GEORGIAN MATHEMATICAL JOURNAL: Vol. 6, No. 2, 1999, 149-168

ON THE INTEGRABILITY OF STRONG MAXIMAL FUNCTIONS CORRESPONDING TO DIFFERENT FRAMES

G. ONIANI

ABSTRACT. For the frame θ in $\mathbb{R}^n$, let $B_2(\theta)(x)$ $(x \in \mathbb{R}^n)$ be a family of all n-dimensional rectangles containing x and having edges parallel to the straight lines of θ, and let $M_{B_2(\theta)}$ be a maximal operator corresponding to $B_2(\theta)$. The main result of the paper is the following

Theorem. *For any function $f \in L(1+\ln^+ L)(\mathbb{R}^n)$ $(n \geq 2)$ there exists a measure preserving and invertible mapping $\omega : \mathbb{R}^n \to \mathbb{R}^n$ such that*

1. $\{x : \omega(x) \neq x\} \subset \operatorname{supp} f$;
2. $\sup\limits_{\theta \in \theta(\mathbb{R}^n)} \int\limits_{\{M_{B_2(\theta)}(f\circ\omega)>1\}} M_{B_2(\theta)}(f \circ \omega) < \infty.$

This theorem gives a general solution of M. de Guzmán's problem that was previously studied by various authors.

1. Definitions and the Notation

Let B be a mapping defined on $\mathbb{R}^n$ such that, for every $x \in \mathbb{R}^n$, $B(x)$ is a family of open bounded sets in $\mathbb{R}^n$ containing x. The maximal operator M_B corresponding to B is defined as follows: for $f \in L_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$

$$M_B(f)(x) = \sup_{R \in B(x)} \frac{1}{|R|} \int\limits_R |f| \quad \text{if} \quad B(x) \neq \varnothing,$$

and

$$M_B(f)(x) = 0 \quad \text{if} \quad B(x) = \varnothing.$$

A frame in $\mathbb{R}^n$ will be called a set whose elements are n pairwise orthogonal straight lines passing through the origin O. Frames will be denoted by θ, $\theta = \{\theta^1, \dots, \theta^n\}$. Under θ_0 will be meant a frame $\{Ox^1, \dots, Ox^n\}$, where $Ox^1, \dots, Ox^n$ are the coordinate axes of $\mathbb{R}^n$. A set of all frames in $\mathbb{R}^n$ will be denoted by $\theta(\mathbb{R}^n)$.

1991 *Mathematics Subject Classification.* 28A15.

Key words and phrases. Strong maximal operator, rectangle, frame.

A set congruent to a set of the form $I_1 \times \cdots \times I_n$, where $I_1, \dots, I_n$ are intervals of positive length on the straight line, will be called an n-dimensional rectangle or simply a rectangle in $\mathbb{R}^n$.

The frame $\theta = \{\theta^1, \dots, \theta^n\}$ for which the sides of the rectangle I are parallel to the corresponding straight lines θ^j $(j = 1, \dots, n)$ will be called the frame of I which will be denoted by $\theta(I)$.

For a nonempty set $E \subset \theta(\mathbb{R}^n)$ we shall denote by $B_2(E)(x)$ $(x \in \mathbb{R}^n)$ a family of all rectangles I in $\mathbb{R}^n$ with the properties $x \in I$, $\theta(I) \in E$. Instead of $B_2(\{\theta\})$ we shall write $B_2(\theta)$ when $E = \{\theta\}$, and B_2 when $\theta = \theta_0$.

Since M_{B_2} is said to be a strong maximal operator, it is natural to call $M_{B_2(\theta)}$ the strong maximal operator corresponding to the frame θ.

By $B_1(x)$ $(x \in \mathbb{R}^n)$ we denote a family of all cubic intervals in $\mathbb{R}^n$ containing x (for $n = 1$ a one-dimensional interval is understood here as a square interval).

The support $\{x \in \mathbb{R}^n : f(x) \neq 0\}$ of the function $f : \mathbb{R}^n \to \mathbb{R}$ will be denoted by $\operatorname{supp} f$.

2. Formulation of the Question and the Main Result

The class $L(1 + \ln^+ L)(\mathbb{R}^n)$ was characterized by Guzmán and Welland ([1, 2], Ch. II, §6) by means of the maximal operator M_{B_1}. In particular, they have shown that for $f \in L(\mathbb{R}^n)$ the following conditions are equivalent:

$$1. \quad f \in L(1 + \ln^+ L)(\mathbb{R}^n),$$

$$2. \quad \int\limits_{\{M_{B_1}(f) > 1\}} M_{B_1}(f) < \infty.$$

From the strong maximal Jessen–Marcinkiewicz–Zygmund's theorem it follows that if

$$f \in L(1 + \ln^+ L)^n(\mathbb{R}^n), \tag{2.1}$$

then

$$\int\limits_{\{M_{B_2}(f) > 1\}} M_{B_2}(f) < \infty. \tag{2.2}$$

Guzmán (see [2], Ch. II, §6) posed the question whether it was possible to characterize the class $L(1 + \ln^+ L)^2(\mathbb{R}^2)$ by the operator M_{B_2} as it was done for the class $L(1 + \ln^+ L)(\mathbb{R}^n)$ using the operator M_{B_1}, i.e., whether conditions (2.1) and (2.2) are equivalent for $f \in L(\mathbb{R}^2)$. Gogoladze [4, 5] and Bagby [6] answered this question in the negative.

It can be easily verified that much more than (2.2) is fulfilled for $f \in L(1+\ln^+ L)^n(\mathbb{R}^n)$, in particular,

$$\sup_{\theta\in\theta(\mathbb{R}^n)} \int\limits_{\{M_{B_2(\theta)}(f)>1\}} M_{B_2(\theta)}(f) < \infty. \tag{2.3}$$

A question arises if it is possible to characterize the class $L(1+\ln^+ L)^n(\mathbb{R}^n)$ by condition (2.3), i.e., if conditions (2.1) and (2.3) are equivalent for $f \in L(\mathbb{R}^n)$ $(n \geq 2)$.

This question was answered in the negative for $n = 2$ in [7]. The answer remains negative for an arbitrary $n > 2$ as well. In particular, the following theorem is valid.

Theorem 1. *For any function $f \in L(1+\ln^+ L)(\mathbb{R}^n)$ $(n \geq 2)$ there exists a measure preserving and invertible mapping $\omega : \mathbb{R}^n \to \mathbb{R}^n$ such that*

1. $\{x : \omega(x) \neq x\} \subset \operatorname{supp} f$,
2. $\displaystyle\sup_{\theta\in\theta(\mathbb{R}^n)} \int\limits_{\{M_{B_2(\theta)}(f\circ\omega)>1\}} M_{B_2(\theta)}(f\circ\omega) < \infty.$

Note that we had to use many new arguments to proceed from the case to $n = 2$ to the case of arbitrary $(n \geq 2)$.

Theorem 1 was first formulated by us in a less general for in [8].

3. Auxiliary Statements

Throughout the discussion preceding Lemma 4 we shall consider the spaces $\mathbb{R}^n$ with $n \geq 2$.

We shall call a strip in $\mathbb{R}^n$ an open set bounded by two different parallel hyperplanes, i.e., a set of the form

$$\{x \in \mathbb{R}^n \ : \ a < \alpha_1 x^2 + \cdots + \alpha_n x^n < b\},$$

where a, b $(a > b)$ and $\alpha_1, \ldots, \alpha_n$ $(\alpha_1^2 + \cdots + \alpha_n^2 > 0)$ are some real numbers, and x^k $(k = 1, \ldots, n)$ here and everywhere below denotes the k-th coordinate of the point $x \in \mathbb{R}^n$. The strip width will be called the distance between the hyperplanes that bound the strip, i.e., the number $b - a$ will be called the strip width.

In the sequel it will always be assumed that χ_A is the characteristic function of the set A.

Lemma 1. *For every $x \in \mathbb{R}^n$ let $B(x)$ be a family of open bounded and convex sets in $\mathbb{R}^n$, containing x, and let S be a strip in $\mathbb{R}^n$ of width δ. Then*

$$M_B(\chi_S)(x) < \frac{2^n\delta}{\operatorname{dist}(x,S)} \quad \textit{when} \quad \operatorname{dist}(x,S) \geq \delta.$$

Proof. Let $\mathrm{dist}(x,S)\ge\delta$ and $R\in B(x)$, $R\cap S\ne\varnothing$.

Among the hyperplanes bounding S we denote by Γ the hyperplane which is the closest to x. It is obvious that $R\cap\Gamma\ne\varnothing$. For every $y\in R\cap\Gamma$ let Δ_y be a segment connecting x and y. It is assumed that $K=\bigcup\limits_{y\in R\cap\Gamma}\Delta_y$. Since R is convex, we have

$$K\subset R. \tag{3.1}$$

Let H be the homothety centered at x and with the coefficient

$$\alpha=\frac{\mathrm{dist}(x,S)+\delta}{\mathrm{dist}(x,S)}\ .$$

Let us show that

$$R\cap S\subset H(K)\backslash K. \tag{3.2}$$

Indeed, assume that $z\in R\cap S$ and denote by y the point at which the segment connecting x and z intersects with Γ. Since $x,z\in R$, by virtue of the convexity of R we have $y\in R$. Therefore $y\in R\cap\Gamma$. By the definitions of the set K and homothety H we easily obtain $z\in H(\Delta_y)\subset H(K)$. $(R\cap S)\cap K=\varnothing$. Therefore $z\notin K$. Thus $z\in H(K)\backslash K$. Thus, since $z\in R\cap S$ is arbitrary, we have proved (3.2).

Using (3.1), (3.2), the definition of H and obvious inequality $\alpha^n-1<\frac{2^n\delta}{\mathrm{dist}(x,S)}$ we can write

$$\frac{1}{|R|}\int\limits_R\chi_S=\frac{|R\cap S|}{|R|}\le\frac{|H(K)\backslash K|}{|K|}=\frac{(\alpha^n-1)|K|}{|K|}<\frac{2^n\delta}{\mathrm{dist}(x,S)},$$

which, obviously, proves the lemma. $\square$

For the rectangle I in $\mathbb{R}^n$ having pairwise orthogonal edges of lengths $\delta_1,\delta_2,\dots,\delta_n$, where $\delta_1\le\delta_2\le\dots\le\delta_n$, we introduce the notation:

(1) $r(I)$ is a number δ_2/δ_1;

(2) when $r(I)>1$, for $h\ge1$, $J(I,h)$ is an open rectangle with the following properties: $J(I,h)$ has the same center and frame as I; the length of the edges of $J(I,h)$ parallel to the edges of I of the length δ_1 is equal to $(2^{n+1}h+1)\delta_1$, while the length of the edges of $J(I,h)$ parallel to the edges of I of length δ_j $(j=2,\dots,n)$ is equal to $3\delta_j$;

(3) for $r(I)>1$, ℓ_I is a straight line passing through O and parallel to the edges of I of length δ_1.

For the straight line ℓ in $\mathbb{R}^n$ and $0<\varepsilon<\pi/4$ we assume

$$E(\ell,\varepsilon)=\big\{\theta\in\theta(\mathbb{R}^n):\angle(\ell,\theta^j)<\pi/2-\varepsilon,\ j=1,\dots,n\big\},$$

where $\angle(\cdot,\cdot)$ is the angle lying between the two straight lines.

Lemma 2. *Let I be a rectangle in $\mathbb{R}^n$, $h > 1$, $0 < \varepsilon < \pi/4$, $r(I) > \frac{nh}{\sin\varepsilon}$, and $E = E(\ell_I, \varepsilon)$. Then*

$$\big\{M_{B_2(E)}(h\chi_{_I}) > 1\big\} \subset J(I, h),$$

and therefore

$$\big|\big\{M_{B_2(E)}(h\chi_{_I}) > 1\big\}\big| \le 9^n h|I|.$$

Proof. Without loss of generality we assume that

$$I = (-\delta_1/2, \delta_1/2) \times \cdot \times (-\delta_n/2, \delta_n/2),$$

where $\delta_1 < \delta_1 \le \cdots \le \delta_n$. We write

$$S_1 = \Big\{x \in \mathbb{R}^n : |x^1| < \Big(2^n h + \frac{1}{2}\Big)\delta_1\Big\},$$
$$S_j = \big\{x \in \mathbb{R}^n : |x^j| < 3\delta_j/2\big\} \quad (j = 2, \dots, n).$$

As is easily seen, $J(I, h)$ is the intersection of the strips $S_1, \dots, S_n$.

Let $S = \{x \in \mathbb{R}^n : |x^1| < \delta_1/2\}$ and $x \in S_1$. Obviously, $\operatorname{dist}(x, S) \ge 2^n h\delta_1$. Now by lemma 1 we write

$$M_{B_2(E)}(h\chi_{_I})(x) = hM_{B_2(E)}(\chi_{_I})(x) \le hM_{B_2(E)}(\chi_{_S})(x) < \frac{h2^n\delta_1}{\operatorname{dist}(x, S)} \le 1.$$

Hence we conclude that

$$\big\{M_{B_2(E)}(h\chi_{_I}) > 1\big\} \subset S_1. \tag{3.3}$$

Consider arbitrary $2 \le j \le n$. Let $x \notin S_j$, $J \in B_2(E)(x)$, and $J \cap I \ne \varnothing$. Obviously, $\operatorname{dist}(x, I) \ge \delta_j$, and we have

$$\operatorname{dist}(x, I) \le \operatorname{diam} I < t_1 + t_2 + \cdots + t_n,$$

where $t_1, t_2, \dots, t_n$ are lengths of orthogonal edges of J. Therefore there exists a side of J with the length greater than δ_j/n. We can represent J as a union of pairwise nonintersecting intervals equal and parallel to above-mentioned edge: $J = \bigcup_{\alpha\in T} \Delta_\alpha$. Obviously,

$$|\Delta_\alpha|_1 > \delta_j/n \quad (\alpha \in T). \tag{3.4}$$

(Here and everywhere below, for the set A contained in some k-dimensional ($k = 1, \dots, n-1$) affine subspace $\mathbb{R}^n$, we denote by $|A|_k$ k-dimensional measure of A.)

Let us prove that

$$\frac{h|\Delta_\alpha \cap I|_1}{|\Delta_\alpha|_1} \le 1 \quad (\alpha \in T). \tag{3.5}$$

Indeed, let ℓ be the straight line containing the segment Δ_α. It is easy to see that $|\ell \cap S|_1 = \delta_1/\cos\angle(\ell, Ox^1)$. $J \in B_2(E)(x)$, Therefore $\angle(\ell, Ox^1) <$

$\pi/2-\varepsilon$. Consequently, $|\ell\cap S|_1\le\frac{\delta_1}{\cos(\pi/2-\varepsilon)}=\frac{\delta_1}{\sin\varepsilon}$, which by virtue of (3.4) and the inequality $\delta_j>r(I)\delta_1\ge\frac{nh\delta_1}{\sin\varepsilon}$ implies

$$\frac{h|\Delta_\alpha\cap I|_1}{|\Delta_\alpha|_1}\le\frac{h|\ell\cap S|_1}{\delta_j/n}\le\frac{h\delta_1}{\sin\varepsilon}\,\frac{\sin\varepsilon}{h\delta_1}=1.$$

It is not difficult to verify that

$$\frac{1}{|J|}\int\limits_J h\chi_{_I}=\frac{h|J\cap I|}{|J|}\le\sup_{\alpha\in T}\frac{h|\Delta_\alpha\cap I|_1}{|\Delta_\alpha|_1}.$$

Hence, by (3.5),

$$\frac{1}{|J|}\int\limits_J h\chi_{_I}\le 1,$$

which, taking into account the arbitrariness of $J\in B_2(E)(x)$, $L\cap I\ne\varnothing$, allowsus to conclude that

$$M_{B_2(E)}(h\chi_{_I})(x)\le 1\quad(x\notin S_j,\quad 2\le j\le n).$$

This and (3.3) imply

$$\big\{M_{B_2(E)}(h\chi_{_I})>1\big\}\subset\bigcap_{j=1}^n S_j=J(I,h).\qquad\square$$

Lemma 3. *If among the pairwise different straight lines $\ell_1,\dots,\ell_k$ ($k\ge n$) in $\mathbb{R}^n$ which pass through the same point none of n lie in the same hyperplane, then there exists $\varepsilon>0$ such that for every straight line ℓ in $\mathbb{R}^n$ and every $1\le k_1<k_2<\dots<k_n\le k$*

$$\min_{1\le j\le n}\angle(\ell,\ell_{kj})<\frac{\pi}{2}-\varepsilon\ .$$

Proof. Let $x_j\in\mathbb{R}^n$, $\|x_j\|=1$ ($\|\cdot\|$ is the norm in $\mathbb{R}^n$, $j=1,\dots,n$) be the direction vector of the straight line ℓ_j. If we assume the contrary to the assertion of the lemma, then for every $m\in\mathbb{N}$ there exist $y_m\in\mathbb{R}^n$, $\|y_m\|=1$, and numbers $1\le k_1(m)<k_2(m)<\cdots k_n(m)\le k$ such that

$$\arccos\big|(y_m,x_{k_j(m)})\big|>\frac{\pi}{2}-\frac{1}{m}$$

for $j=1,\dots,n$, where $(\cdot,\cdot)$ is the scalar product in $\mathbb{R}^n$. Hence by the compactness of the unit sphere in $\mathbb{R}^n$ and the continuity of the scalar product there exist $y\in\mathbb{R}^n$, $\|y\|=1$, and $1\le k_1<k_2<\cdots<k_n\le k$ such that

$$(y,x_{k_j})=0$$

for $j=1,\dots,n$. This implies that the points $x_{k_1},\dots,x_{k_n}$ belong to the hyperplane which is orthogonal to y. Thus the straight lines $\ell_{k_1},\dots,\ell_{k_n}$ lie in the same hyperplane which contradicts the condition of the lemma. $\square$

Lemma 4. *Let f be a continuous function on $\mathbb{R}^n$, $\theta \in \theta(\mathbb{R}^n)$, $\lambda > 0$, and an open set G contain $\{M_{B_2(\theta)}(f) > \lambda\}$. If for the rectangle I in $\mathbb{R}^n$ with $\theta(I) = \theta$, $I \backslash G \neq \varnothing$, then*

$$\int\limits_{I \cap G} |f| \leq \lambda |I \cap G|.$$

Proof. We prove the lemma by induction with respect to n. For $n = 1$ the proof is obvious. Consider the passage from $n-1$ to n.

Without loss of generality we assume that $\theta = \theta_0$ and I is closed.

Introduce the notation:

$$\begin{aligned} \Gamma_t &= \{x \in \mathbb{R}^n : x^1 = t\}, \\ I_t &= I \cap \Gamma_t, \quad G_t = G \cap \Gamma_t, \end{aligned} \quad (t \in \mathbb{R})$$
$$\begin{aligned} J &= \{t \in \mathbb{R}^n : I_t \neq \varnothing\}, \\ S_1 &= \{t \in J : I_t \subset G_t\}, \\ S_2 &= \{t \in J : I_t \backslash G_t \neq \varnothing\}. \end{aligned}$$

It is easy to see that S_1 is open by the natural topology on the interval J. Therefore S_1 divides into pairwise nonintersecting intervals $\{\delta_k\}_{k \in T \subset \mathbb{N}}$. Obviously, the n-dimensional rectangles $\Delta_k = \bigcup\limits_{t \in \delta_k} I_t$ $(k \in T)$ satisfy the conditions

$$\partial \Delta_k \cap \partial G \neq \varnothing \quad (k \in T), \tag{3.6}$$

where $\partial \Delta_k$ and ∂G are the boundaries of Δ_k and G, respectively;

$$\theta(\Delta_k) = \theta(I) = \theta_0 \quad \text{and} \quad \Delta_k \subset I \cap G \quad (k \in T), \tag{3.7}$$
$$\Delta_k \cap \Delta_m = \varnothing \quad (k \neq m). \tag{3.8}$$

By the conditions of the lemma, $M_{B_2}(f)(x) \leq \lambda$ for $x \in \partial G$ and therefore, with (3.6) and (3.7) taken into account, we have

$$\int\limits_{\Delta_k} |f| \leq \lambda |\Delta_k| \quad (k \in T),$$

which on account to (3.8) implies

$$\int\limits_{\bigcup\limits_{k \in T} \Delta_k} |f| \leq \lambda \Big| \bigcup_{k \in T} \Delta_k \Big|. \tag{3.9}$$

Estimate now the integral of $|f|$ on $(I\cap G)\setminus \bigcup\limits_{k\in T} \Delta_k$. Let M be an $(n-1)$-dimensional strong maximal operator. For each $t\in\mathbb{R}$ consider the function $g_t(y)=f(t,y)$ $(y\in\mathbb{R}^{n-1})$ and assume that

$$F(t,y)=M(g_t)(y) \quad (t\in\mathbb{R}, \quad y\in\mathbb{R}^{n-1}).$$

For $t\in S_2$ we have

$$\big\{F(t,\cdot)>\lambda\big\}\subset G_t. \tag{3.10}$$

Indeed, assume the contrary, i.e., there exist $t_0\in S_2$, $y_0\in\mathbb{R}^{n-1}$ and an $(n-1)$-dimensional interval R such that $(t_0,y_0)\notin G_{t_0}$, $R\ni y_0$, and

$$\int\limits_R |g_{t_0}(y)|\,dy\geq \lambda|R|_{n-1}.$$

Then by the continuity of f, for a sufficiently small one-dimensional interval $\Delta\ni t_0$ we shall have

$$\int\limits_{\Delta\times R} |f(t,y)|\,dt\,dy>\lambda|\Delta\times R|.$$

Hence $M_{B_2}(f)(t_0,y_0)>\lambda$. On the other hand, since $(t_0,y_0)\notin G_{t_0}$, we have $(t_0,y_0)\notin G\supset\{M_{B_2}(f)>\lambda\}$. The obtained contradiction proves (3.10).

By virtue of (3.10) and the induction assumption we easily obtain

$$\int\limits_{I_t\cap G_t} |f(t,y)|\,dy\leq\lambda|I_t\cap G_t|_{n-1}$$

for $t\in S_2$.

Thus we can immediately write

$$\int\limits_{I\cap G\setminus\bigcup\limits_{k\in T}\Delta_k} |f| = \int\limits_{S_2}\Bigg[\int\limits_{I_t\cap G_t} |f(t,y)|\,dy\Bigg]dt\leq$$
$$\leq\int\limits_{S_2}\lambda|I_t\cap G_t|\,dt=\lambda\Big|(I\cap G)\setminus\bigcup\limits_{k\in T}\Delta_k\Big|.$$

whence by (3.7) and (3.9) we conclude that Lemma 4 is valid. $\square$

Denote by $\overline{L}(\mathbb{R}^n)$ a class of all functions $f\in L(\mathbb{R}^n)$ for each of which there exists, for $\varepsilon>0$, a continuous function $g\in L(\mathbb{R}^n)$ on $\mathbb{R}^n$ such that $|g(x)|\leq|f(x)|$ almost everywhere on $\mathbb{R}^n$, and $\|f-g\|_1<\varepsilon$.

Lemma 5. *Let $f \in \overline{L}(\mathbb{R}^n)$, $\theta \in \theta(\mathbb{R}^n)$, $\lambda > 0$, and then open set G contain $\{M_{B_2(\theta)}(f) > \lambda\}$. If for the rectangle I in $\mathbb{R}^n$ with $\theta(I) = \theta$, $I \backslash G \neq \varnothing$, then*

$$\int\limits_{I \cap G} |f| \leq \lambda |I \cap G|.$$

Proof. $f \in \overline{L}(\mathbb{R}^n)$. Therefore for arbitrarily given $\varepsilon > 0$ there exists a continuous function $g \in L(\mathbb{R}^n)$ on $\mathbb{R}^n$ such that $|g(x)| \leq |f(x)|$ almost everywhere on $\mathbb{R}^n$, and $\|f - g\|_1 < \varepsilon$. It is obvious that

$$\big\{M_{B_2(\theta)}(g) > \lambda\big\} \subset \big\{M_{B_2(\theta)}(f) > \lambda\big\} \subset G.$$

Now by Lemma 4

$$\int\limits_{I \cap G} |g| \leq \lambda |I \cap G|,$$

and therefore

$$\int\limits_{I \cap G} |f| - \varepsilon \leq \lambda |I \cap G|,$$

whence by the arbitrariness of $\varepsilon > 0$ we conclude that Lemma 5 is valid. □

Lemma 6. *Let $f_k \in \overline{L}(\mathbb{R}^n)$, $f_k \geq 0$ $(k \in \mathbb{N})$, $E \subset \theta(\mathbb{R}^n)$, $E \neq \varnothing$, $\lambda > 0$, and let for $k, m \in \mathbb{N}$ and $k \neq m$ the following conditions be fulfilled:*

$$\begin{aligned}
&\operatorname{supp} f_k \cap \operatorname{supp} f_m = \varnothing,\\
&\operatorname{supp} f_k \cap \big\{M_{B_2(E)}(f_m) > \lambda\big\} = \varnothing,\\
&\big\{M_{B_2(E)}(f_k) > \lambda\big\} \cap \big\{M_{B_2(E)}(f_m) > \lambda\big\} = \varnothing.
\end{aligned}$$

Then

$$\bigg\{M_{B_2(E)}\Big(\sum_{k=1}^{m} f_k\Big) > \lambda\bigg\} = \bigcup_{k=1}^{\infty} \big\{M_{B_2(E)}(f_k) > \lambda\big\}.$$

Proof. Denote $G_k = \big\{M_{B_2(E)}(f_k) > \lambda\big\}$, $k \in \mathbb{N}$. For each $k \in \mathbb{N}$

$$f_k(x) \leq \lambda \quad \text{almost everywhere on} \quad \mathbb{R}^n \backslash G_k. \tag{3.11}$$

Indeed, otherwise, since the differential bases $B_2(\theta)$, $\theta \in \theta(\mathbb{R}^n)$, are dense (see, for e.g., [2], Ch.II, §3), for arbitrary $\theta \in E$ and $A_j = (\mathbb{R}^n \backslash G_k) \cap \{f_k > \lambda + 1/j\}$ $(j \in \mathbb{N})$ we shall have

$$\lim_{I \in B_2(\theta)(x),\, \operatorname{diam} I \to 0} \frac{|I \cap A_j|}{|I|} \quad \text{for almost all} \quad x \in A_j.$$

Hence $M_{B_2(E)}(f_k) \geq M_{B_2(\theta)}(f_k) > \lambda$ for almost all $x \in (\mathbb{R}^n \backslash G_k) \cap \{f_k > \lambda\}$, which contradicts the definition of G_k.

By (3.11) and the condition of the lemma we write

$$\sum_{k=1}^{\infty} f_k(x) \le \lambda \quad \text{for almost all} \quad x \notin \bigcup_{k=1}^{\infty} G_k.$$

Hence, by the conditions of the lemma and by Lemma 5, we find that for every $x \notin \bigcup_{k=1}^{\infty} G_k$ and $I \in B_2(E)(x)$

$$\int_I \sum_{k=1}^{\infty} f_k \le \sum_{k=1}^{\infty} \int_{I\cap G_k} f_k + \int_{I\setminus \bigcup_{k=1}^{\infty} G_k} \sum_{k=1}^{\infty} f_k \le$$

$$\le \sum_{k=1}^{\infty} \lambda |I \cap G_k| + \lambda \Big| I \setminus \bigcup_{k=1}^{\infty} G_k \Big| = \lambda |I|.$$

Therefore

$$M_{B_2(E)}\Big(\sum_{k=1}^{\infty} f_k\Big)(x) \le \lambda \quad \text{for} \quad x \notin \bigcup_{k=1}^{\infty} G_k. \qquad \square$$

The next assertion belongs to Jessen, Marcinkiewicz, and Zygmund and is referred to as the strong maximal theorem (see [3] or [2], Ch. II, §3).

Theorem. *If $f \in L(1+\ln^+ L)^{n-1}(\mathbb{R}^n)$, then*

$$\big|\big\{M_{B_2}(f) > \lambda\big\}\big| \le c_1 \int_{\mathbb{R}^n} \frac{|f|}{\lambda} \Big(1 + \ln^+ \frac{|f|}{\lambda}\Big)^{n-1} \quad (\lambda > 0),$$

where c_1 is the constant depending only on n.

The foollowing lemma is a simple improvement of this result.

Lemma 7. *If $f \in L(1+\ln^+ L)^{n-1}(\mathbb{R}^n)$, then for every $\theta \in \theta(\mathbb{R}^n)$*

$$\big|\big\{M_{B_2(\theta)}(f) > \lambda\big\}\big| \le c_2 \int_{\{|f|>\lambda/2\}} \frac{|f|}{\lambda} \Big(1 + \ln \frac{2|f|}{\lambda}\Big)^{n-1} \quad (\lambda > 0),$$

where the constant c_2 depends only on n.

Proof. For arbitrary fixed $\lambda > 0$ assume $f_* = f\chi_{\{|f|\le\lambda/2\}}$ and $f^* = f\chi_{\{|f|>\lambda/2\}}$. $f = f_* + f^*$. Therefore $M_{B_2}(f) \le M_{B_2}(f_*) + M_{B_2}(f^*)$. Hence

$$\big\{M_{B_2} > \lambda\big\} \subset \big\{M_{B_2}(f_*) > \lambda/2\big\} \cup \big\{M_{B_2}(f^*) > \lambda/2\big\}.$$

But $\{M_{B_2}(f_*) > \lambda/2\} = \varnothing$ and therefore by the strong maximal theorem

$$\big|\big\{M_{B_2}(f) > \lambda\big\}\big| \le \big|\big\{M_{B_2}(f^*) > \lambda/2\big\}\big| \le c_1 \int\limits_{\mathbb{R}^n} \frac{2|f^*|}{\lambda}\Big(1 + \ln^+ \frac{2|f^*|}{\lambda}\Big)^{n-1} \le$$

$$\le 2c_1 \int\limits_{\{|f|>\lambda/2\}} \frac{|f|}{\lambda}\Big(1 + \ln \frac{2|f|}{\lambda}\Big)^{n-1}. \tag{3.12}$$

Let γ_θ, $\theta \in \theta(\mathbb{R}^n)$, be a rotation such that $\theta = \{\gamma_\theta(Ox^1), \dots, \gamma_\theta(Ox^n)\}$. In view of the fact that the rotation is a measure preserving mapping, we readily obtain

$$M_{B_2(\theta)}(f)(x) = M_{B_2}(f \circ \gamma_\theta)\big(\gamma_\theta^{-1}(x)\big) \quad (x \in \mathbb{R}^n) \tag{3.13}$$

Therefore

$$\big|\big\{M_{B_2(\theta)}(f) > \lambda\big\}\big| = \big|\big\{M_{B_2}(f \circ \gamma_\theta) > \lambda\big\}\big| \quad (\lambda > 0).$$

By this and (3.12) we conclude that the lemma is valid. $\square$

Lemma 8. *If $f \in L(1 + \ln^+ L)^n(\mathbb{R}^n)$, then for every $\theta \in \theta(\mathbb{R}^n)$*

$$\int\limits_{\{M_{B_2(\theta)}(f)>\lambda\}} M_{B_2(\theta)}(f) \le c_3 \int\limits_{\mathbb{R}^n} |f|\Big(1 + \ln^+ \frac{|f|}{\lambda}\Big)^n \quad (\lambda > 0),$$

where the constant c_3 depends only on n.

Proof. Let $f \in L(1 + \ln^+ L)^n(\mathbb{R}^n)$ and $\lambda > 0$. We have

$$\int\limits_{\{M_{B_2(\theta)}(f)>\lambda\}} M_{B_2}(f) = -\int\limits_{\lambda}^{\infty} t\,dF(t) = \big[-tF(t)\big]_\lambda^\infty + \int\limits_{\lambda}^{\infty} F(t)\,dt,$$

where $F(t) = |\{M_{B_2}(f) > t\}|$ $(t > 0)$. By Lemma 7

$$tF(t) \le c_2 \int\limits_{\{|f|>t/2\}} |f|\Big(1 + \ln \frac{2|f|}{t}\Big)^{n-1} \quad (t > 0). \tag{3.14}$$

Hence

$$\int\limits_{\{M_{B_2(\theta)}(f)>\lambda\}} M_{B_2}(f) = \lambda F(\lambda) + \int\limits_{\lambda}^{\infty} F(t)\,dt. \tag{3.15}$$

Lemma 7 yields

$$\int_\lambda^\infty F(t)\,dt \le c_2 \int_\lambda^\infty \int_{\{|f(x)|>t/2\}} \frac{|f(x)|}{t}\Big(1+\ln\frac{2|f(x)|}{t}\Big)^{n-1} dx\,dt =$$

$$= c_2 \int_{\{|f(x)|>\lambda/2\}} \int_\lambda^{2|f(x)|} \frac{|f(x)|}{t}\Big(1+\ln\frac{2|f(x)|}{t}\Big)^{n-1} dt\,dx \le$$

$$\le c_2 \int_{\{|f(x)|>\lambda/2\}} \int_\lambda^{2|f(x)|} \frac{|f(x)|}{t}\Big(1+\ln\frac{2|f(x)|}{\lambda}\Big)^{n-1} dt\,dx \le$$

$$\le c_2 \int_{\{|f(x)|>\lambda/2\}} |f(x)|\Big(1+\ln\frac{2|f(x)|}{\lambda}\Big)^{n} dx,$$

whence with regard for (3.14) and (3.15) we obtain

$$\int_{\{M_{B_2}(f)>\lambda\}} M_{B_2}(f) \le 2c_2 \int_{\mathbb{R}^n} |f|\Big(1+\ln^+\frac{2|f|}{\lambda}\Big)^n \quad (\lambda>0) \tag{3.16}$$

for $f\in L(1+\ln^+ L)^n(\mathbb{R}^n)$.

(3.13) readily implies

$$\int_{\{M_{B_2(\theta)}(f)>\lambda\}} M_{B_2(\theta)}(f) = \int_{\{M_{B_2}(f\circ\gamma_\theta)>\lambda\}} M_{B_2}(f\circ\gamma_\theta) \quad (\lambda>0) \tag{3.17}$$

for $f\in L(1+\ln^+ L)^n(\mathbb{R}^n)$ and $\theta\in\theta(\mathbb{R}^n)$.

Since the rotation is the measure preserving mapping, by (3.16) and (3.17) we immediately conclude that Lemma 8 is valid. $\square$

Lemma 9. *Let $f\in L(1+\ln^+ L)(\mathbb{R}^n)$, $g:\mathbb{R}^n\to\mathbb{R}$ be a measurable function, and $a,b>0$ and $\lambda\ge 0$. If*

$$\big|\{|g|>t\}\big| \le \frac{a}{t}\int_{\{|f|>bt\}} |f| \quad (t\ge\lambda), \tag{3.18}$$

then

$$\int_{\{|g|>\lambda\}} |g| \le a\int_{\mathbb{R}^n} |f|\Big(1+\ln^+\frac{|f|}{b\lambda}\Big).$$

Proof. We have

$$\int\limits_{\{|g|>\lambda\}} |g| = -\int\limits_{\lambda}^{\infty} t\,dF(t) = \big[-tF(t)\big]_{\lambda}^{\infty} + \int\limits_{\lambda}^{\infty} F(t)\,dt,$$

where $F(t) = |\{|g| > t\}|$ $(t \geq 0)$. By (3.18)

$$tF(t) \leq a \int\limits_{\{|f|>bt\}} |f| \quad (t \geq \lambda). \tag{3.19}$$

Hence

$$\int\limits_{\{|g|>\lambda\}} |g| = \lambda F(\lambda) + \int\limits_{\lambda}^{\infty} F(t)\,dt. \tag{3.20}$$

By (3.18)

$$\int\limits_{\lambda}^{\infty} F(t)\,dt \leq a \int\limits_{\lambda}^{\infty} \frac{1}{t} \int\limits_{\{|f(x)|>bt\}} |f(x)|\,dx\,dt =$$

$$= a \int\limits_{\{|f(x)|>b\lambda\}} |f(x)| \int\limits_{\lambda}^{|f(x)|/b} \frac{dt}{t}\,dx = a \int\limits_{\{|f(x)|>b\lambda\}} |f(x)| \ln \frac{|f(x)|}{b\lambda}\,dx.$$

Hence with (3.19) and (3.20) taken into account, we conclude that Lemma 9 is valid. □

Lemma 10. *Let f_1 and f_2 be the nonnegative measurable functions defined on $\mathbb{R}^n$. Then*

$$\int\limits_{\{f_1+f_2>2\lambda\}} (f_1 + f_2) \leq (1+\lambda)\Big(\int\limits_{\{f_1>\lambda\}} f_1 + \int\limits_{\{f_2>\lambda\}} f_2\Big) \quad (\lambda \geq 0).$$

Proof. The validity of the lemma follows from the following relations easy to verify:

(1) $$\int\limits_{\{f_1+f_2>2\lambda\}} (f_1+f_2) \leq \int\limits_{\{f_1>\lambda\}\cup\{f_2>\lambda\}} (f_1+f_2);$$

(2) $$\int\limits_{\{f_1>\lambda\}\cup\{f_2>\lambda\}} f_j \leq \int\limits_{\{f_j>\lambda\}} f_j + \lambda|\{f_i > \lambda\}|,$$ where $j, i \in \overline{1,2}$ and $j \neq i$. □

The set $E \subset \mathbb{R}^n$ is called elementary if it is a union of a finite number of n-dimensional intervals.

Lemma 11. *Let A be a subset of $\mathbb{R}^n$ of positive measure. Then for each $\delta_k > 0$ $(k \in \mathbb{N})$ with $\sum\limits_{k=1}^{\infty} \delta_k < |A|$ and $\varepsilon_k > 0$ $(k \in \mathbb{N})$ there exist pairwise nonintersecting elementary sets G_k $(k \in \mathbb{N})$ such that*

$$|G_k| = \delta_k \quad \text{and} \quad |G_k \backslash A| < \varepsilon_k.$$

Proof. Let us construct the sequence $\{G_k\}$ with the needed properties. For this we shall need the following simple facts:

(1) For each measurable set E and number δ with $0 \le \delta \le |E|$ there exists a measurable set $E' \subset E$ with $|E'| = \delta$;

(2) For each open set $E \subset \mathbb{R}^n$ and number δ with $0 < \delta < |E|$ there exists an elementary set $E' \subset E$ with $|E'| = \delta$.

By virtue of (1), there exists $E \subset A$ with $|E| = \delta_1$. Let an open set Q be such that $Q \supset E$, $|Q| > |E| = \delta_1$ and $|Q \backslash E| < \varepsilon_1$. According to (2), there exists an elementary set $G_1 \subset Q$ with $|G_1| = \delta_1$. Obviously, $|G_1 \backslash A| \le |Q \backslash E| < \varepsilon_1$.

Suppose the pairwise nonintersecting elementary sets $G_1, \dots, G_k$ with the properties

$$|G_j| = \delta_j \quad \text{and} \quad |G_j \backslash A| < \varepsilon_j \quad (j \in \overline{1,k})$$

have already been constructed. Then

$$\Big| A \backslash \bigcup_{j=1}^{k} (\overline{G}_j \cap A) \Big| \ge |A| - \sum_{j=1}^{k} \delta_j > \delta_{k+1},$$

where $\overline{G}_j$ is the closure of G_j. Therefore by (1), there exists

$$E \subset A \backslash \bigcup_{j=1}^{k} (\overline{G}_j \cap A)$$

with $|E| = \delta_{k+1}$. We can easily obtain an open set $Q \supset E$ with the properties

$$Q \cap \bigcup_{j=1}^{k} \overline{G}_j = \varnothing, \quad |Q| > |E| = \delta_{k+1}, \quad |Q \backslash E| < \varepsilon_{k+1}.$$

By (2), we can choose an elementary set $G_{k+1} \subset Q$ such that $|G_{k+1}| = \delta_{k+1}$. By virtue of the properties of Q we have

$$|G_{k+1} \backslash A| \le |Q \backslash E| < \varepsilon_{k+1},$$

$$G_{k+1} \cap \bigcup_{j=1}^{k} G_j = \varnothing,$$

which obviously proves Lemma 11. $\square$

We shall need the following simple lemma (see [2], Ch. III, §1).

Lemma 12. *Let G be an open bounded set in $\mathbb{R}^n$, and K be a compact set in $\mathbb{R}^n$ with $|K|>0$. Then there exists a sequence $\{K_k\}$ of pairwise nonintersecting sets, homothetic to K, contained in G and such that $\big|G\setminus \bigcup_k K_k\big|=0$.*

We shall also need the following well-known fact from the measure theory (see, e.g., [9], Ch. "Uniform Approximation").

Lemma 13. *For every measurable sets $A_1, A_2 \subset \mathbb{R}^n$, $|A_1|=|A_2|$, there exists a measure preserving and invertible mapping $\omega: A_1 \to A_2$.*

4. Proof of Theorem 1

Without loss of generality we assume that $f\geq 0$ and $f \notin (1+\ln^+ L)^n(\mathbb{R}^n)$. Denote

$$
\begin{aligned}
&G=\operatorname{supp} f, \quad A_k=\{k-1\leq f<k\} \ \ (k\in\mathbb{N}),\\
&k_0=\min\Big\{k\geq 2 \ : \ \sum_{m=k}^{\infty} 9^n m|A_m|<|G|\Big\},\\
&N=\{k\geq k_0 : |A_k|>0\}.
\end{aligned}
$$

Choose natural numbers $m_k\geq n$ $(k\in N)$ such that

$$
\sum_{k\in N}\frac{k(\ln k)^n|A_k|}{m_k}<1\ . \tag{4.1}
$$

For $k\in N$, let $\ell_{k,1},\dots,\ell_{k,m_k}$ be the straight lines passing through the origin with none of n lying in the same hyperplane. Then by Lemma 2 there exists $\varepsilon_k>0$ such that

$$
\min_{1\leq j\leq n}\angle(\ell,\ell_{k,\nu_j})<\frac{\pi}{2}-\varepsilon_k \tag{4.2}
$$

for every $1\leq\nu_1<\nu_2<\cdots<\nu_n\leq m_k$ and for every straight line ℓ.

For every $k\in N$ and $m\in\overline{1,m_k}$ let us consider the rectangle $I_{k,m}$ with the properties:

$$
r(I_{k,m})\geq\frac{4kn}{\sin\varepsilon_k}, \quad |I_{k,m}|=\frac{|A_k|}{m_k}, \quad \ell_{I_{k,m}}=\ell_{k,m}. \tag{4.3}
$$

Denote $J_{k,m}=J(I_{k,m},4k)$, $E_{k,m}=E(\ell_{k,m},\varepsilon_k)$ $(k\in N,\ m\in\overline{1,m_k})$. By Lemma 2

$$
\big\{M_{B_2(E_{k,m})}(4k\chi_{I_{k,m}})>1\big\}\subset J_{k,m}.
$$

From the definition of k_0 and $J_{k,m}$ and from (4.3), we conclude by virtue of Lemma 11 that there exist pairwise nonintersecting open sets $Q_{k,m}$ such that

$$
|Q_{k,m}|=|J_{k,m}| \quad\text{and}\quad |Q_{k,m}\setminus G|<\frac{1}{2^k m_k}\ .
$$

For each $k \in N$ and $m \in \overline{1, m_k}$ we complete $Q_{k,m}$ with pairwise nonintersecting rectangles $\{J_{k,m,q}\}$ which are homothetic to the rectangle $J_{k,m}$ (see Lemma 12), i.e.,

$$\begin{aligned} &J_{k,m,q} = H_{k,m,q}(J_{k,m}), \quad \text{where } H_{k,m,q} \text{ is the homothety } (q \in \mathbb{N}), \\ &J_{k,m,q} \subset Q_{k,m} \quad (q \in \mathbb{N}), \\ &J_{k,m,q} \cap J_{k,m,q'} = \varnothing \quad (q \neq q'), \\ &\Big| Q_{k,m} \setminus \bigcup_{q \in \mathbb{N}} J_{k,m,q} \Big| = 0. \end{aligned}$$

Let $I_{k,m,q} = H_{k,m,q}(I_{k,m})$ ($k \in N$, $m \in \overline{1, m_k}$, $q \in \mathbb{N}$). Because of the homothety properties we can easily see that

$$J_{k,m,q} = J(I_{k,m,q}, 4k), \tag{4.4}$$

$$\big\{ M_{B_2(E_{k,m})}(4k\chi_{I_{k,m,q}}) > 1 \big\} \subset J_{k,m,q} \tag{4.5}$$

for $k \in N$, $m \in \overline{1, m_k}$, $q \in \mathbb{N}$, and

$$\sum_{q \in \mathbb{N}} |I_{k,m,q}| = |I_{k,m}| = \frac{|A_k|}{m_k} \tag{4.6}$$

for $k \in N$, $m \in \overline{1, m_k}$.

Denote

$$\begin{aligned} g_{k,m} &= \sup \big\{ k\chi_{I_{k,m,q}} : q \in \mathbb{N} \big\} \quad (k \in N, \ \ m \in \overline{1, m_k}), \\ g &= \sup \big\{ g_{k,m} : k \in N, \ \ m \in \overline{1, m_k} \big\}, \end{aligned}$$

and prove that

$$\sup_{\theta \in \theta(\mathbb{R}^n)} \int\limits_{\{M_{B_2(\theta)}(g) > 1/2\}} M_{B_2(\theta)}(g) < \infty. \tag{4.7}$$

The following estimate is valid:

$$\operatorname{card} S_{\theta,k} < n^2 \quad (\theta \in \theta(\mathbb{R}^n), \quad k \in N), \tag{4.8}$$

where $S_{\theta,k} = \{m \in \overline{1, m_k} : \theta \notin E_{k,m}\}$. Indeed, let us assume the contrary, i.e., that $\operatorname{card} S_{\theta,k} \geq n^2$ for some $\theta \in \theta(\mathbb{R}^n)$ and $k \in N$. Then there exist $1 \leq \nu_1 < \cdots < \nu_{n^2} \leq m_k$ such that $\theta \in E_{k,\nu_j}$ ($j \in \overline{1, n^2}$), i.e., $\max\limits_{1 \leq i \leq n} \angle(\theta^i, \ell_{k,\nu_j}) \geq \frac{\pi}{2} - \varepsilon_k$ ($j \in \overline{1, n^2}$). Hence there exist a straight line $\theta^i \in \theta$ and indices $\nu_1', \dots, \nu_n' \in \{\nu_1, \dots, \nu_{n^2}\}$ such that $\angle(\theta^i, \ell_{k,\nu_j'}) \geq \frac{\pi}{2} - \varepsilon_k$ ($j \in \overline{1, n}$), which contradicts (4.2). Therefore (4.8) is proved.

Let us consider an arbitrary frame θ. Suppose

$$g_\theta = \begin{cases} \sup\{g_{k,m} : k \in N,\ m \in S_{\theta,k}\} & \text{if } \bigcup\limits_{k\in N} S_{\theta,k} \neq \varnothing, \\ 0 & \text{if } \bigcup\limits_{k\in N} S_{\theta,k} = \varnothing. \end{cases}$$

By Lemma 8, (4.1), (4.3), (4.6) and (4.8) we have

$$\int\limits_{\{M_{\mathbf{I}(\theta)}(g_\theta)>1/4\}} M_{\mathbf{I}(\theta)}(g_\theta) \le c_3 \int\limits_{\mathbb{R}^n} g_\theta (1+\ln^+ 4g_\theta)^n <$$

$$< c_3 \sum_{k\in N} n^2 k(1+\ln 4k)^n \frac{|A_k|}{m_k} < 5^n n^2 c_3 \sum_{k\in N} \frac{k(\ln k)^n |A_k|}{m_k} < 5^n n^2 c_3. \quad (4.9)$$

Denote

$$T = \big\{(k,m,q) : k \in N,\ \ m \in \overline{1,m_k}\setminus S_{\theta,k},\ \ q\in\mathbb{N}\big\},$$
$$J_{k,m,q}(\lambda) = J(I_{k,m,q}, k/\lambda) \quad \text{for} \quad (k,m,q)\in T \ \text{ and } \ 1/4 \le \lambda < k.$$

Obviously,

$$r(I_{k,m,q}) > \frac{4kn}{\sin\varepsilon_k} \ge \frac{kn}{\lambda \sin\varepsilon_k}$$

for $(k,m,q)\in T$ and $1/4 \le \lambda < k$, whence on account of (4.4), (4.5) and Lemma 2

$$\big\{M_{B_2(E_{k,m})}(k\chi_{I_{k,m,q}}) > \lambda\big\} =$$
$$= \Big\{M_{B_2(E_{k,m})}\Big(\frac{k}{\lambda}\,\chi_{I_{k,m,q}}\Big) > 1\Big\} \subset J_{k,m,q}(\lambda) \subset J_{k,m,q}.$$

Consequently, since $\theta\in E_{k,m}$, we have

$$\big\{M_{B_2(\theta)}(k\chi_{I_{k,m,q}}) > \lambda\big\} \subset J_{k,m,q}(\lambda) \subset J_{k,m,q}. \quad (4.10)$$

On the other hand, it is clear that

$$\big\{M_{B_2(\theta)}(k\chi_{I_{k,m,q}}) > \lambda\big\} = \varnothing \quad (4.11)$$

for $(k,m,q)\in T$ and $\lambda \ge k$.

It is easy to see that the functions $k\chi_{I_{k,m,q}}$ belong to the class $\overline{L}(\mathbb{R}^n)$ and therefore, keeping in mind that the rectangles $J_{k,m,q}$ are pairwise nonintersecting and using (4.10), (4.11) and Lemma 6 we have

$$\big\{M_{B_2(\theta)}(g-g_\theta) > \lambda\big\} \subset \bigcup_{(k,m,q)\in T,\ k>\lambda} J_{k,m,q}(\lambda) \quad \text{for} \quad \lambda \ge 1/4.$$

The above inequality, (4.3), (4.6) and the definition of g imply that

$$\big|\{M_{B_2(\theta)}(g-g_\theta)>\lambda\}\big| \le \sum_{(k,m,q)\in T,\ k>\lambda} |J_{k,m,q}(\lambda)| \le$$

$$\le \sum_{k\in N,\ k>\lambda} \sum_{m=1}^{m_k}\sum_{q=1}^{\infty} 9^n\,\frac{k}{\lambda}\,|I_{k,m,q}| = \frac{9^n}{\lambda}\sum_{k\in N,\ k>\lambda} k|A_k| \le \frac{2\cdot 9^n}{\lambda}\int\limits_{\{f>\lambda/2\}} f$$

for $\lambda \ge 1/4$.

Consequently, by Lemma 9 we obtain

$$\int\limits_{\{M_{B_2(\theta)}(g-g_\theta)>1/4\}} M_{B_2(\theta)}(g-g_\theta) \le 2\cdot 9^n \int\limits_{\mathbb{R}^n} f(1+\ln^+ 8f). \tag{4.12}$$

From (4.9), (4.12) and Lemma 10 we find that

$$\int\limits_{\{M_{B_2(\theta)}(g)>1/2\}} M_{B_2(\theta)}(g) < 2\cdot 5^n n^2 c_3 + 4\cdot 9^n \int\limits_{\mathbb{R}^n} f(1+\ln^+ 8f),$$

whence by virtue of the arbitrariness of θ we conclude that (4.7) is valid.

Denote

$$P_k = \bigcup_{m=1}^{m_k}\bigcup_{q=1}^{\infty}(I_{k,m,q}\cap G) \quad (k\in N).$$

By our choice of sets $Q_{k,m}$ we easily see that

$$0 \le |A_k| - |P_k| < \frac{1}{2^k} \quad (k\in N). \tag{4.13}$$

Let $A'_k \subset A_k$ $(k\in N)$ be some measurable set with $|A'_k| = |P_k|$. By Lemma 13 there exists a measure preserving and invertible mapping $\omega : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\omega(P_k) = A'_k \quad (k\in N), \quad \omega\Big(G\setminus\bigcup_{k\in N} P_k\Big) = G\setminus\bigcup_{k\in N} A'_k,$$
$$\omega(x) = x \quad (x\in\mathbb{R}^n\setminus G). \tag{4.14}$$

Suppose

$$\varphi_1 = (f\circ\omega)\chi_{\bigcup\limits_{k\in N} P_k} \quad \text{and} \quad \varphi_2 = (f\circ\omega)\chi_{\mathbb{R}^n\setminus\bigcup\limits_{k\in N} P_k}.$$

Obviously, $f\circ\omega = \varphi_1+\varphi_2$. We have

$$\int\limits_{\mathbb{R}^n} \varphi_2(1+\ln^+\varphi_2)^n = \int\limits_{\mathbb{R}^n\setminus\bigcup\limits_{k\in N} A'_k} f(1+\ln^+ f)^n =$$

$$= \int\limits_{\{0\le f<k_0-1\}} f(1+\ln^+ f)^n + \sum_{k\in N} \int\limits_{A_k\setminus A'_k} f(1+\ln^+ f)^n = \alpha_1+\alpha_2.$$

It can be seen that $\alpha_1<\infty$, and by (4.13)

$$\alpha_2 \le \sum_{k\in N} \frac{k(1+\ln k)^n}{2^k} < \infty.$$

Thus $\varphi_2 \in L(1+\ln^+ L)^n(\mathbb{R}^n)$. Therefore, by the obvious inequality $\varphi_1 \le g$, (4.7) and Lemmas 8 and 10, we conclude that

$$\sup_{\theta\in\theta(\mathbb{R}^n)} \int\limits_{\{M_{B_2(\theta)}(f\circ\omega)>1\}} M_{B_2(\theta)}(f\circ\omega) < \infty,$$

which together with (4.14) completes the proof of Theorem 1.

5. Remarks

(1) By the equality $M_{B_2(\theta)}(\alpha f) = \alpha M_{B_2(\theta)}(f)$ $(\alpha>0)$, we can easily verify that Theorem 1 remains valid if instead of $\{M_{B_2(\theta)}(f\circ\omega)>1\}$ we shall take the integrals on $\{M_{B_2(\theta)}(f\circ\omega)>\lambda\}$, where $\lambda>0$ is an arbitrarily fixed number.

(2) Theorem 1 immediately yields the following improvement:

Theorem 2. *For every function $f \in L(1+\ln^+ L)(\mathbb{R}^n)$ $(n\ge 2)$ and measurable sets $G_1, G_2 \in \mathbb{R}^n$ such that $f\chi_{\mathbb{R}^n\setminus G_1} \in L(1+\ln^+ L)^n(\mathbb{R}^n)$ and $|G_1| = |G_2|$ there exists a measure preserving and invertible mapping $\omega : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$1)\quad \omega(G_1) = G_2 \quad \textit{and} \quad \{x : \omega(x)\ne x\} \subset G_1\cup G_2,$$

$$2)\quad \sup_{\theta\in\theta(\mathbb{R}^n)} \int\limits_{\{M_{B_2(\theta)}(f\circ\omega)>1\}} M_{B_2(\theta)}(f\circ\omega) < \infty.$$

Proof. Let $\omega_1 : \mathbb{R}^n\to\mathbb{R}^n$ be a measure preserving and invertible mapping such that (see Lemma 12) $\omega_1(G_1) = G_2$ and $\{x : \omega(x)\ne x\} \subset G_1\cup G_2$. Consider the function $g = (f\circ\omega_1)\chi_{G_2}$. Then $\operatorname{supp} g \subset G_2$, and by virtue of Theorem 1 (see Remark (1)) there exists a measure preserving and invertible mapping $\omega_2 : \mathbb{R}^n\to\mathbb{R}^n$ such that

$$\{x : \omega_2(x)\ne x\} \subset \operatorname{supp} g \subset G_2 \ \text{ and } \ \sup_{\theta\in\theta(\mathbb{R}^n)} \int\limits_{\{M_{B_2(\theta)}(g\circ\omega_2)>1/2\}} M_{B_2(\theta)}(g\circ\omega_2) < \infty.$$

Obviously, $(f\circ\omega_1)\chi_{\mathbb{R}^n\setminus G_2} \in L(1+\ln^+ L)^n(\mathbb{R}^n)$. Therefore, by Lemmas 8 and 10, one can take $\omega_2\circ\omega_1$ as ω. $\square$

(3) For arbitrary $\varepsilon > 0$, a mapping ω "correcting" the function $f \in L(1+\ln^+ L)(\mathbb{R}^n)$ can be chosen so that

$$|\{f \circ \omega \neq f\}| < \varepsilon.$$

For this it is enough in Theorem 2 to take G_1 and G_2 with measures less than $\varepsilon/2$.

(4) When $G_1 = \{|f| > 1\}$, and G_2 is a cubic interval, Theorem 2 has been proved for $n = 2$ in [7] and announced for $n \geq 2$ in [8].

References

1. M. de Guzmán and G. V. Welland, On the differentiation of integrals. *Rev. Un. Mat. Argentina* **25**(1971), 253–276.

2. M. de Guzmán, Differentiation of integrals in $\mathbb{R}^n$. *Lecture Notes in Mathematics* 481, *Springer,* 1975.

3. B. Jessen, J. Marcinkiewicz, and A. Zygmund, Note on the differentiability of multiple integrals. *Fund. Math.* **25**(1935), 217–234.

4. L. D. Gogoladze, On maximal Hardy–Littlewood functions. (Russian) *Soobshch. Akad. Nauk Gruz. SSR* **3**(1983), 257–259.

5. L. D. Gogoladze, On the summation of multiple trigonometric series and conjugate functions. (Russian) *Dissertation for the Doctor of Science Degree, Tbilisi,* 1984.

6. R. J. Bagby, A note on the strong maximal function. *Proc. Amer. Math. Soc.* **88**(1983), 648–650.

7. G. Oniani, On strong maximal operators corresponding to different frames. *Georgian Math. J.* **3**(1996), No. 1, 81–96.

8. G. Oniani, On the integrability of maximal functions $M_{B_2(\theta)}(f)$. *Georgian Bull. Acad. Sci.* **156**(1997), No. 1, 5–6.

9. P. R. Halmos, Lectures on the ergodic theory. *Math. Soc. Japan, Tokyo,* 1956.

(Received 20.06.1997)

Author's address:
A. Tsereteli Kutaisi State University
55, Queen Tamar St., Kutaisi 384000
Georgia