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ON A CLASS OF COVARIANCE OPERATORS

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Abstract. This paper is the continuation of [1] in which complex symmetries of distributions and their covariance operators are investigated. Here we also study the most general quaternion symmetries of random vectors. Complete classification theorems on these symmetries are proved in terms of covariance operator spectra.

Let $\xi = (\xi', \xi'')$ be a random vector with components from the separable real Hilbert space $H_{\mathbb{R}}$ with a scalar product $(\cdot|\cdot)_{\mathbb{R}}$. Thus ξ takes values from the direct sum $H_0 = H_{\mathbb{R}} \oplus H_{\mathbb{R}}$ whose scalar product has the form

$$(h|g)_0 = (h'|g')_{\mathbb{R}} + (h''|g'')_{\mathbb{R}} \ \ \forall\, h = (h'|h''),\ g = (g', g'') \in H_0.$$

Each element of the space $L_0 = L(H_0)$ of linear bounded operators on H_0 can be represented as a block matrix

$$A = (a_{ij})_{i,j=1}^2 = \begin{pmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{pmatrix}, \quad a_{ij} \in L(H_{\mathbb{R}}), \quad i, j = 1, 2. \tag{1}$$

Let us consider the operator $U(x', x'') = (-x'', x')\ \forall (x', x'') \in H_0$ whose representation (1) can be written as

$$U = \begin{pmatrix} 0, & -I \\ I, & 0 \end{pmatrix}, \tag{2}$$

where I is the unit operator from $L(H_{\mathbb{R}})$. The operator U possesses the important property $U^*U = UU^* = I$, implying that U is orthogonal and

$$U^2 = -I. \tag{3}$$

Definition 1. A random vector ξ is called symmetric if (ξ', ξ'') and $(-\xi'', \xi')$ are equally distributed, i.e., if ξ and $U\xi$ are such.

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Clearly, if ξ is symmetric, then ξ, $-\xi$, $U\xi$, $-U\xi$ are equally distributed.

Proposition 1. *Let ξ_1, ξ_2, ξ_2, ξ_4 be the independent copies of a random vector ξ. Then a random vector*

$$\xi_a = \frac{1}{2}[\xi_1 - \xi_2 + U\xi_3 - U\xi_4]$$

is symmetric.

The proof immediately follows from (3).

Definition 2. ξ_a is called an additive symmetrization of a random vector ξ.

Proposition 2. *Let $\mathfrak{M}$ be a random operator taking the values I, $-I$, U, $-U$, each with probability $\frac{1}{4}$. Then a random vector*

$$\xi_m = \mathfrak{M}\xi$$

is symmetric if ξ and $\mathfrak{M}$ are independent.

Proof. It follows from (3) that the operator $U\mathfrak{M}$ has the same distribution as $\mathfrak{M}$, and does not depend on ξ, so that ξ_m and $U\xi_m$ are equally distributed. □

Definition 3. ξ_m is called a multiplicative symmetrization.

Proposition 3. *Let a symmetric random vector ξ have a strong moment of second order. Then it is centered and its covariance K_0 satisfies the relation*

$$UK_0 = K_0U. \tag{4}$$

Proof. Since ξ and $-\xi$ are equally distributed, ξ is centered. Since ξ and $U\xi$ are equally distributed, we have $K_0 = U^*K_0U$. But in that case (3) implies (4).

Denote by L_1 the commutant of the operator U, i.e., $L_1 = \{A|A \in L_0, AU = UA\}$. Then (4) implies that $K_0 \in L_1$. □

Proposition 4. *The operator A belongs to L_1 if and only if its matrix representation has the form*

$$A = \begin{pmatrix} a, & -b \\ b, & a \end{pmatrix}, \quad a, b \in L(H_{\mathbb{R}}). \tag{5}$$

The proof follows immediately.

In [1], operators of form (5) are called proper and, accordingly, the centered random vector ξ possessing the covariance K_0 is called proper for $K_0 \in L_1$, i.e., for

$$K_0 = \begin{pmatrix} R & -T \\ T & R \end{pmatrix}. \tag{6}$$

Generally speaking, if $\xi = (\xi', \xi'')$ is the centered random vector with covariance K_0, then K_0 has the form

$$K_0 = \begin{pmatrix} R_1 & T^* \\ T & R_2 \end{pmatrix}, \tag{7}$$

where R_1 and R_2 are the covariances of ξ' and ξ'', respectively, and T is their mutual covariance. Thus the properity of ξ implies that

$$R_1 = R_2 = R, \quad T^* = -T. \tag{8}$$

Definition 4. For $A \in L_0$ we call the expression

$$\overbracket{A} = \frac{1}{2}[A + U^* A U]$$

the averaging of the operator A.

It is easy to verify that the block representation of the averaging $\overbracket{K_0}$ of the operator K_0 from (7) is written as

$$\overbracket{K_0} = \frac{1}{2}\begin{pmatrix} R_1 + R_2 & T^* - T \\ T - T^* & R_1 + R_2 \end{pmatrix}.$$

Proposition 5. *The operator $A \in L_0$ is proper, i.e., belongs to L_1 if and only if $\overbracket{A} = A$.*

The proof is obvious.

Let ξ be a centered random vector with covariance K_0. Denote by K_a and K_m the covariances of additive and multiplicative symmetrizations of ξ.

Proposition 6. *The equalities*

$$K_a = K_m = \overbracket{K_0}$$

are valid.

Proof. The equality $K_a = \overline{K_0}$ immediately follows from Proposition 1. Indeed, since $\xi_a = \frac{1}{2}(\xi_1 - \xi_2 + U\xi_3 - U\xi_4)$, we have

$$K_a = \frac{1}{4}(K_0 + K_0 + U^*K_0U + U^*K_0U) = \frac{1}{2}(K_0 + U^*KU) = \overline{K_0}.$$

Let us prove the equality $K_m = \overline{K_0}$. We have

$$\begin{aligned}(K_m h|h)_0 &= E(\xi_m|h)_0^2 = E(\mathfrak{M}\xi|h)_0^2 = E(\xi|\mathfrak{M}^*h)_0^2 = E_{\mathfrak{M}}E_\xi(\xi|\mathfrak{M}^*h)_0^2 = \\ &= E_m(K_0\mathfrak{M}^*h|\mathfrak{M}^*h)_0 = \frac{1}{4}((K_0h|h)_0 + (K_0(-h)|(-h))_0 + \\ &+ (K_0U^*h|U^*h)_0 + (K_0(-U^*h)|(-U^*h))_0) = \\ &= \frac{1}{2}((K_0h|h)_0 + (UK_0U^*h|h)) = \frac{1}{2}((K_0 + U^*K_0U)h|h)_0.\end{aligned}$$

The latter equality follows from (3), which gives $U^* = -U$. □

As shown above (Proposition 4), a symmetric random vector ξ with covariance is proper. The converse statement, generally speaking, does not hold, but we have

Proposition 7. *A random proper Gaussian vector is symmetric.*

Proof. For the characteristic functional of a proper Gaussian vector we have

$$\begin{aligned}\varphi(Ut) &= e^{-\frac{1}{2}(K_0Ut|Ut)_0} = e^{-\frac{1}{2}(U^*K_0Ut|t)_0} = \\ &= e^{-\frac{1}{2}(U^*UK_0t|t)_0} = e^{-\frac{1}{2}(K_0t|t)_0} = \varphi(t),\end{aligned}$$

from which it follows that ξ and $U\xi$ are equally distributed, i.e., ξ is symmetric. □

Thus if ξ is a proper Gaussian vector, then ξ and ξ_a are equally distributed. Moreover, we have

Theorem 1. *The distributions of ξ and ξ_a coincide if and only if ξ is a random Gaussian vector.*

This theorem is a corollary of a more general theorem proved by N. Vakhania in [2].

The averaging $\overline{K_0}$ of the covariance operator K_0 naturally appears as the covariance operator of ξ from some complex Hilbert space H_1. Indeed, let us define the multiplication of vectors $x \in H_0$ by scalars $\lambda \in \mathbb{C}$ as

$$\lambda x = (\lambda' + i\lambda'')x = (\lambda' I + \lambda'' U)x \quad \forall\, x \in H_0, \lambda \in \mathbb{C} \tag{9}$$

and its Hermitian form as

$$(x|y)_1 = (x|y)_0 + i(x|Uy)_0 \quad \forall\, x, y \in H_0. \tag{10}$$

As a result, the real space H_0 becomes the complex Hilbert space H_1 with the Hermitian form $(\cdot|\cdot)_1$ and algebra of $\mathbb{C}$-linear operators L_1. Hence our random vector ξ becomes the complex random vector $\xi = \xi' + i\xi''$, which by virtue of (9) implies that $\xi = (\xi', 0) + U(\xi'', 0)$, and its covariance K_1 with respect to the Hermitian form $(\cdot|\cdot)_1$ is defined by the relation

$$(K_1 h|h)_1 = \frac{1}{2} E|(\xi|h)_1|^2. \tag{11}$$

Proposition 8. *The equality*

$$K_1 = \overbracket{K_0}$$

is valid.

Proof.

$$(K_1 h|h)_1 = \frac{1}{2} E|(\xi|h)_1|^2 = \frac{1}{2} E\big((\xi|h)_0^2 + (\xi|Uh)_0^2\big) = \frac{1}{2}(K_0 h|h)_0 + \\ + \frac{1}{2}(U^* K_0 U h|h)_0 = \Big(\frac{1}{2}(K_0 + U^* K_0 U) h|h\Big)_0 = (\overbracket{K_0} h|h)_0.$$

Since by (11) $(K_1 h|h)_1$ is real, by (10) $(K_1 h|h)_1 = (K_1 h|h)_0$ and therefore $K_1 = \overbracket{K_0}$. $\square$

This important equality is thoroughly investigated in [1].

The above reasoning can be summarized as

Theorem 2. *The following conditions are equivalent:* (a) ξ *is a proper random vector* (*see* (6) *and* (8)); (b) $K_0 = \overbracket{K_0}$; (c) $K_0 = K_a$; (d) $K_0 = K_m$; (e) $K_0 = K_1$.

Let ξ be centered and have the covariance K_0. Recall that K_0 is a completely continuous, positive-definite operator. Denote by M_i $(i = 1, 2, \dots)$ the eigensubspaces of K_0 which correspond to the eigenvalues μ_i, $(i = 1, 2, \dots)$, $\mu_1 > \mu_2 > \cdots$; $\mu_i \to 0$. If the operator K_0 is degenerate, denote $M_0 = \operatorname{Ker} K_0$. Let us now describe proper covariance operators in terms of their eigenvalues.

Definition 5. A random vector ξ is called orthogonally proper if there exists an orthogonal operator O such that $O\xi$ is a proper random vector.

Theorem 3. *A random vector* ξ *is orthogonally proper if and only if* $\dim M_k$ *is even for all* $k \geq 0$.

Proof. Sufficiency. Let $\dim M_k$ be even for all $k \geq 0$. Then there exists a basis of the space H_0 consisting of eigenvectors of the operator K_0 : $f_1, f_2, \ldots$ such that for $K_0 f_{2p-1} = \mu f_{2p-1}$ we have $K_0 f_{2p} = \mu f_{2p}$, $p = 1, 2, \ldots$. Let further g_p $(p = 1, 2, \ldots)$ be a basis of $H_{\mathbb{R}}$. Then a sequence of vectors $e_1, e_2, \ldots$ from H_0, for which $e_{2p-1} = (g_p, 0)$, $e_{2p} = (0, g_p)$, $p = 1, 2, \ldots$, is the orthonormal basis H_0. By (2) $Ue_{2p-1} = e_{2p}$, $Ue_{2p} = -e_{2p-1}$ $(p = 1, 2, \ldots)$. Let us now define the orthogonal operator O as $Oe_k = f_k$ $(k = 1, 2, \ldots)$. The covariance $O\xi$ has the form $\Gamma = O^{-1} K_0 O$. Next we shall prove that $\Gamma U = U\Gamma$. Indeed, we have

$$\begin{aligned} U\Gamma e_{2p-1} &= UO^{-1}K_0 O e_{2p-1} = UO^{-1}K_0 f_{2p-1} = UO^{-1}\mu_k f_{2p-1} = \\ &= \mu_k UO^{-1} f_{2p-1} = \mu_k e_{2p-1} = \mu_k e_{2p}; \\ \Gamma U e_{2p-1} &= O^{-1}K_0 e_{2p} = O^{-1}K_0 f_{2p} = \mu_k O^{-1} f_{2p} = \mu_k e_{2p}, \end{aligned}$$

$p = 1, 2, \ldots$. The same reasoning is true for e_{2p} $(p = 1, 2, \ldots)$.

Necessity. Let O be an orthogonal operator such that $O\xi$ is proper. Then $M'_k = O^{-1} M_k$ $(k \geq 0)$ are the eigensubspaces of Γ and μ_k are the corresponding eigenvalues. Indeed, if $x \in M_k$, then $\Gamma x = O^{-1} K_0 O x$. Since $Ox \in M_k$, we have $\Gamma x = O^{-1} \mu_k O x = \mu_k x$. Conversely, assume that $\Gamma x = \mu_k x$. Then, on the one hand, $\mu_k x = \Gamma x = O^{-1}(K_0 O x)$, and, on the other, $O^{-1}(\mu_k O x) = \mu_k x$. Hence it follows that $K_0 O x = \mu_k O x$ and therefore $Ox \in M_k$, i.e., $x \in M'_k$. Since Γ is a proper operator, we have $\Gamma U = U\Gamma$. Thus we conclude that M'_k are invariant for U as well. Indeed, assuming that $x \in M'_k$, we obtain $\Gamma U x = U \Gamma x = U \mu_k x = \mu_k U x$. But this implies that, together with x, Ux also belongs to M'_k. But this is possible only when $\dim M'_k$ is even. Therefore $\dim M_k$ will be even too. (Throughout the paper it is assumed that "infinity" may have any multiplicity). □

Remark 1. The orthogonal property is understood as a property with respect to $U_1 = OUO^{-1}$ in the sense that U_1 is orthogonal, $U_1^2 = -I$, and $U_1 K_0 = K_0 U_1$.

Since K_0 is the self-conjugate operator, H_0 can be represented as an orthogonal sum $H_0 = \overline{\operatorname{im} K_0} + \operatorname{Ker} K_0$. $e_1, e_2, \ldots$ is the orthonormal basis of the subspace $\overline{\operatorname{im} K_0}$ consisting of the eigenvectors of K_0 corresponding to the values $\lambda_1 \geq \lambda_2 \geq \cdots$.

Theorem 4. *A self-conjugate invertible operator S, permutable with K_0 and such that a random vector $\eta = S\xi$ is orthogonally proper, exists if and only if $\inf_{p \geq 1} \frac{\lambda_{2p}}{\lambda_{2p-1}} = c_0 > 0$ and $\operatorname{Ker} K_0$ is even.*

Proof. Sufficiency. We write S as $S_{e_k} = \sigma_k e_k$, $S_x = x$ $\forall x \in \operatorname{Ker} K_0$, where σ_k are defined by the equalities $\sigma_{2p} = 1$, $\sigma_{2p-1} = (\lambda_{2p}/\lambda_{2p-1})^{1/2}$ $p = 1, 2, \ldots$. It is obvious that S is permutable with K_0 and self-conjugate. We shall prove that it is invertible. Note that $\sqrt{c_0} \leq \sigma_k \leq 1$ for all x.

The right-hand side of the inequality follows from the definition of σ_k and from $\lambda_{2p} \leq \lambda_{2p-1}$ $(p = 1, 2, \dots)$, while the left-hand side is provided by the condition of the theorem $\inf_{p\geq 1} \frac{\lambda_{2p}}{\lambda 2p-1} = c_0 > 0$. Therefore S is invertible. Finally, let us prove that the random vector $\eta = S\xi$ is orthogonally proper. Since S is self-conjugate and permutable with K_0, the covariance η is S^*K_0S, we obtain $S^*K_0S = S^*SK_0 = S^2K_0$. Thus the covariance η in the basis $e_1, e_2, \dots$ can be represented in the diagonal form $\sigma_1^2\lambda_1, \sigma_2^2\lambda_2, \dots$. But $\sigma_{2p}^2\lambda_{2p} = \lambda_{2p}$, $\sigma_{2p-1}^2\lambda_{2p-1} = (\lambda_{2p}/\lambda_{2p-1})\lambda_{2p-1} = \lambda_{2p}$, $p = 1, 2, \dots$. Therefore all eigensubspaces S^*K_0S (including $\operatorname{Ker} K_0$) are even, i.e., $\eta = S\xi$ is orthogonally proper.

Necessity. Let S satisfy the conditions of the theorem: S is invertible, self-conjugate, permutable with K_0, and the dimensions of the eigensubspaces S^*K_0S are even. Hence it immediately follows that $\operatorname{Ker} K_0 = \operatorname{Ker} S^*K_0S$ and $\operatorname{im} S^*K_0S$. Clearly, $S_{e_k} = \sigma_k e_k$, $k = 1, 2, \dots$, $\alpha_0 \leq \sigma_k \leq \alpha_1$, for all k and $\sigma_k^2\lambda_k = \beta_k$ are such that $\beta_{2p-1} = \beta_{2p}$ $p = 1, 2, \dots$. Further, $\lambda_{2p}/\lambda_{2p-1} = (\sigma_{2p-1}/\sigma_{2p})^2\beta_{2p}/\beta_{2p-1} \geq (\frac{\alpha_0}{\alpha_1})^2 > 0$. □

We shall now give an example of a random vector which does not belong to the above-mentioned class. Let ξ be a random vector with covariance K_0 whose eigenvalues $\lambda_k = e^{-k^\alpha}$ $(\alpha > 1)$ $(k = 1, 2, \dots)$. It is assumed that the conditions of Theorem 4 are fulfilled for $\alpha = 1$ and for all orders less than e^{-k}.

Proposition 9. *The operators from* $L_0 = L(H_0)$

$$U = \begin{pmatrix} 0, & -I \\ I, & 0 \end{pmatrix}, \quad V = \begin{pmatrix} v, & 0 \\ 0, & -v \end{pmatrix}, \tag{12}$$

where v is an orthogonal operator in $H_{\mathbb{R}}$ such that $v^2 = -I$ (for the existence of such an operator see [3], [4]*), give, in H_0, an additional structure (see* [5]*, Ch. 11, §2), i.e., U and V are orthogonal in H_0 and*

$$U^2 = V^2 = -I, \quad UV = -VU. \tag{13}$$

Proof. Follows immediately. □

Theorem 5. *The covariance K_0 of a proper random vector (see* (6)*,* (8)*) is permutable with V if and only if*

$$Rv = vR, \quad T_v = -vT.$$

Proof. Follows immediately. □

Definition 6. A proper random vector ξ is called twice proper if its covariance operator K_0 is permutable with V.

The operators defined by formulas (12), (13) allow us to introduce, in the space H_0, the so-called Hermitian form defining one more covariance K for ξ. We shall see below that for twice proper random vectors K and K_0 coincide.

Consider the set

$$\mathbb{H} = \big\{A|A \in L_0 A = aI + bU + cV + dW,\ a,b,c,d \in \mathbb{R}\big\},$$

where $W = UV$. Note that W is an orthogonal operator such that $WU = -UW$, $WV = -VW$. It is easy to verify that $\mathbb{H}$ is a four-dimensional $\mathbb{R}$-algebra with involution with respect to standard operations $A+B$, AB, λA, A^*. Moreover, $\mathbb{H}$ is a body whose every nonzero element is invertible. Indeed, for any $A = aI+bU+cV+dW$ we have $AA^* = (a^2+b^2+c^2+d^2)I$, which means that if $a \neq 0$, then A^{-1} exists and $A^{-1} = (a^2+b^2+c^2+d^2)^{-1}A^*$. The set $\mathbb{H}$ acts on H_0 in a natural way: if $A \in \mathbb{H}$, $x \in H_0$, then its action is simply Ax. Finally, the mapping $(\cdot|\cdot) : H_0 \times H_0 \to \mathbb{H}$ defined by the formula

$$\begin{aligned}\forall (x,y) \in H_0 \times H_0\ (x|y) = (x|y)_0 I + (x|Uy)_0 U + \\ +(x|Vy)_0 V + (x|Wy)_0 W\end{aligned} \tag{14}$$

gives the Hermitian form. Indeed, it is easy to verify that

$$(x+y|z) = (x|z) + (y|z)\ \forall\ \ x,y \in H_0,$$
$$(Ax|y) = A(x|y)\ \forall A \in \mathbb{H},\ x,y \in H_0;\ (x|y)^* = (y|x)\ \forall x,y \in H_0,$$

and, finally, $(x|x) \geq 0\ \forall x \in \mathbb{H}\ (x|x) = 0 \Leftrightarrow x = 0$, where $A \geq 0$ means that the operator A is nonnegative. But since $|(|x)| = (x|x)_0$, the topologies generated by the Hermitian forms $\cdot|\cdot)$ and $\cdot|\cdot)_0$ in H_0 coincide.

For the centered random vector ξ with covariance K_0 we define the covariance K with respect to the Hermitian form (14) by the formula $(Kh|h) = \frac{1}{4}E|(\xi|h)|^\alpha$.

Proposition 10. *The covariance K has the form*

$$K = \frac{1}{4}(K_0 + U^*K_0U + V^*K_0V + W^*K_0W). \tag{15}$$

Proof. (14) implies

$$\begin{aligned}K &= \frac{1}{4}\big[E(\xi|h)_0^2 + E(\xi|Uh)_0^2 + E(\xi|Vh)_0^2 + E(\xi|Wh)_0^2\big] = \\ &= \frac{1}{4}\big[(K_0h|h)_0 + (K_0Uh|Uh)_0 + (K_0Vh|Vh)_0 + (K_0Wh|Wh)_0\big] = \\ &= \frac{1}{4}\big((K_0 + U^*K_0U + V^*K_0Vh + W^*K_0W)h|h\big)_0.\end{aligned}$$

Since, by definition, $(Kh|h)$ coincides with $(Kh|h_0)_0$, we obtain (15). □

Proposition 11. *If a random vector is twice proper, then*

$$K_0 = K.$$

So far we have been concerned with introducing new structures in the space H_0. A similar situation occurs in the case of the narrowing of a scalar field when a rich structure is given for the initial space and we are interested in the behavior of a random vector in different structures. The case $\mathbb{R} \subset \mathbb{C}$ is treated in [1]. Here we shall consider the restriction of scalars in the quaternion case. Recall that the quaternion body $\mathbb{H}$ is a four-dimensional $\mathbb{R}$-algebra with the basis $1, i, j, k$ and multiplication table

	i	j	k
i	-1	k	-j
j	-k	-1	i
k	j	-i	-1

The conjugate quaternion $\overline{\mu}$ to $\mu = \mu' + i\mu'' + j\mu''' + k\mu^{IV}$ is defined as $\overline{\mu} = \mu' - i\mu'' - j\mu''' - k\mu^{IV}$. If $\lambda = \lambda' + i\lambda'' + j\lambda''' + k\lambda^{IV}$, then the expression $(\lambda|\mu) = \lambda\overline{\mu}$ defines the quaternion Hermitian form, while $(\lambda|\mu)_0 = \operatorname{Re}\lambda\overline{\mu}$ gives a Euclidean scalar product $\mathbb{H}$ over $\mathbb{R}^0 : (\lambda|\mu)_0 = \lambda'\mu' + \lambda''\mu'' + \lambda'''\mu''' + \lambda^{IV}\mu^{IV}$ with the norm $|\lambda\mu|_0 = |\lambda\mu| = |\lambda|_0|\mu|_0$, where, as usual, it is assumed that $|\cdot|_0 = (\cdot|\cdot)_0^{1/2}$. Note that $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$. The quaternion Hilbert space H is an Abelian group with respect to the sum $x + y$ of the vectors from H, where the product λx of scalars from $\mathbb{H}$ and of scalars from H is defined. Moreover, this product is complete and separable with respect to the following Hermitian form with values in H given on $\mathbb{H} : \forall x, y \in H$:

$$(x|y) = (x|y)_0 + (x|iy)_0 i + (x|jy)_0 j + (x|ky)_0 k.$$

Here

$$(\cdot|\cdot)_0 = \operatorname{Re}(\cdot|\cdot)$$

is the usual real scalar product. Thus, in the case of the restriction of the scalars from $\mathbb{H}$ to $\mathbb{R}$, the space H transforms to the real Hilbert space H_0, while in the case of the restriction to $\mathbb{C}$, it transforms to the complex Hilbert space H_1 with the Hermitian form

$$\big(\cdot\,|\,\cdot\big) = \big(\cdot\,|\,\cdot\big)_0 + \big(\cdot\,|i(\cdot)\big)_0 i.$$

Since $|\cdot| = |\cdot|_1 = |\cdot|_0$, we find that H^0, H_1, H_0, being topological Abelian groups, completely coincide.

Denote by L, L_1, L_0 the spaces of bounded linear operators $\mathbb{H}$, $\mathbb{C}$, $\mathbb{R}$, respectively. The restriction of the scalars gives the natural embeddings

$L \subset L_1 \subset L_0$. Now we introduce in H_0 the operators U and V

$$U = iI, \quad V = jI.$$

which, as is easy to verify, satisfy (13). Therefore all the above arguments hold for the random vector ξ with values from H_0. Thus Theorem 3 will read as follows:

Theorem 6. *A random vector ξ is twice orthogonally proper if and only if $\dim M_k$ is a multiple of four for all $k \geq 0$.*

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