots dt_n\biggr)dt_2+$$

$$+\,x_1^{-1}\int\limits_{x_2^{1/\beta}}^{1}\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1}(t_2\cdots t_n)^{-2}u(x_1,t_2^{\beta},\dots,t_n^{\beta})\,dt_2\cdots dt_n=$$

$$=x_1^{-1}(x_2^{-1}-x_2^{\frac{1}{\beta}})\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1}(t_3\cdots t_n)^{-2}u(x_1,x_2,t_3^{\beta},\dots,t_n^{\beta})\,dt_3\cdots dt_n+$$

$$+\,x_1^{-1}\int\limits_{x_2^{1/\beta}}^{1}\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1}(t_2\cdots t_n)^{-2}u(x_1,t_2^{\beta},\dots,t_n^{\beta})\,dt_2\cdots dt_n\leq$$

$$\leq(x_1x_1)^{-1}\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1}(t_3\cdots t_n)^{-2}u(x_1,x_2,t_3^{\beta},\dots,t_n^{\beta})\,dt_3\cdots dt_n+$$

$$+\,x_1^{-1}\int\limits_{x_2^{1/\beta}}^{1}\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1}(t_2\cdots t_n)^{-2}u(x_1,t_2^{\beta},\dots,t_n^{\beta})\,dt_2\cdots dt_n=$$

$$=(x_1x_2)^{-1}\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1}(t_3\cdots t_n)^{-2}u(x_1,x_2,t_3^{\beta},\dots,t_n^{\beta})\,dt_3\cdots dt_n+$$

$$+\,(\beta x_1)^{-1}\int\limits_{x_2}^{1}\cdots\int\limits_{x_n}^{1} t_2^{-\frac{1}{\beta}}(t_2\cdots t_n)^{-2}u(x_1,t_2,t_3^{\beta},\dots,t_n^{\beta})t_2^{\frac{1}{\beta}-1}\,dt_2\cdots dt_n =$$

$$=(x_1x_2)^{-1}\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1}(t_3\cdots t_n)^{-2}u(x_1,x_2,t_3^{\beta},\dots,t_n^{\beta})\,dt_3\cdots dt_n +$$

$$+\,(\beta x_1)^{-1}\int\limits_{x_2}^{1}\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1} t_2^{-1}(t_3\cdots t_n)^{-2}u(x_1,t_2,t_3^{\beta},\dots,t_n^{\beta})t_2^{-\frac{1}{\beta}}\,dt_2\cdots dt_n.$$

Again, using the property $(*)$ of the function u we have

$$\mathcal{I}_1 \le (x_1x_2)^{-1}\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1}(t_3\cdots t_n)^{-2}u(x_1,x_2,t_3^{\beta},\dots,t_n^{\beta})\,dt_3\cdots dt_n +$$

$$+(x_1\beta)^{-1}\int\limits_{x_2}^{1}\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1} x_2^{-1}u(x_1,x_2,t_3^{\beta},\dots,t_n^{\beta})t^{-\frac{1}{\beta}}(t_3\cdots t_n)^{-2}dt_2\cdots dt_n =$$

$$=(x_1x_2)^{-1}\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1}(t_3\cdots t_n)^{-2}u(x_1,x_2,t_3^{\beta},\dots,t_n^{\beta})\,dt_3\cdots dt_n +$$

$$+(\beta x_1x_2)^{-1}\int\limits_{x_2}^{1}t_2^{-\frac{1}{\beta}}\,dt_2\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1}(t_3\cdots t_n)^{-2}u(x_1,x_2,t_3^{\beta},\dots,t_n^{\beta})dt_3\cdots dt_n =$$

$$=(x_1x_2)^{-1}\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1}(t_3\cdots t_n)^{-2}u(x_1,x_2,t_3^{\beta},\dots,t_n^{\beta})\,dt_3\cdots dt_n +$$

$$+\,(\beta x_1x_2)^{-1}(1-\beta^{-1})^{-1}[t_2^{1-\frac{1}{\beta}}]_{x_2}^{1}\,\times$$

$$\times\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1}(t_3\cdots t_n)^{-2}u(x_1,x_2,t_3^{\beta},\dots,t_n^{\beta})\,dt_3\cdots dt_n \le$$

$$\le (x_1x_2)^{-1}\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1}(t_3\cdots t_n)^{-2}u(x_1,x_2,t_3^{\beta},\dots,t_n^{\beta})\,dt_3\cdots dt_n +$$

$$+(\beta-1)^{-1}(x_1x_2)^{-1}\int\limits_{x_3}^{1}\cdots\int\limits_{x_n}^{1}(t_3\cdots t_n)^{-2}u(x_1,x_2,t_3^{\beta},\dots,t_n^{\beta})dt_3\cdots dt_n =$$

$$= c(x_1x_2)^{-1}\int_{x_3}^{1}\cdots\int_{x_n}^{1}(t_3\cdots t_n)^{-2}u(x_1,x_2,t_3^{\beta},\dots,t_n^{\beta})\,dt_3\cdots dt_n.$$

Therefore

$$\int_{x_1}^{1}\cdots\int_{x_n}^{1}(t_1\cdots t_n)^{-2}B(t_1,t_2,\dots,t_n)\,dt_3\cdots dt_n \le$$

$$\le c(x_1x_2)^{-1}\int_{x_3}^{1}\cdots\int_{x_n}^{1}(t_3\cdots t_n)^{-2}u(x_1,x_2,t_3^{\beta},\dots,t_n^{\beta})\,dt_3\cdots dt_n.$$

Clearly, continuing a similar reasoning in a step by step manner, we arrive at the inequality

$$\int_{x_1}^{1}\cdots\int_{x_n}^{1}(t_1\cdots t_n)^{-2}B(t_1,t_2,\dots,t_n)\,dt_3\cdots dt_n \le$$

$$\le c(x_1x_2\cdots x_n)^{-1}u(x_1,x_2,\dots,x_n)$$

and thus prove the validity of relation (3). □

Note that by assuming that

$$B_{m_1,m_2,\dots,m_n} = B(m_1^{-1},m_2^{-1},\dots,m_n^{-1}) \quad (m_k = 1,2,\dots, \quad k=1,2,\dots,n),$$

where B is the function from Theorem 1, we obtain

Theorem 2. *Let the function u satisfy the conditions of Theorem 1. Then we have a sequence $(B_{m_1,\dots,m_n})_{m_1\ge1,\dots,m_n\ge1}$ such that*

$$B_{m_1,\dots,m_k,\dots,m_n} - B_{m_1,\dots,m_k+1,\dots,m_n} \ge 0$$

for each $m_k = 1,2,\dots$ $(k=1,2,\dots,n)$,

$$\sum_{m_1=1}^{N_1}\sum_{m_2=1}^{N_2}\cdots\sum_{m_n=1}^{N_n}B_{m_1,m_2,\dots,m_n} \le cN_1N_2\cdots N_n u(N_1^{-1},N_2^{-1},\dots,N_n^{-1})$$

for every $N_k = 1,2,\dots$ $(k=1,2,\dots,n)$, and

$$B_{m_1,m_2,\dots,m_n} \le u(m_1^{-1},m_2^{-1},\dots,m_n^{-1})$$

for every $m_k = 1,2,\dots$ $(k=1,2,\dots,n)$.

Also note that if $u = \omega$ is a mixed modulus of continuity, then the sequence $(B_{m_1,\dots,m_n})$ will additionally satisfy the condition

$$\sum_{k_1=0}^{1}\sum_{k_2=0}^{1}\cdots\sum_{k_n=0}^{1}(-1)^{k_1+k_2+\cdots k_n}B_{m_1+k_1,m_2+k_2,\dots,m_n+k_n}\geq 0$$

for all $m_k \geq 1$ $(k = 1, 2, \dots, n)$.

Note finally that Theorems 1 and 2 have applications in the theory of embedding of classes of functions of several variables, in the theory of conjugate functions of several variables, and in other problems of multidimensional analysis.

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