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ON THE ABSOLUTE SUMMABILITY OF SERIES WITH RESPECT TO BLOCK-ORTHONORMAL SYSTEMS

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Abstract. Theorems determining Weyl's multipliers for the summability almost everywhere by the $|c,1|$ method of the series with respect to block-orthonormal systems are proved. In particular, it is stated that if the sequence $\{\omega(n)\}$ is the Weyl multiplier for the summability almost everywhere by the $|c,1|$ method of all orthogonal series, then there exists a sequence $\{N_k\}$ such that $\{\omega(n)\}$ will be the Weyl multiplier for the summability almost everywhere by the $|c,1|$ method of all series with respect to the Δ_k-orthonormal systems.

The present paper deals with the summability almost everywhere (a.e.) by the $|c,\alpha|$ method of series with respect to block-orthonormal systems. Under the summability by the $|c,\alpha|$ method of the series

$$\sum_{n=1}^{\infty} a_n$$

is understood the convergence of the series

$$\sum_{n=1}^{\infty} \left|\sigma_{n+1}^{(\alpha)} - \sigma_n^{(\alpha)}\right|,$$

where

$$\sigma_n^{(\alpha)} = \frac{1}{A_n^\alpha} \sum_{k=1}^{\infty} A_{n-k}^\alpha \, a_k$$

are the Cesàro (c,α)-means.

The problem of the summability a.e. by the $|c,\alpha|$ method of orthogonal series was considered by P.L. Ul'yanov [1]. In particular, he proved that if

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the condition

$$\sum_{n=1}^{\infty} \frac{1}{n\,\omega(n)} < \infty, \tag{1}$$

is fulfilled for a positive nondecreasing sequence $\{\omega(n)\}$, then the convergence of the series

$$\sum_{n=1}^{\infty} a_n^2\,\omega(n)$$

guarantees the summability a.e. on $(0,1)$ by the $|c,\alpha|$ method ($\alpha > \frac{1}{2}$) of the series

$$\sum_{n=1}^{\infty} a_n\,\varphi_n(x) \tag{2}$$

for every orthonormal system from $L^2(0,1)$.

If however

$$\sum_{n=1}^{\infty} \frac{1}{n\,\omega(n)} = \infty,$$

then there exists an even function $f(x) \in \bigcap\limits_{p\ge 1} L^p[0,2\pi]$ such that its Fourier series

$$f(x) \sim \sum_{n=1}^{\infty} c_n \cos nx$$

converges a.e. on $[0,2\pi]$ and for every fixed $\alpha > 0$ is not $|c,\alpha|$ summable a.e. on $[0,2\pi]$ though

$$\sum_{n=1}^{\infty} c_n^2\,\omega(n) < \infty.$$

Definition 1 (see [2]). Let $\{N_k\}$ be an increasing sequence of natural numbers, $\Delta_k = (N_k, N_{k+1}]$, $k = 1, 2, \dots$, and $\{\varphi_n\}$ be a system of functions from $L^2(0,1)$. The system $\{\varphi_n\}$ will be called a Δ_k-orthonormal system (Δ_k-ONS) if:

(1) $\|\varphi_n\|_2 = 1$, $n = 1, 2, \dots$;

(2) $(\varphi_i, \varphi_j) = 0$ for $i, j \in \Delta_k$, $i \neq j$, $k \geq 1$.

Definition 2 (see [1]). A positive nondecreasing sequence $\{\omega(n)\}$ will be called the Weyl multiplier for the summability a.e. of series with respect to the Δ_k-ONS $\{\varphi_n\}$ if the condition

$$\sum_{n=1}^{\infty} a_n^2\,\omega(n) < \infty \tag{3}$$

guarantees the summability a.e. by the $|c, \alpha|$ method of the corresponding series (2).

Below we shall quote the theorem showing that if the sequence $\{\omega(n)\}$ is the Weyl multiplier for the summability a.e. by the $|c, 1|$ method of all orthogonal series (2), then it will be the Weyl multiplier for the summability a.e. by the $|c, 1|$ method of all series (2) with respect to the Δ_k-ONS for the increasing sequence of natural numbers $\{N_k\}$.

Theorem 1. *If a positive nondecreasing sequence $\{\omega(n)\}$ is the Weyl multiplier for the summability a.e. by the $|c, 1|$ method of all orthonormal series* (2)*, then there exists an increasing sequence of natural numbers $\{N_k\}$ such that $\{\omega(n)\}$ is the Weyl multiplier for the summability a.e. by the $|c, 1|$ method of all series* (2) *with respect to the $\Delta_k = (N_k, N_{k+1}]$-ONS.*

Proof. We prove this theorem by the Wang–Ul'yanov's scheme (see [1]) modifying it accordingly. Let the positive nondecreasing sequence $\{\omega(n)\}$ be the Weyl multiplier for the summability a.e. by the $|c, 1|$ method of all orthogonal series (2). Then condition (1) is fulfilled.

As is known (see [1]), for the positive nondecreasing on $[n_0, +\infty)$ function $\omega(x)$ the series

$$\sum_{m=n_0}^{\infty} \frac{1}{m\,\omega(m)} \quad \text{and} \quad \sum_{m=n_0^2}^{\infty} \frac{1}{m\,\omega(\sqrt{m})}$$

converge or diverge simultaneously. Therefore, taking into account (1), we have

$$\sum_{n=1}^{\infty} \frac{1}{n\,\omega(\sqrt[4]{n})} < \infty.$$

Then

$$\sum_{n=1}^{\infty} \frac{R(n)}{n\,\omega(\sqrt[4]{n})} < \infty, \tag{4}$$

where

$$R(n) = \frac{\left(\sum\limits_{k=2}^{\infty} \frac{1}{k\,\omega(\sqrt[4]{k})}\right)^{\frac{1}{2}}}{\left(\sum\limits_{k=n+1}^{\infty} \frac{1}{k\,\omega(\sqrt[4]{k})}\right)^{\frac{1}{2}}}.$$

Obviously, $R(1) = 1$, $R(n) < R(n+1)$ and $\lim_{n\to\infty} R(n) = +\infty$.

Define the sequence $k(n)$ by the recursion formula

$$k(1) = 0, \quad k(n+1) = \begin{cases} k(n) + 1 & \text{if } R(n+1) \geq k(n) + 1, \\ k(n) & \text{if } R(n+1) < k(n) + 1, \end{cases} \quad n \geq 1.$$

Thus we obtain the nondecreasing sequence of nonnegative integers for which

$$k(n) \le R(n), \quad n = 1, 2, \dots . \tag{5}$$

Note that for the sequence $k(n)$ there exists an increasing sequence of natural numbers $\{N_k\}$ (it is assumed that $N_0 = 0$) which is defined by the formula

$$k(n) = \max\{k : N_k < n\}.$$

Then, taking into account (4) and (5), we find that

$$\sum_{n=1}^{\infty} \frac{k(n)}{n\,\omega(\sqrt[4]{n})} < \infty. \tag{6}$$

Let $\{\varphi_n\}$ be a block-orthonormal system with $\Delta_k = (N_k, N_{k+1}]$ and condition (3) be fulfilled. Then for the corresponding series (2) we have

$$\sigma_n(x) - \sigma_{n-1}(x) = \frac{1}{n(n-1)} \sum_{i=1}^{\infty} a_i (i-1)\,\varphi_i(x), \quad n \ge 2.$$

Denoting by c the absolute positive constants which, generally speaking, may have different values in different inequalities and using (6), we find that

$$\sum_{n=2}^{\infty} \int_0^1 \big|\sigma_n(x) - \sigma_{n-1}(x)\big| dx \le \sum_{n=2}^{\infty} \Big(\int_0^1 \big|\sigma_n(x) - \sigma_{n-1}(x)\big|^2 dx \Big)^{\frac{1}{2}} \le$$

$$\le c \sum_{n=2}^{\infty} \frac{1}{n^2} \Big(\int_0^1 \Big| \sum_{i=1}^{n} a_i (i-1) \varphi_i(x) \Big|^2 dx \Big)^{\frac{1}{2}} \le$$

$$\le c \sum_{n=1}^{\infty} \frac{1}{n^2} \Big(\int_0^1 \Big| \sum_{i=1}^{N_{k(n)}} a_i (i-1) \varphi_i(x) \Big|^2 + \int_0^1 \Big| \sum_{i=N_{k(n)}+1}^{n} a_i (i-1) \varphi_i(x) \Big|^2 dx \Big)^{\frac{1}{2}} \le$$

$$\le c \sum_{n=1}^{\infty} \frac{1}{n^2} \Big(k(n) \sum_{i=1}^{N_{k(n)}} a_i^2 i^2 + \sum_{i=N_{k(n)}+1}^{n} a_i^2 i^2 \Big)^{\frac{1}{2}} \le$$

$$\le c \sum_{n=1}^{\infty} \frac{1}{n^2} \Big(k(n) \sum_{i=1}^{n} i^2 a_i^2 \Big)^{\frac{1}{2}} \le c \sum_{n=1}^{\infty} \frac{\sqrt{k(n)}}{n^2} \Big[\Big(\sum_{i=1}^{[\sqrt[4]{n}]} i^2 a_i^2 \Big)^{\frac{1}{2}} +$$

$$+ \Big(\sum_{i=[\sqrt[4]{n}]+1}^{n} i^2 a_i^2 \Big)^{\frac{1}{2}} \Big] \le c \Big(\sum_{n=1}^{\infty} \frac{\sqrt{k(n)} (\sqrt{n}\,\sqrt[4]{n})^{\frac{1}{2}}}{n^2} +$$

$$+\sum_{n=1}^{\infty}\frac{\sqrt{k(n)}}{n^2}\frac{n\omega(\sqrt[4]{n}))^{\frac12}}{n\omega(\sqrt[4]{n}))^{\frac12}}\Big(\sum_{i=[\sqrt[4]{n}]+1}^{n}i^2a_i^2\Big)^{\frac12}\Big)\le$$

$$\le\Big(\sum_{n=1}^{\infty}\frac{1}{n^{\frac98}}+\Big(\sum_{n=1}^{\infty}\frac{\omega(\sqrt[4]{n})}{n^3}\sum_{i=[\sqrt[4]{n}]+1}^{n}i^2a_i^2\Big)^{\frac12}\Big(\sum_{n=1}^{\infty}\frac{k(n)}{k\omega(\sqrt[4]{n})}\Big)^{\frac12}\Big)\le$$

$$\le c+c\Big(\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{i=1}^{\infty}i^2a_i^2\omega(i)\Big)^{\frac12}\le c+c\Big(\sum_{i=1}^{\infty}a_i^2\omega(i)\Big)^{\frac12}<\infty,$$

whence by Levy's theorem

$$\sum_{n=2}^{\infty}\big|\sigma_n(x)-\sigma_{n-1}(x)\big|<\infty\quad\text{a.e. on}\quad(0,1).\qquad\square$$

The theorem below makes it possible to determine the Weyl multipliers for the summability a.e. of the series (2) with respect to the Δ_k-ONS for regularly increasing sequences $\{N_k\}$.

Theorem 2. *Let an increasing sequence of natural numbers* $\{N_k\}$ *be given, for which the condition*

$$\sum_{k=n}^{\infty}\frac{1}{N_k^2}=O\Big(\frac{n}{N_n^2}\Big)\quad(n\to\infty)\tag{7}$$

is fulfilled, and let

$$k(n)=\max\{k:N_k<n\}.$$

If for the positive nondecreasing sequence $\{\omega(n)\}$ *condition* (1) *is fulfilled, then for every* Δ_k*-ONS* $\{\varphi_n\}$ *the condition*

$$\sum_{n=1}^{\infty}a_n^2\,\omega(n)\,k(n)<\infty\tag{8}$$

guarantees the summability a.e. by the $|c,1|$ *method of the corresponding series* (2).

Proof. Let conditions (1), (7) and (8) be fulfilled. Then for the corresponding series (2) we have

$$\sum_{n=2}^{\infty}\int_0^1\big|\sigma_n(x)-\sigma_{n-1}(x)\big|dx\le\sum_{n=2}^{\infty}\Big(\int_0^1\big|\sigma_n(x)-\sigma_{n-1}(x)\big|^2dx\Big)^{\frac12}\le$$

$$\le c\sum_{n=1}^{\infty}\frac{1}{n^2}\Big(\int_0^1\Big|\sum_{i=1}^{N_{k(n)}}a_i(i-1)\varphi_i(x)\Big|^2dx+\int_0^1\Big|\sum_{i=N_{k(n)}+1}^{n}a_i(i-1)\varphi_i(x)\Big|^2dx\Big)^{\frac12}\le$$

$$\leq c\sum_{n=1}^{\infty}\frac{1}{n^2}\Big(k(n)\sum_{i=1}^{\infty}i^2a_i^2\Big)^{\frac12}\leq c\sum_{n=1}^{\infty}\frac{\sqrt{k(n)}}{n^2}\bigg[\Big(\sum_{i=n}^{[\sqrt[4]{n}]}i^2a_i^2\Big)^{\frac12}+$$

$$+\Big(\sum_{i=[\sqrt[4]{n}]+1}^{n}i^2a_i^2\Big)^{\frac12}\bigg]\leq c\bigg(\sum_{n=1}^{\infty}\frac{\sqrt{k(n)}n^{\frac38}}{n^2}+$$

$$+\sum_{n=1}^{\infty}\frac{\sqrt{k(n)}}{n^2}\frac{(n\omega(\sqrt[4]{n}))^{\frac12}}{(n\omega(\sqrt[4]{n}))^{\frac12}}\Big(\sum_{i=[\sqrt[4]{n}]+1}^{n}i^2a_i^2\Big)^{\frac12}\bigg)\leq$$

$$\leq c+c\Big(\sum_{n=1}^{\infty}\frac{k(n)}{n^3}\sum_{i=1}^{\infty}i^2a_i^2\omega(i)\Big)^{\frac12}\Big(\sum_{n=1}^{\infty}\frac{1}{n\omega(\sqrt[4]{n})}\Big)^{\frac12}\leq$$

$$\leq c+c\Big(\sum_{i=1}^{\infty}i^2a_i^2\omega(i)\sum_{i=1}^{\infty}\frac{k(n)}{n^3}\Big)^{\frac12}=$$

$$=c+c\bigg(\sum_{i=1}^{\infty}i^2a_i^2\omega(i)\bigg(\sum_{n=i}^{N_{k(i)+1}}\frac{k(i)}{n^3}+\sum_{j=k(i)+1}^{\infty}j\sum_{n=N_j+1}^{N_{j+1}}\frac{1}{n^3}\bigg)\bigg)^{\frac12}\leq$$

$$\leq c+c\bigg(\sum_{i=1}^{\infty}i^2a_i^2\omega(i)\bigg(\frac{k(i)}{i^2}+(k(i)+1)\sum_{n=N_{k(i)+1}+1}^{\infty}\frac{1}{n^3}+$$

$$+\sum_{j=k(i)+1}^{\infty}\frac{1}{N_j^2}\bigg)\bigg)^{\frac12}\leq c+c\Big(\sum_{i=1}^{\infty}i^2a_i^2\omega(i)k(i)\Big)^{\frac12}<\infty,$$

whence by Levy's theorem

$$\sum_{n=2}^{\infty}\big|\sigma_n(x)-\sigma_{n-1}(x)\big|<\infty \quad \text{a.e. on} \quad (0,1). \qquad \square$$

Remark 1. In Theorem 2, the Weyl multipliers defined by conditions (1) and (8) can be assumed to be exact on the set of sequences $\{N_k\}$ with condition (7) in the sense that if condition (1) is violated, then one can construct a sequence $\{N_k\}$ for which condition (7) is fulfilled and also there exists a trigonometric series

$$\sum_{n=1}^{\infty}b_n\cos nx,$$

which is nonsummable by the $|c,\alpha|$ method for almost all $x\in[0,2\pi]$ (for every fixed $\alpha>0$) though

$$\sum_{n=1}^{\infty}b_n^2\,\omega(n)\,k(n)<\infty.$$

Indeed, let the condition

$$\sum_{n=1}^{\infty} \frac{1}{n\,\omega(n)} = \infty$$

be fulfilled for the sequence $\{\omega(n)\}$.

We construct an increasing sequence of natural numbers $\{N_k\}$ in such a way that the condition

$$k = O\Big(\sum_{n=1}^{N_k} \frac{1}{n\,\omega(n)}\Big)^{\beta}, \quad 0 < \beta \le \frac{1}{2},$$

be fulfilled and the sequence $\frac{N_k}{k}$ be increasing.

Clearly, condition (7) is fulfilled (see [3], Remark 2).

Take

$$s_k = \sum_{n=1}^{k} \frac{1}{n\,\omega(n)}, \quad k = 1, 2, \dots ,$$

and

$$c_m = \frac{1}{\sqrt{m}\,\omega(m)\,(s_m)^{\beta+\frac{1}{2}}} \quad m = 1, 2, \dots .$$

Then for arbitrary $\varepsilon_m = \pm 1$ we have

$$\sum_{m=1}^{\infty} (\varepsilon_m c_m)^2 \omega(m) k(m) = \sum_{m=1}^{\infty} \frac{k(m)}{m\,\omega(m)(s_m)^{2\beta+1}} =$$

$$= \sum_{k=0}^{\infty} \sum_{m=N_k+1}^{N_{k+1}} \frac{k(m)}{m\,\omega(m)(s_m)^{2\beta+1}} \le c \sum_{k=0}^{\infty} \sum_{m=N_k+1}^{N_{k+1}} \frac{(s_{N_k})^{\beta}}{m\,\omega(m)(s_m)^{2\beta+1}} \le$$

$$\le c \sum_{k=0}^{\infty} \sum_{m=N_k+1}^{N_{k+1}} \frac{1}{m\,\omega(m)(s_m)^{1+\beta}} \le c \sum_{m=1}^{\infty} \frac{1}{m\,\omega(m)(s_m)^{1+\beta}} < \infty.$$

On the other hand,

$$\sum_{n=0}^{\infty} \Big\{ \sum_{m=2^n+1}^{2^{n+1}} c_m^2 \Big\}^{\frac{1}{2}} \ge \sum_{n=0}^{\infty} \Big\{ \sum_{m=2^n+1}^{2^{n+1}} \frac{1}{m(\omega(m))^2 (s_m)^{1+2\beta}} \Big\}^{\frac{1}{2}} \ge$$

$$\ge \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\omega(2^n)(s_{2^n})^{\frac{1}{2}+\beta}} = \infty.$$

Therefore by Billard's theorem [1], for almost all choices of $\varepsilon_k = \pm 1$ the series

$$\sum_{m=1}^{\infty} \varepsilon_m\, c_m \cos mx$$

is $|c,\alpha|$-nonsummable ($\alpha > 0$) at almost every point $x \in [0, 2\pi]$ though

$$\sum_{n=1}^{\infty} b_n^2 \, \omega(n) \, k(n) < \infty,$$

where $b_n = \varepsilon_n c_n$.

Remark 2. The above theorems remain also valid for $|c,\alpha|$ methods with $c\alpha > \frac{1}{2}$.

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