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APPLICATION OF ANALOGUES OF GENERAL KOLOSOV–MUSKHELISHVILI REPRESENTATIONS IN THE THEORY OF ELASTIC MIXTURES

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Abstract. The existence and uniqueness of a solution of the first, the second and the third plane boundary value problem are considered for the basic homogeneous equations of statics in the theory of elastic mixtures. Applying the general Kolosov–Muskhelishvili representations from [1], these problems can be splitted and reduced to the first and the second boundary value problem for an elliptic equation which structurally coincides with the equation of statics of an isotropic elastic body.

Introduction

Analogues of Kolosov–Muskhelishvili representation formulas were obtained in [1] for equations of statics in the theory of elastic mixtures. These formulas have various applications. In this paper they will be used to reduce the first, the second and the third boundary value problems of statics in the theory of elastic mixtures [2] to the first and the second boundary value problem for an elliptic equation which structurally coincides with an equation of statics of an isotropic elastic body. It will be shown that the theory and methods of solving the boundary value problems developed in [3] can be extended to the plane boundary value problems of statics in the theory of elastic mixtures.

§ 1. First Boundary Value Problem

As is known [2], the first boundary value problem is considered with a displacement vector given on the boundary. To split this problem we have

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to use the general representations of displacement vector components [1] having the form

$$\begin{aligned} u_1+iu_2&=m_1\varphi_1(z)+m_2\varphi_2(z)+\frac{z}{2}\big[l_4\overline{\varphi_1'(z)}+l_5\overline{\varphi_2'(z)}\big]+\overline{\psi_1(z)},\\ u_3+iu_4&=m_2\varphi_1(z)+m_3\varphi_2(z)+\frac{z}{2}\big[l_5\overline{\varphi_1'(z)}+l_6\overline{\varphi_2'(z)}\big]+\overline{\psi_2(z)}, \end{aligned} \tag{1.1}$$

where $u=(u_1,u_2,u_3,u_4)$ is a four-dimensional displacement vector; $\varphi_1(z)$, $\varphi_2(z)$, $\psi_1(z)$, $\psi_2(z)$ are arbitrary analytic functions, and

$$\begin{aligned} &m_1=e_1+\frac{l_4}{2},\quad m_2=e_2+\frac{l_5}{2},\quad m_3=e_3+\frac{l_6}{2},\\ &l_1=\frac{a_2}{d_2},\quad l_2=-\frac{c}{d_2},\quad l_3=\frac{a_1}{d_2},\quad d_2=a_1a_2-c^2>0,\\ &l_1+l_4=\frac{a_2+b_2}{d_1},\quad l_2+l_5=-\frac{c+d}{d_1},\quad l_3+l_6=\frac{a_1+b_1}{d_1},\\ &d_1=(a_1+b_1)(a_2+b_2)-(c+d)^2>0 \end{aligned} \tag{1.2}$$

the coefficients a_1, b_1, a_2, b_2, c, d are contained in the basic homogeneous equations of statics in the theory of elastic mixtures which are written as [1]:

$$\begin{aligned} a_1\Delta u'+b_1\operatorname{grad}\theta'+c\Delta u''+d\operatorname{grad}\theta''=0,\\ c\Delta u'+d\operatorname{grad}\theta'+a_2\Delta u''+b_2\operatorname{grad}\theta''=0, \end{aligned} \tag{1.3}$$

where $u'=(u_1,u_2)$, $u''=(u_3,u_4)$ are partial displacements of an elastic mixture and

$$\theta'=\operatorname{div}u',\quad \theta''=\operatorname{div}u''. \tag{1.4}$$

To reduce the first boundary value problem of statics in the theory of elastic mixtures to the first boundary value problem of statics of an isotropic elastic body we rewrite (1.1) as

$$\begin{aligned} u_1+Xu_3+i(u_2+Xu_4)&=(m_1+Xm_2)\varphi_1(z)+(m_2+Xm_3)\varphi_2(z)+\\ &+\frac{z}{2}\big[(l_4+Xl_5)\overline{\varphi_1'(z)}+(l_5+Xl_6)\overline{\varphi_2'(z)}\big]+\overline{\psi_1(z)}+X\overline{\psi_2(z)}, \end{aligned} \tag{1.5}$$

where X is an arbitrary real constant. We define the unknown X by the equation

$$\frac{m_2+Xm_3}{m_1+Xm_2}=\frac{l_5+Xl_6}{l_4+Xl_5}. \tag{1.6}$$

Using the formulas [1,2]

$$\begin{aligned}
&2\Delta_0\varepsilon_1 = l_5m_2 - l_4m_3, \quad 2\Delta_0\varepsilon_3 = l_4m_2 - l_5m_1,\\
&2\Delta_0\varepsilon_2 = l_6m_2 - l_5m_3, \quad 2\Delta_0\varepsilon_4 = l_5m_2 - l_6m_1,\\
&\Delta_0 = m_1m_3 - m_2^2 > 0
\end{aligned} \tag{1.7}$$

and their equivalent formulas

$$\begin{aligned}
&\delta_0\varepsilon_1 = 2(a_2b_1 - cd) + b_1b_2 - d^2, \quad \delta_0\varepsilon_3 = 2(da_2 - cb_2),\\
&\delta_0\varepsilon_2 = 2(da_1 - cb_1), \quad \delta_0\varepsilon_4 = 2(a_1b_2 - cd) + b_1b_2 - d^2,\\
&\delta_0 = (2a_1 + b_1)(2a_2 + b_2) - (2c + d)^2 = 4\Delta_0d_1d_2 > 0,
\end{aligned} \tag{1.8}$$

by (1.6) we obtain the quadratic equation with respect to X

$$\varepsilon_2X^2 - (\varepsilon_4 - \varepsilon_1)X - \varepsilon_3 = 0. \tag{1.9}$$

Note that ε_2 and ε_3 do not vanish simultaneously. Indeed, if the equality $\varepsilon_2 = \varepsilon_3 = 0$ is satisfied, then by virtue of (1.8) we obtain

$$\frac{b_1}{a_1} = \frac{d}{c} = \frac{b_2}{a_2} = \lambda. \tag{1.10}$$

The constant $\lambda \neq 0$, since for $\lambda = 0$ equality (1.10) implies $b_1 = d = b_2 = 0$ and (1.3) gives

$$a_1\Delta u' + c\Delta u'' = 0, \quad c\Delta u' + a_2\Delta u'' = 0.$$

Hence, taking into account that $d_2 = a_1a_2 - c^2 > 0$ [2], we have

$$\Delta u' = 0, \quad \Delta u'' = 0.$$

Thus we have obtained a trivial case of an elastic mixture.

Now, substituting (1.10) into (1.3), we have

$$\begin{aligned}
&a_1(\Delta u' + \lambda \operatorname{grad}\theta') + c(\Delta u'' + \lambda \operatorname{grad}\theta'') = 0,\\
&c(\Delta u' + \lambda \operatorname{grad}\theta') + a_2(\Delta u'' + \lambda \operatorname{grad}\theta'') = 0.
\end{aligned}$$

Hence, again taking into account that $a_1a_2 - c^2 > 0$, we find

$$\Delta u' + \lambda \operatorname{grad}\theta' = 0, \quad \Delta u'' + \lambda \operatorname{grad}\theta'' = 0, \tag{1.11}$$

i.e., we have splitted the first boundary value problem. For u' and u'' the splitted problems are investigated by the same technique as the first boundary value problem of statics of an isotropic elastic body. Since $1+\lambda = \frac{a_1+b_1}{a_1} = \frac{a_2+b_2}{a_2} > 0$, equations (1.11) are elliptic.

Thus we have shown that ε_2 and ε_3 do not vanish simultaneously.

In what follows it will be assumed that $\varepsilon_2 \neq 0$. This assumption can be made without loss of generality. Indeed, if $\varepsilon_3 \neq 0$ and $\varepsilon_2 = 0$, then equation

(1.9) has only one root, which is not sufficient for our further investigation. So we have to combine (1.1) and (1.5) as follows:

$$u_3 + Yu_1 + i(u_4 + Yu_2).$$

By repeating the above arguments we obtain the quadratic equation

$$\varepsilon_3 Y^2 - (\varepsilon_4 - \varepsilon_1)Y = 0,$$

which yields

$$Y_1 = 0, \quad Y_2 = \frac{\varepsilon_4 - \varepsilon_1}{\varepsilon_3}.$$

Thus we have derived two roots, which enables us to accomplish our task.

It is important that the roots of equation (1.9) be real values. The discriminant of equation (1.9) can be written as

$$(\varepsilon_4 - \varepsilon_1)^2 + 4\varepsilon_2\varepsilon_3 = \frac{4}{\delta_0^2 a_1 a_2}\big\{[a_2(da_1 - cb_1) + a_1(da_2 - cb_2)]^2 + d_2(a_1b_2 - a_2b_1)^2\big\}.$$

The latter expression vanishes only if conditions (1.10) are fulfilled. In what follows this trivial case will be omitted.

Thus equation (1.9) has two different real roots:

$$\begin{aligned} X_1 &= \frac{\varepsilon_4 - \varepsilon_1 + \sqrt{(\varepsilon_4 - \varepsilon_1)^2 + 4\varepsilon_2\varepsilon_3}}{2\varepsilon_2}, \\ X_2 &= \frac{\varepsilon_4 - \varepsilon_1 - \sqrt{(\varepsilon_4 - \varepsilon_1)^2 + 4\varepsilon_2\varepsilon_3}}{2\varepsilon_2}. \end{aligned} \tag{1.12}$$

Rewriting condition (1.5) for X_1 and X_2 separately, we have

$$\begin{aligned} & u_1 + X_1u_3 + i(u_2 + X_1u_4) = (m_1 + X_1m_2)\varphi_1(z) + \\ & \quad + (m_2 + X_1m_3)\varphi_2(z) - k_1 z\big[(m_1 + X_1m_2)\overline{\varphi_1'(z)} + \\ & \quad + (m_2 + X_1m_3)\overline{\varphi_2'(z)}\big] + \overline{\psi_1(z)} + X_1\overline{\psi_2(z)}, \\ & u_1 + X_2u_3 + i(u_2 + X_2u_4) = (m_1 + X_2m_2)\varphi_1(z) + \\ & \quad + (m_2 + X_2m_3)\varphi_2(z) - k_2 z\big[(m_1 + X_2m_2)\overline{\varphi_1'(z)} + \\ & \quad + (m_2 + X_2m_3)\overline{\varphi_2'(z)}\big] + \overline{\psi_1(z)} + X_2\overline{\psi_2(z)}, \end{aligned} \tag{1.13}$$

where we have introduced the notation

$$k_j = -\frac{l_4 + X_j l_5}{2(m_1 + X_j m_2)}, \quad j = 1, 2. \tag{1.14}$$

By virtue of (1.7) and (1.9) we readily obtain

$$k_j = \varepsilon_1 + X_j\varepsilon_2, \quad j = 1, 2.$$

Therefore k_1 and k_2 have the form

$$\begin{aligned} 2k_1 &= \varepsilon_1 + \varepsilon_4 + \sqrt{(\varepsilon_4 - \varepsilon_1)^2 + 4\varepsilon_2\varepsilon_3}, \\ 2k_2 &= \varepsilon_1 + \varepsilon_4 - \sqrt{(\varepsilon_4 - \varepsilon_1)^2 + 4\varepsilon_2\varepsilon_3}. \end{aligned} \tag{1.15}$$

Introducing the notation

$$u_1 + X_j u_3 = v_1^{(j)}, \quad u_2 + X_j u_4 = v_2^{(j)}, \tag{1.16}$$

$$\begin{gathered} (m_1 + X_j m_2)\varphi_1(z) + (m_2 + X_j m_3)\varphi_2(z) = \Phi_j(z), \\ \psi_1(z) + X_j \psi_2(z) = \Psi_j(z), \quad j = 1, 2, \end{gathered} \tag{1.17}$$

we can rewrite (1.13) as

$$v_1^{(j)} + i v_2^{(j)} = \Phi_j(z) - k_j z \overline{\Phi_j'(z)} + \overline{\Psi_j(z)}, \quad j = 1, 2, \tag{1.18}$$

where $\Phi_1(z)$, $\Phi_2(z)$, $\Psi_1(z)$, $\Psi_2(z)$ are new analytic functions.

Note that structurally (1.18) coincides with the general Kolosov–Muskhelishvili representation for displacement vector components.

It is obvious by (1.16) that if u_1, u_2, u_3, u_4 are given, this will mean that $v_1^{(j)}$ and $v_2^{(j)}$, $j = 1, 2$, are given, too. It is likewise obvious that if $v_1^{(j)}$ and $v_2^{(j)}$, $j = 1, 2$, are found, then

$$\begin{aligned} u_1 &= \frac{-X_2 v_1^{(1)} + X_1 v_1^{(2)}}{X_1 - X_2}, & u_2 &= \frac{-X_2 v_2^{(1)} + X_1 v_2^{(2)}}{X_1 - X_2} \\ u_3 &= \frac{v_1^{(1)} - v_1^{(2)}}{X_1 - X_2}, & u_4 &= \frac{v_2^{(1)} - v_2^{(2)}}{X_1 - X_2}. \end{aligned} \tag{1.19}$$

(1.17) immediately implies that when $\Phi_j(z)$ and $\Psi_j(z)$, $j = 1, 2$, are known, we can define $\varphi_1(z)$, $\varphi_2(z)$, $\psi_1(z)$, $\psi_2(z)$ uniquely and write them as

$$\begin{aligned} \varphi_1(z) &= \frac{-(m_2 + X_2 m_3)\Phi_1(z) + (m_2 + X_1 m_3)\Phi_2(z)}{(X_1 - X_2)\Delta_0}, \\ \varphi_2(z) &= \frac{(m_1 + X_2 m_2)\Phi_1(z) - (m_1 + X_1 m_2)\Phi_2(z)}{(X_1 - X_2)\Delta_0}, \\ \psi_1(z) &= \frac{-X_2 \Psi_1 + X_1 \Psi_2}{X_1 - X_2}, \quad \psi_2(z) = \frac{\Psi_1 - \Psi_2}{X_1 - X_2}. \end{aligned} \tag{1.20}$$

Next, we shall show which equation is satisfied by the vector $v^{(j)} = (v_1^{(j)}, v_2^{(j)})$, $j = 1, 2$. Let this equation have the form

$$\Delta v^{(j)} + M_j \operatorname{grad} \operatorname{div} v^{(j)} = 0, \quad j = 1, 2, \tag{1.21}$$

and define M_j depending on k_j.

By simple calculations we find from (1.18) that

$$\operatorname{div} v^{j)} = 2(1-k_j)\operatorname{Re}\Phi_j'(z), \quad \Delta v_1^{(j)} = -4k_j \operatorname{Re}\Phi_j''(z), \quad \Delta v_2^{(j)} = 4k_j \operatorname{Im}\Phi_j''(z).$$

After substituting the latter expression into (1.21) we obtain

$$M_j = \frac{2k_j}{1-k_j}, \quad j=1,2. \tag{1.22}$$

Therefore we have proved that the vector $v^{(j)} = (v_1^{(j)}, v_2^{(j)})$ defined by (1.18) satisfies equation (1.21) if M_j is given by (1.22). It will be shown below that $|k_j| < 1$, $1+M_j > 0$, $j=1,2$. The latter inequality is a sufficient condition for system (1.21) to be elliptic.

Thus in the theory of elastic mixtures the first boundary value problem for an equation of statics is splitted, in the general case, into two problems for equation (1.21) which structurally coincides with an equation of statics of an isotropic elastic body.

Similarly to the first boundary value problem for an equation of statics of an isotropic elastic body [3] we can derive here an integral Fredholm equation of second order for equation (1.21) using the boundary conditions

$$v_1^{(j)} + iv_2^{(j)} = F_j(t), \quad t \in S, \quad j=1,2. \tag{1.23}$$

Indeed, if we define the functions $\Phi_j(z)$ and $\Psi_j(z)$, $j=1,2$, from (1.18) by means of the potentials

$$\begin{aligned} \Phi_j(z) &= \frac{1}{2\pi i}\int_S g_j(\zeta)\frac{\partial \ln\sigma}{\partial s(y)}ds, \\ \Psi_j(z) &= \frac{1}{2\pi i}\int_S \overline{g}_j(\zeta)\frac{\partial \ln\sigma}{\partial s(y)}ds - \frac{k_j}{2\pi i}\int_S g_j(\zeta)\frac{\partial}{\partial s(y)}\frac{\overline{\zeta}}{\sigma}ds, \end{aligned} \tag{1.24}$$

then

$$\overline{\Phi_j'(z)} = -\frac{1}{2\pi i}\int_S \overline{g_j(\zeta)}\frac{\partial}{\partial s(y)}\frac{1}{\overline{\sigma}}ds$$

and

$$v_1^{(j)} + iv_2^{(j)} = \frac{1}{2\pi i}\int_S g_j(\zeta)\,\frac{\partial}{\partial s(y)}\,\ln\frac{\sigma}{\overline{\sigma}}ds + \frac{k_j}{2\pi i}\int_S \overline{g_j(\zeta)}\,\frac{\partial}{\partial s(y)}\,\frac{\sigma}{\overline{\sigma}}ds. \tag{1.25}$$

In (1.24) and (1.25) $g_j(\zeta)$ are the desired functions of the point $\zeta = y_1 + iy_2$, where y_1 and y_2 are the coordinates of the point $y \in s$, $\sigma = z - \zeta$, $\overline{\sigma} = \overline{z} - \overline{\zeta}$, $z = x_1 + ix_2$, and

$$\frac{\partial}{\partial s(y)} = n_1(y)\frac{\partial}{\partial y_2} - n_2(y)\frac{\partial}{\partial y_1}, \tag{1.26}$$

where $n = (n_1(y), n_2(y))$ is the external (with respect to the finite domain D^+) normal unit vector at the point y, while s is the Lyapunov curve.

Passing to the limit as $z \to t \in s$, externally or internally, to define g_j we obtain the integral Fredholm equation of second kind

$$\pm g_j(t) + \frac{1}{2\pi i} \int_S g_j(\zeta) \frac{\partial}{\partial s(y)} \ln \frac{t-\zeta}{\overline{t}-\overline{\zeta}} ds + \\ + \frac{k_j}{2\pi i} \int_S \overline{g_j(\zeta)} \frac{\partial}{\partial s(y)} \frac{t-\zeta}{\overline{t}-\overline{\zeta}} ds = F_j(t), \tag{1.27}$$

where

$$F_j(t) = f_1 + X_j f_3 + i(f_2 + X_j f_4), \quad j = 1, 2, \tag{1.28}$$

and $f_k(t) = (u_k)^{\pm}, \quad k = \overline{1, 4}$.

Equations (1.27) have a simple form and are very helpful both for theoretical investigations and for an effective solution of the first boundary value problem. These equations actually coincide with the Sherman–Lauricella equation [3].

Let us investigate the parameters k_1 and k_2 in (1.27) which are defined by (1.15).

Formula (1.15) gives rise to

$$k_1 + k_2 = \varepsilon_1 + \varepsilon_4, \quad k_1 k_2 = \varepsilon_1 \varepsilon_4 - \varepsilon_2 \varepsilon_3, \tag{1.29}$$

which implies

$$\begin{aligned} &1 - k_1 + 1 - k_2 = 2 - \varepsilon_1 - \varepsilon_4, \\ &(1 - k_1)(1 - k_2) = 1 - \varepsilon_1 - \varepsilon_4 + \varepsilon_1 \varepsilon_4 - \varepsilon_2 \varepsilon_3. \end{aligned} \tag{1.30}$$

By easy calculations it follows from (1.8) that

$$\begin{aligned} &\delta_0(\varepsilon_1 + \varepsilon_4) = 2(a_1 b_2 + a_2 b_1 - 2cd + b_1 b_2 - d^2), \\ &\delta_0(\varepsilon_1 \varepsilon_4 - \varepsilon_2 \varepsilon_3) = b_1 b_2 - d^2. \end{aligned} \tag{1.31}$$

Substituting the latter expression into (1.30) and performing some simple transformations, we obtain

$$\begin{aligned} 1 - k_1 + 1 - k_2 = \frac{2}{\delta_0} \Big\{ & 4\Delta_1 - 3\lambda_5(\mu_1 + \mu_2 + 2\mu_3) + \\ & + \frac{1}{a_1(b_2 - \lambda_5)} \big[a_1(b_2 - \lambda_5) - c(d + \lambda_3)\big]^2 + \\ & + \frac{d_2(b_1 - \lambda_5)(b_2 - \lambda_5) + c^2[(b_1 - \lambda_5)(b_2 - \lambda_5) - (d + \lambda_5)^2]}{a_1(b_2 - \lambda_5)} \Big\}, \\ (1 - k_1)(1 - k_2) = 1 - & \frac{2(a_1 b_2 + a_2 b_1 - 2cd + b_1 b_2 - d^2)}{\delta_0} + \\ & + \frac{b_1 b_2 - d^2}{\delta_0} = \frac{4 d_2}{\delta_0}. \end{aligned} \tag{1.32}$$

Since the potential energy of an elastic mixture is positive definite [2], we have

$$\begin{aligned}&\Delta_1=\mu_1\mu_2-\mu_3^2>0,\quad \lambda_5<0,\quad \mu_1+\mu_2+2\mu_3>0,\\&d_2=a_1a_2-c^2=\Delta_1-\lambda_5(\mu_1+\mu_2+2\mu_3)>0,\\&a_1>0,\quad b_2-\lambda_5>0,\quad (b_1-\lambda_5)(b_2-\lambda_5)-(d+\lambda_5)^2>0.\end{aligned}\tag{1.33}$$

With (1.33) taken into account, (1.32) implies that the sum and the product of two values $1-k_1$ and $1-k_2$ are greater than zero. Hence we conclude that each value is positive.

Therefore

$$k_1<1,\quad k_2<1.\tag{1.34}$$

In quite a similar manner, with (1.15) taken into account, we obtain

$$\begin{aligned}&1+k_1+1+k_2=2+\varepsilon_1+\varepsilon_4,\\&(1+k_1)(1+k_2)=1+\varepsilon_1+\varepsilon_4+\varepsilon_1\varepsilon_4-\varepsilon_2\varepsilon_3.\end{aligned}$$

Now using (1.31) and performing cumbersome calculations, we derive

$$\begin{aligned}1+k_1+1+k_2=\frac{2}{\delta_0}\Big\{&4\Delta_1-\lambda_5(\mu_1+\mu_2+2\mu_3)+\\&+2\big[(b_1-\lambda_5)(b_2-\lambda_5)-(d+\lambda_5)^2\big]+\\&+\frac{1}{(3\mu_1-\lambda_5)(b_2-\lambda_5)}\big[(3\mu_1-\lambda_5)(b_2-\lambda_5)-(3\mu_3+\lambda_5)(d+\lambda_5)\big]^2+\\&+\frac{[(b_1-\lambda_5)(b_2-\lambda_5)-(d+\lambda_5)^2](3\mu_1-\lambda_5)(3\mu_2-\lambda_5)}{(3\mu_1-\lambda_5)(b_2-\lambda_5)}+\\&+3(d+\lambda_5)^2\big[3\Delta_1-\lambda_5(\mu_1+\mu_2+2\mu_3)\big]\Big\},\\(1+k_1)(1+k_2)=\frac{4}{\delta_0\mu_1}\Big\{&(a_1+b_1)\Delta_1+\frac{\mu_1(b_2-\lambda_5)+\mu_3^2}{b_2-\lambda_5}\times\\&\times\big[(b_1-\lambda_5)(b_2-\lambda_5)-(d+\lambda_3)^2\big]+\frac{1}{b_2-\lambda_5}\big[\mu_1(b_2-\lambda_5)-\mu_3(d+\lambda_5)\big]^2\Big\}.\end{aligned}$$

Hence, as above, we conclude that $1+k_1$ and $1+k_2$ are greater than zero, i. e., which together with (1.34) leads to

$$-1<k_j<1,\quad j=1,2.\tag{1.35}$$

It is interesting to note that for the parameter k_1 one can obtain a more narrow change interval. Indeed, by virtue of (1.32) formula (1.22) for $j=1$

can be rewritten as

$$M_1 = \frac{2k_1(1-k_2)\delta_0}{4d_2}.$$

By virtue of (1.15) and (1.31) the latter formula takes the form

$$M_1 = \frac{a_1b_2 + a_2b_1 - 2cd}{2d_2} + \frac{\delta_0}{4d_2}\sqrt{(\varepsilon_4-\varepsilon_1)^2 + 4\varepsilon_2\varepsilon_3},$$

which implies

$$M_1 + \frac{1}{2} = \frac{a_1(b_2-\lambda_5) + a_2(b_1-\lambda_5) - 2c(d+\lambda_5)\Delta_1}{2d_2} + \\ + \frac{\delta_0}{4d_2}\sqrt{(\varepsilon_4-\varepsilon_1)^2 + 4\varepsilon_2\varepsilon_3} > 0.$$

Thus we have obtained

$$M_1 > -\frac{1}{2}. \tag{1.36}$$

Using now (1.22), it is easy to establish that $k_1 > -\frac{1}{3}$ and thus we obtain the interval

$$k_1 \in \Big] -\frac{1}{3}, 1\Big[. \tag{1.37}$$

Note that though inequality (1.36) does not hold for M_2, we can rewrite M_2 similarly to M_1 as follows:

$$M_2 = \frac{a_1b_2 + a_2b_1 - 2cd}{2d_2} - \frac{\delta_0}{4d_2}\sqrt{(\varepsilon_4-\varepsilon_1)^2 + 4\varepsilon_2\varepsilon_3}.$$

Applying (1.35), we find from (1.22) that

$$1 + M_j = \frac{1+k_j}{1-k_j} > 0, \quad j = 1, 2.$$

Therefore equation (1.21) is elliptic.

Let us now rewrite (1.21) as

$$\Delta v + M \operatorname{grad}\operatorname{div} v = 0. \tag{1.38}$$

where $v = v^{(j)}$, $j = 1, 2$, when $M = M_j$.

For equation (1.38) we introduce the generalized stress vector [2]:

$$\overset{\varkappa}{T}v = (1+\varkappa)\frac{\partial v}{\partial n} + (M-\varkappa)n \operatorname{div} v + \varkappa s\Big(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big),$$

where $n = (n_1, n_2)$ is an arbitrary unit vector, $s = -(n_2, n_1)$, and $\varkappa$ an arbitrary constant.

For the generalized stress vector we choose a particular case with $\varkappa = \frac{M}{M+2}$.

Then $\overset{\varkappa}{T} \equiv N$ and the Green formula for the finite domain D^+ and the infinite domain D^- will respectively have the form

$$\int_{D^+} N(v,v)dy_1dy_2 = \int_S vNv\;ds, \tag{1.39}$$

$$\int_{D^-} N(v,v)dy_1dy_2 = -\int_S vNv\;ds, \tag{1.40}$$

where

$$N(v,v) = \frac{(1+M)^2}{M+2}\Big(\frac{\partial v_1}{\partial y_1}+\frac{\partial v_2}{\partial y_2}\Big)^2 + \frac{1}{M+2}\Big(\frac{\partial v_2}{\partial y_1}-\frac{\partial v_1}{\partial y_2}\Big)^2 + \\ +\frac{M+1}{M+2}\Big[\Big(\frac{\partial v_1}{\partial y_2}+\frac{\partial v_2}{\partial y_1}\Big)^2 + \Big(\frac{\partial v_1}{\partial y_1}-\frac{\partial v_2}{\partial y_2}\Big)^2\Big]. \tag{1.41}$$

For equation (1.38) $1+M>0$, which means that $N(v,v)$ defined by (1.41) is positive definite.

Formulae (1.39) and (1.40) hold when v is a regular vector [2]. Moreover, for the infinite domain D^- the vector v satisfies the conditions

$$v = O(1), \quad \frac{\partial v}{\partial y_k} = O(R^{-2}), \quad k=1,2,$$

where $R^2 = y_1^2 + y_2^2$.

The operator N plays an important role in the investigation of the first boundary value problem.

Our further discussion and effective solution of the first boundary value problem proceed exactly in the same way as for an equation of statics of an isotropic elastic body.

§ 2. Second Boundary Value Problem

The second boundary value problem is investigated with the vector Tu given on the boundary. The projections of this vector are defined as follows [1]:

$$\begin{aligned} (Tu)_2 - i(Tu)_1 &= \frac{\partial}{\partial s(x)}\Big\{(A_1-2)\varphi_1(z) + A_2\varphi_2(z) + \\ &\quad + z\big[B_1\overline{\varphi_1'(z)} + B_2\overline{\varphi_2'(z)}\big] + 2\mu_1\overline{\psi_1(z)} + 2\mu_3\overline{\psi_2(z)}\Big\}, \\ (Tu)_4 - i(Tu)_3 &= \frac{\partial}{\partial s(x)}\Big\{A_3\varphi_1(z) + (A_4-2)\varphi_2(z) + \\ &\quad + z\big[B_3\overline{\varphi_1'(z)} + B_4\overline{\varphi_2'(z)}\big] + 2\mu_3\overline{\psi_1(z)} + 2\mu_2\overline{\psi_2(z)}\Big\}, \end{aligned} \tag{2.1}$$

where $\varphi_k(z)$ and $\psi_k(z)$ $(k=1,2)$ are arbitrary analytic functions; the constants A_k and B_k $(k=\overline{1,4})$ have the values:

$$\begin{aligned}
&A_1=2(\mu_1 m_1+\mu_3 m_2)=2+B_1+2\lambda_5\frac{a_2+c}{d_2}, \quad B_1=\mu_1 l_4+\mu_3 l_5,\\
&A_2=2(\mu_1 m_2+\mu_3 m_3)=B_2-2\lambda_5\frac{a_1+c}{d_2}, \quad B_2=\mu_1 l_5+\mu_3 l_6,\\
&A_3=2(\mu_3 m_1+\mu_2 m_2)=B_3-2\lambda_5\frac{a_2+c}{d_2}, \quad B_3=\mu_3 l_4+\mu_2 l_5,\\
&A_4=2(\mu_3 m_2+\mu_2 m_3)=2+B_4+2\lambda_5\frac{a_1+c}{d_2}, \quad B_4=\mu_3 l_5+\mu_2 l_6,
\end{aligned} \tag{2.2}$$

the operator $\frac{\partial}{\partial s(x)}$ is defined by (1.26).

Representations (2.1) can be rewritten equivalently as

$$\begin{aligned}
&F_2-iF_1+c_1=(A_1-2)\varphi_1(z)+A_2\varphi_2(z)+\\
&\quad +z\big[B_1\overline{\varphi_1'(z)}+B_2\overline{\varphi_2'(z)}\big]+2\mu_1\overline{\psi_1(z)}+2\mu_3\overline{\psi_2(z)},\\
&F_4-iF_3+c_2=A_3\varphi_1(z)+(A_4-2)\varphi_2(z)+\\
&\quad +z\big[B_3\overline{\varphi_1'(z)}+B_4\overline{\varphi_2'(z)}\big]+2\mu_3\overline{\psi_1(z)}+2\mu_2\overline{\psi_2(z)},
\end{aligned} \tag{2.3}$$

where c_1 and c_2 are arbitrary constants and

$$F_k=\int_0^{S(x)}(Tu)_k ds, \quad k=\overline{1,4}. \tag{2.4}$$

Now we combine (2.3) as follows:

$$\begin{aligned}
F_2+XF_4-i(F_1+XF_3)+c_1+Xc_2=(A_1-2+XA_3)\varphi_1(z)+\\
+[A_2+X(A_4-2)]\varphi_2(z)+z\big[(B_1+XB_3)\overline{\varphi_1'(z)}+\\
+(B_2+XB_4)\overline{\varphi_2'(z)}\big]+2(\mu_1+X\mu_3)\overline{\psi_1(z)}+2(\mu_3+X\mu_2)\overline{\psi_2(z)},
\end{aligned} \tag{2.5}$$

where X is the unknown constant value.

Define X by the equation

$$\frac{B_2+XB_4}{B_1+XB_3}=\frac{A_2+X(A_4-2)}{A_1-2+XA_3}. \tag{2.6}$$

Using the notation [2]

$$\begin{aligned}
&H_1=B_1(2-A_4)+B_2A_3, \quad H_2=B_1A_2+B_2(2-A_1),\\
&H_3=B_3(2-A_4)+B_4A_3, \quad H_4=B_3A_2+B_4(2-A_1),
\end{aligned} \tag{2.7}$$

we can rewrite (2.6) as

$$H_3X^2-(H_4-H_1)X-H_2=0. \tag{2.8}$$

By substituting the coefficients from (2.2) and (1.2) into (2.7) we obtain, for H_k $(k=\overline{1,4})$, the new expressions

$$\begin{aligned} H_1 &= -\Delta_2 + \frac{2\lambda_5}{d_2}\big[(a_1+c)A_3 + (a_2+c)(A_4-2)\big],\\ H_2 &= \frac{2\lambda_5}{d_2}\big[(a_1+c)(2-A_1) - (a_2+c)A_2\big],\\ H_3 &= \frac{2\lambda_5}{d_2}\big[(a_2+c)(2-A_4) - (a_1+c)A_3\big],\\ H_4 &= -\Delta_2 - \frac{2\lambda_5}{d_2}\big[(a_1+c)(2-A_1) - (a_2+c)A_2\big], \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} &\Delta_2 = (2-A_1)(2-A_4) - A_2A_3,\\ &H_1+H_3 = -\Delta_2, \quad H_2+H_4 = -\Delta_2. \end{aligned} \tag{2.10}$$

We shall show that the condition $\Delta_2 \neq 0$ is fulfilled. To this end, using (2.2), we rewrite Δ_2 as

$$\begin{aligned} \Delta_2 &= \Big(B_1 + 2\lambda_5\frac{a_2+c}{d_2}\Big)\Big(B_4 + 2\lambda_5\frac{a_1+c}{d_2}\Big) - \\ &\quad -\Big(B_2 - 2\lambda_5\frac{a_1+c}{d_2}\Big)\Big(B_3 - 2\lambda_5\frac{a_2+c}{d_2}\Big) = \\ &= B_1B_4 - B_2B_3 + \frac{2\lambda_5}{d_2}\big[(a_1+c)(B_1+B_3) + (a_2+c)(B_2+B_4)\big]. \end{aligned} \tag{2.11}$$

Note that, by virtue of (1.2), from (2.2) we readily obtain

$$B_1B_4 - B_2B_3 = \Delta_1\frac{b_1b_2-d^2}{d_1d_2}, \tag{2.12}$$

$$\begin{aligned} B_1+B_3 &= -\frac{1}{d_1}(b_1b_2 - d^2 + a_2b_1 - cd + da_2 - cb_2),\\ B_2+B_4 &= -\frac{1}{d_1}(b_1b_2 - d^2 + a_1b_2 - cd + da_1 - cb_1). \end{aligned} \tag{2.13}$$

Using the identity

$$(a_1+c)(a_2b_1-cd+da_2-cb_2)+(a_2+c)(a_1b_2-cd+da_1-cb_1) = d_2(b_1+b_2+2d),$$

which is easy to prove, and taking into account (2.12) and (2.13), expression (2.11) can be rewritten as

$$\begin{aligned} \Delta_2d_1d_2 &= \big[\Delta_1 - 2\lambda_5(a_1+a_2+2c)\big](b_1b_2-d^2) - 2\lambda_5d_2(b_1+b_2+2d) = \\ &= \big[\Delta_1 - 2\lambda_5(a_1+a_2+2c)\big]\big[(b_1-\lambda_5)(b_2-\lambda_5) - (d+\lambda_5)^2\big] - \\ &\quad -\lambda_5(b_1+b_2+2d)\Delta_1. \end{aligned} \tag{2.14}$$

Since $d_1 > 0$, $d_2 > 0$, $\lambda_5 < 0$, $a_1 + a_2 + 2c \equiv \mu_1 + \mu_2 + 2\mu_3 > 0$, $(b_1-\lambda_5)(b_2-\lambda_5)-(d+\lambda_5)^2 > 0$ and $b_1+b_2+2d \equiv b_1-\lambda_5+b_2-\lambda_5+2(d+\lambda_5) > 0$, we have $\Delta_2 > 0$.

Now we shall show that H_2 and H_3 do not vanish simultaneously. Indeed, if it is assumed that H_2 and H_3 vanish simultaneously, then for $\lambda_5 \neq 0$ and $d_2 \neq 0$ (2.9) will imply $(a_1+c)(2-A_1)-(a_2+c)A_2 = 0$, $(a_2+c)(2-A_4)-(a_1+c)A_3 = 0$, i. e.,

$$\frac{a_2+c}{a_1+c} = \frac{2-A_1}{A_2} = \frac{A_3}{2-A_4}.$$

Hence we conclude that $\Delta_2 = 0$, which is impossible.

Without loss of generality we can take $H_3 \neq 0$. This can be shown in the same manner as a similar statement in §1.

By solving equation (2.8) we obtain

$$X_1 = 1, \quad X_2 = -\frac{H_2}{H_3} = H_0. \tag{2.15}$$

Rewriting now (2.5) for X_1 and X_2 separately, we have

$$F_2+F_4-i(F_1+F_3)+c_1+c_2=\Phi(z)+z\overline{\Phi'(z)}+\overline{\Psi(z)}, \tag{2.16}$$

$$F_2+H_0F_4-i(F_1+H_0F_3)+c_1+H_0c_2=\Phi_0(z)+k_0z\overline{\Phi_0'(z)}+\overline{\Psi_0(z)}, \tag{2.17}$$

where

$$k_0 = \frac{B_1+H_0B_3}{A_1-2+H_0A_3}, \tag{2.18}$$

and the functions $\Phi(z)$, $\Psi(z)$, $\Phi_0(z)$, $\Psi_0(z)$ are defined as

$$\begin{aligned} &\Phi(z) = (B_1+B_3)\varphi_1(z) + (B_2+B_4)\varphi_2(z), \\ &\Phi_0(z)=(A_1-2+H_0A_3)\varphi_1(z)+(A_2+H_0A_4-2H_0)\varphi_2(z), \end{aligned} \tag{2.19}$$

$$\begin{aligned} &\Psi(z) = 2(\mu_1+\mu_3)\psi_1(z) + 2(\mu_3+\mu_2)\psi_2(z), \\ &\Psi_0(z) = 2(\mu_1+H_0\mu_3)\psi_1(z) + 2(\mu_3+H_0\mu_2)\psi_2(z). \end{aligned} \tag{2.20}$$

We next substitute the value of H_0 from (2.15) and the values of H_2 and H_3 from (2.7) into (2.18). After performing some simple transformations we obtain

$$k_0 = \frac{\Delta_1(b_1b_2-d^2)}{\Delta_2d_1d_2}. \tag{2.21}$$

Let us show that the value of k_0 lies in the interval $]-1,1[$. Indeed, using (2.14), from (2.21) we have

$$1-k_0 = \frac{\Delta_2d_1d_2-\Delta_1(b_1b_2-d^2)}{\Delta_2d_1d_2} =$$

$$=-2\lambda_5\frac{(b_1+b_2+2d)\Delta_1+(a_1+a_2+2c)[(b_1-\lambda_5)(b_2-\lambda_5)-(d+\lambda_5)^2]}{\Delta_2 d_1 d_2}>0.$$

Hence $k_0 < 1$. (2.21) now readily implies

$$1+k_0=\frac{2}{\Delta_2 d_1}\big[(b_1-\lambda_5)(b_2-\lambda_5)-(d+\lambda_5)^2\big].$$

Thus we have found that

$$-1 < k_0 < 1. \tag{2.22}$$

Note that $H_0 \neq 1$. Indeed, if $H_0 = 1$, then (2.18) implies $k_0 = 1$, which is impossible because (2.22) holds.

After $\Phi(z)$, $\Phi_0(z)$, $\Psi(z)$ and $\Psi_0(z)$ are found, using (2.19) and (2.20) we define $\varphi_k(z)$ and $\psi_k(z)$, $k = 1, 2$, uniquely, since the determinants of the respective transformations are equal to $(H_0 - 1)\Delta_2$ and $(H_0 - 1)\Delta_1$.

Applying arguments similar to those used for representation (1.18), the desired functions $\Phi(z)$, $\Psi(z)$, $\Phi_0(z)$ and $\Psi_0(z)$ from (2.16) and (2.17) must be sought for in the form

$$\begin{aligned}
\Phi(z) &= \frac{1}{2\pi i}\int_S g(\zeta)\frac{\partial \ln\sigma}{\partial s(y)}ds,\\
\Psi(z) &= \frac{1}{2\pi i}\int_S \overline{g(\zeta)}\frac{\partial \ln\sigma}{\partial s(y)}ds+\frac{1}{2\pi i}\int_S g(\zeta)\frac{\partial}{\partial s(y)}\frac{\overline{\zeta}}{\sigma}ds,\\
\Phi_0(z) &= \frac{1}{2\pi i}\int_S g_0(\zeta)\frac{\partial \ln\sigma}{\partial s(y)}ds,\\
\Psi_0(z) &= \frac{1}{2\pi i}\int_S \overline{g_0(\zeta)}\frac{\partial \ln\sigma}{\partial s(y)}ds+\frac{k_0}{2\pi i}\int_S g_0(\zeta)\frac{\partial}{\partial s(y)}\frac{\overline{\zeta}}{\sigma}ds.
\end{aligned}$$

In that case (2.16) and (2.17) can be rewritten as

$$\begin{aligned}
&F_2+F_4-i(F_1+F_3)+c_1+c_2=\frac{1}{2\pi i}\int_S g(\zeta)\frac{\partial}{\partial s(y)}\ln\frac{\sigma}{\overline{\sigma}}ds-\\
&\qquad-\frac{1}{2\pi i}\int_S \overline{g(\zeta)}\frac{\partial}{\partial s(y)}\frac{\sigma}{\overline{\sigma}}ds,\\
&F_2+H_0F_4-i(F_1+H_0F_3)+c_1+H_0c_2=\\
&\qquad=\frac{1}{2\pi i}\int_S g_0(\zeta)\frac{\partial}{\partial s(y)}\ln\frac{\sigma}{\overline{\sigma}}ds-\frac{k_0}{2\pi i}\int_S \overline{g_0(\zeta)}\frac{\partial}{\partial s(y)}\frac{\sigma}{\overline{\sigma}}ds.
\end{aligned} \tag{2.23}$$

Now passing to the limit in (2.23) as $z \to t \in s$, internally or externally, to define g and g_0 we obtain the integral Fredholm equations of second order:

$$\pm g(t)+\frac{1}{2\pi i}\int_S g(\zeta)\frac{\partial}{\partial s(y)}\ln\frac{t-\zeta}{\overline{t}-\overline{\zeta}}ds-$$

$$-\frac{1}{2\pi i}\int_S \overline{g(\zeta)}\frac{\partial}{\partial s(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}ds = f(t), \tag{2.24}$$

$$\pm g_0(t) + \frac{1}{2\pi i}\int_S g_0(\zeta)\frac{\partial}{\partial s(y)}\ln\frac{t-\zeta}{\overline{t}-\overline{\zeta}}ds - $$
$$-\frac{k_0}{2\pi i}\int_S \overline{g_0(\zeta)}\frac{\partial}{\partial s(y)}\frac{t-\zeta}{\overline{t}-\overline{\zeta}}ds = F(t), \tag{2.25}$$

where

$$f(t) = \big[F_2 + F_4 - i(F_1+F_3)\big]^{\pm} + c_1 + c_2,$$
$$F(t) = \big[F_2 + H_0F_4 - i(F_1+H_0F_3)\big]^{\pm} + c_1 + H_0c_2.$$

One can investigate equations (2.24) and (2.25) in the same manner as the equation of the basic biharmonic problem and the first boundary value problem of statics of an isotropic elastic body [3].

Thus the second boundary value problem of statics in the theory of elastic mixtures is reduced to the second and the first plane boundary value problem of statics of an isotropic elastic body.

§ 3. Third Boundary Value Problem

As is known [2], the third boundary value problem is considered with the values u_3-u_1, u_4-u_2, $F_1+F_3+c_1$, $F_1+F_4+c_2$, given on the boundary, where c_1 and c_2 are arbitrary constants; u_1, u_2, u_3, u_4 are the projections of the four-dimensional vector u, and $F_k(x)$ $(k=\overline{1,4})$ is defined by (2.4).

By virtue of (1.1), (2.2) and (2.3) the conditions of the third boundary value problem can be written as follows:

$$\begin{aligned}
&u_3-u_1+i(u_4-u_2) = (m_2-m_1)\varphi_1(z) + (m_3-m_2)\varphi_2(z) + \\
&\quad + \frac{z}{2}\big[(l_5-l_4)\overline{\varphi_1'(z)} + (l_6-l_5)\overline{\varphi_2'(z)}\big] + \overline{\psi_2(z)} - \overline{\psi_1(z)}, \\
&F_2+F_4-i(F_1+F_3)+c_1+c_2 = (B_1+B_3)\varphi_1(z) + \\
&\quad + (B_2+B_4)\varphi_2(z) + z\big[(B_1+B_3)\overline{\varphi_1'(z)} + \\
&\quad + (B_2+B_4)\overline{\varphi_2'(z)}\big] + 2(\mu_1+\mu_3)\overline{\psi_1(z)} + 2(\mu_2+\mu_3)\overline{\psi_2(z)}.
\end{aligned} \tag{3.1}$$

Introducing the notattion

$$\begin{aligned}
&(m_2-m_1)\varphi_1(z) + (m_3-m_2)\varphi_2(z) = \Phi_3(z), \\
&\psi_2(z) - \psi_1(z) = \Psi_3(z), \\
&(B_1+B_3)\varphi_1(z) + (B_2+B_4)\varphi_2(z) = \Phi(z), \\
&2(\mu_1+\mu_3)\psi_1(z) + 2(\mu_2+\mu_3)\psi_2(z) = \Psi(z),
\end{aligned} \tag{3.2}$$

we obtain

$$\varphi_1(z)=\frac{(B_2+B_4)\Phi_3+(m_2-m_3)\Phi}{\Delta_3},$$
$$\varphi_2(z)=\frac{-(B_1+B_3)\Phi_3+(m_2-m_1)\Phi}{\Delta_3}, \tag{3.3}$$
$$\psi_1(z)=-\frac{\mu_2+\mu_3}{\beta}\Psi_3+\frac{1}{2\beta}\Psi,\quad \psi_2(z)=\frac{\mu_1+\mu_3}{\beta}\Psi_3+\frac{1}{2\beta}\Psi,$$

where

$$\Delta_3=2\Delta_0(\alpha-\beta),\quad \alpha=\frac{m_1+m_3-2m_2}{\Delta_0},\quad \beta=\mu_1+\mu_2+2\mu_3, \tag{3.4}$$

and $\Delta_0>0$ is given by (1.7)

Since $\Delta_0>0$ and $\Delta_1=\mu_1\mu_2-\mu_3^2>0$, the constants α and β are greater than zero. We shall show that $\Delta_3>0$. For this it is sufficient to prove that $\alpha-\beta>0$. By virtue of (3.4) and (1.2) we have

$$\alpha-\beta=\frac{a_1+a_2+2c}{2\Delta_0 d_2}+\frac{a_1+a_2+2c+b_1+b_2+2d}{2\Delta_0 d_1}-(a_1+a_2+2c)=$$
$$=\frac{1}{\delta_0}\big[2(a_1+a_2+2c)(d_1+d_2)+2(b_1+b_2+2d)d_2-\delta(a_1+a_2+2c)\big],$$

where δ_0 is defined by formula (1.8) which can be rewritten as

$$\delta_0=d_1+d_2+a_2(a_1+b_1)+a_1(a_2+b_2)-2c(c+d).$$

Substituting this value of δ_0 into the preceding formula, we obtain

$$\alpha-\beta=\frac{1}{\delta_0}\big[2(b_1+b_2+2d)d_2+(a_1+a_2+2c)(b_1b_2-d^2)\big].$$

Hence, after some simple transformations, we readily have

$$\alpha-\beta=\frac{1}{\delta_0}\big\{(a_1+a_2+2c)\big[(b_1-\lambda_5)(b_2-\lambda_5)-(d+\lambda_5)^2\big]+$$
$$+(b_1+b_2+2d)\big[2\Delta_1-\lambda_5(a_1+a_2+2c)\big]\big\}.$$

Thus we have shown that $\Delta_3>0$.

Now, using (3.3), (1.7) and (2.2), we obtain

$$(l_5-l_4)\overline{\varphi_1'(z)}+(l_6-l_5)\overline{\varphi_2'(z)}=-2k_3\overline{\Phi_3'(z)}-$$
$$-\frac{2\Delta_0}{\Delta_3}(\varepsilon_1+\varepsilon_3-\varepsilon_2-\varepsilon_4)\overline{\Phi'(z)}, \tag{3.5}$$

where

$$k_3=\frac{\beta(b_1b_2-d^2)}{2\Delta_3 d_1 d_2}. \tag{3.6}$$

Let us show that the parameter k_3 changes in the interval $]-1,1[$. Taking into account the fact that the inequality $\Delta_3>0$ holds, performing some obvious transformations and applying the formulae

$$1-k_3=\frac{b_1+b_2+2d}{\Delta_3 d_1}=\frac{(b_1+d)^2+(b_1-\lambda_5)(b_2-\lambda_5)-(d+\lambda_5)^2}{(b_1-\lambda_5)\Delta_3 d_1}>0,$$
$$1+k_3=\frac{\beta[(b_1-\lambda_5)(b_2-\lambda_5)-(d+\lambda_5)^2]+(b_1+b_2+2d)\Delta_1}{\Delta_3 d_1 d_2}>0,$$

we conclude that $-1<k_3<1$.

Taking into account (3.5) and substituting (3.2) into (3.1) we obtain

$$\begin{aligned}u_3-u_1+i(u_4-u_2)&=\Phi_3(z)-k_3 z\overline{\Phi_3'(z)}+\overline{\Psi_3(z)}-\\&-\frac{\Delta_0}{\Delta_3}(\varepsilon_1+\varepsilon_3-\varepsilon_2-\varepsilon_4)z\overline{\Phi'(z)},\\F_2+F_4-i(F_1+F_3)+c_1+c_2&=\Phi(z)+z\overline{\Phi'(z)}+\overline{\Psi(z)}.\end{aligned}\tag{3.7}$$

Applying the arguments of the preceding paragraphs, we must seek for the desired functions having the form

$$\begin{aligned}\Phi(z)&=\frac{1}{2\pi i}\int_S g(\zeta)\frac{\partial\ln\sigma}{\partial s(y)}ds,\\\Psi(z)&=\frac{1}{2\pi i}\int_S\overline{g(\zeta)}\frac{\partial\ln\sigma}{\partial s(y)}ds+\frac{1}{2\pi i}\int_S g(\zeta)\frac{\partial}{\partial s(y)}\frac{\overline{\zeta}}{\sigma}ds,\\\Phi_3(z)&=\frac{1}{2\pi i}\int_S g_3(\zeta)\frac{\partial\ln\sigma}{\partial s(y)}ds,\\\Psi_3(z)&=\frac{1}{2\pi i}\int_S\overline{g_3(\zeta)}\frac{\partial\ln\sigma}{\partial s(y)}ds+\frac{k_3}{2\pi i}\int_S g_3(\zeta)\frac{\partial}{\partial s(y)}\frac{\overline{\zeta}}{\sigma}ds,\\&+\frac{\Delta_0}{2\pi i\Delta_3}(\varepsilon_1+\varepsilon_3-\varepsilon_2-\varepsilon_4)\int_S g(\zeta)\frac{\partial}{\partial s(y)}\frac{\overline{\zeta}}{\sigma}ds.\end{aligned}\tag{3.8}$$

After substituting (3.8) into (3.7) and performing some simple transformations we obtain

$$\begin{aligned}u_3-u_1+i(u_4-u_2)&=\frac{1}{2\pi i}\int_S g_3(\zeta)\frac{\partial}{\partial s(y)}\ln\frac{\sigma}{\overline{\sigma}}ds+\frac{k_3}{2\pi i}\int_S\overline{g_3(\zeta)}\frac{\partial}{\partial s(y)}\frac{\sigma}{\overline{\sigma}}ds+\\&+\frac{\Delta_0}{2\pi i\Delta_3}(\varepsilon_1+\varepsilon_3-\varepsilon_2-\varepsilon_4)\int_S\overline{g(\zeta)}\frac{\partial}{\partial s(y)}\frac{\sigma}{\overline{\sigma}}ds,\\F_2+F_4-i(F_1+F_3)+c_1+c_2&=\frac{1}{2\pi i}\int_S g(\zeta)\frac{\partial}{\partial s(y)}\ln\frac{\sigma}{\overline{\sigma}}ds-\\&-\frac{1}{2\pi i}\int_S\overline{g(\zeta)}\frac{\partial}{\partial s(y)}\frac{\sigma}{\overline{\sigma}}ds.\end{aligned}$$

Passing to the limit in this formula as $z\to t\in s$, internally or externally, to define g_3 and g we obtain the integral Fredholm equations of second order

$$\pm g_3(t)+\frac{1}{2\pi i}\int_S g_3(\zeta)\frac{\partial}{\partial s(y)}\ln\frac{t-\zeta}{\bar t-\bar\zeta}ds+\frac{k_3}{2\pi i}\int_S\overline{g_3(\zeta)}\frac{\partial}{\partial s(y)}\frac{t-\zeta}{\bar t-\bar\zeta}ds+$$
$$+\frac{\Delta_0}{2\pi i\Delta_3}(\varepsilon_1+\varepsilon_3-\varepsilon_2-\varepsilon_4)\int_S\overline{g(\zeta)}\frac{\partial}{\partial s(y)}\frac{t-\zeta}{\bar t-\bar\zeta}ds=f(t), \tag{3.9}$$

$$\pm g(t)+\frac{1}{2\pi i}\int_S g(\zeta)\frac{\partial}{\partial s(y)}\ln\frac{t-\zeta}{\bar t-\bar\zeta}ds-$$
$$-\frac{1}{2\pi i}\int_S\overline{g(\zeta)}\frac{\partial}{\partial s(y)}\frac{t-\zeta}{\bar t-\bar\zeta}ds=F(t), \tag{3.10}$$

where

$$f(t)=\big[u_3-u_1+i(u_4-u_2)\big]^{\pm},\quad F(t)=\big[F_2+F_4-i(F_1+F_3)\big]^{\pm}+c_1+c_2.$$

Equation (3.10) is the integral Fredholm equation of the basic biharmonic problem. By solving this equation and substituting the found value of g into (3.9) we obtain the integral Fredholm equation with respect to the desired function g_3. This equation is investigated as the equation of the first boundary value problem of statics of an isotropic elastic body.

Thus in the theory of elastic mixtures the third boundary value problem of statics is splitted into two boundary value problems, of which one is the basic biharmonic problem and the other is the first boundary value problem of statics of an isotropic elastic body.

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