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FIRST BOUNDARY VALUE PROBLEM OF ELECTROELASTICITY FOR A TRANSVERSALLY ISOTROPIC PLANE WITH CURVILINEAR CUTS

L. BITSADZE

ABSTRACT. The first boundary value problem of electroelasticity for a transversally isotropic plane with curvilinear cuts is investigated. The solvability of a system of singular integral equations is proved by using the potential method and the theory of singular integral equations.

In this paper the first boundary value problem of electroelasticity is investigated for a transversally isotropic plane with curvilinear cuts. The second boundary value problem of electroelasticity for this kind medium was solved in [1] using the theory of analytic functions and singular integral equations. The boundary value problems of electroelasticity for transversally isotropic media were studied in [2] (Ch. VI), while the boundary value problems of elasticity for anisotropic media with cuts were considered in [3], [4]. The uniqueness theorems for boundary value problems of electroelasticity are given in [5], [6]. Here we shall be concerned with the plane problem of electroelasticity (it is assumed that the second component of the three-dimensional displacement vector is equal to zero, while the components u_1, u_3 and the electrostatic potential u_4 depend only on the variables x_1 and x_3 [2]).

Let an electroelastic transversally isotropic plane be weakened by curvilinear cuts $l_j = a_j b_j$, $j = 1, \dots, p$. Assume that l_j, $j = 1, \dots, p$, are simple nonintersecting open Lyapunov arcs. The direction from a_j to b_j is taken as the positive one on l_j. The normal to l_j will be drawn to the right relative to motion in the positive direction. Denote by D^- the infinite plane with curvilinear cuts l_j, $j = 1, \dots, p$, $l = \bigcup\limits_{j=1}^{p} l_j$. Suppose that D^- if filled with some material.

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We introduce the notation $z = x_1 + ix_3$, $\zeta_k = y_1 + \alpha_k y_3$, $\tau_k = t_1 + \alpha_k t_3$, $z_k = x_1 + \alpha_k x_3$, $\tau = t_1 + it_3$ and $\sigma_k = z_k - \zeta_k$.

The basic homogeneous equations of the plane theory of electroelasticity have the form [2]

$$C(\partial_x)U = 0, \tag{1}$$

where $C(\partial_x) = \|C_{ij}\|_{3\times 3}$,

$$\begin{aligned}
&C_{11} = c_{11}\frac{\partial^2}{\partial x_1^2} + c_{44}\frac{\partial^2}{\partial x_3^2}, \quad C_{12} = (c_{13} + c_{44})\frac{\partial^2}{\partial x_1 \partial x_3},\\
&C_{13} = (e_{13}+e_{15})\frac{\partial^2}{\partial x_1 \partial x_3}, \quad C_{22} = c_{44}\frac{\partial^2}{\partial x_1^2} + c_{33}\frac{\partial^2}{\partial x_3^2},\\
&C_{23} = e_{15}\frac{\partial^2}{\partial x_1^2} + e_{33}\frac{\partial^2}{\partial x_3^2}, \quad C_{33} = -\varepsilon_{11}\frac{\partial^2}{\partial x_1^2} - \varepsilon_{33}\frac{\partial^2}{\partial x_3^2},\\
&\quad U = U(u_1, u_3, u_4),
\end{aligned}$$

u_1, u_3 are the components of the displacement vector, u_4 is the electrostatic potential, c_{11}, $c_{44}, \dots$ are the constants characterizing the medium under consideration.

The first boundary value problem is formulated as follows: let the boundary values of the displacement vector and electrostatic potential be given on both edges of the arc l_j. Further, assume that at infinity we have the principal vector of external forces acting on l, stress, the electrostatic potential and the induction vector. It is required to define the deformed state of the plane.

If we denote by u^+ (u^-) the limit of u on l from the left (right), then the boundary conditions of the problem will take the form

$$u^+ = f^+, \quad u^- = f^-, \tag{2}$$

where f^+ and f^- are the known vector-functions on l of the Hölder class H which have derivatives in the class H^* (for the definitions of the classes H and H^* see [7]) and satisfying, at the ends a_j and b_j of l_j, the conditions

$$f^+(a_j) = f^-(a_j), \quad f^+(b_j) = f^-(b_j).$$

It is obvious that displacement vector discontinuities along the cut l_j generate a singular stress field in the medium. Hence it is of interest for us to study the solution behavior in the neighborhood of the cuts.

We seek for a solution of the problem in the form a double-layer potential [8], [9]

$$U(z) = \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{3} E_{(k)} \int\limits_l \frac{\partial \ln \sigma_k}{\partial s}[g(s) + ih(s)]ds +$$

$$+\sum_{j=1}^{p} V_j(z) + A^{\infty}\,\mathrm{Re}\,z + B^{\infty}\,\mathrm{Im}\,z + M^{p+1}, \tag{3}$$

where s is the arc coordinate,

$$E_{(k)} = -\|A_{pq}^{(k)}\|_{3\times 3}\,\|h_{pq}\|_{3\times 3}\,\frac{ib_3}{A_1 h_{11}}$$

and $\mathrm{Im}\sum\limits_{k=1}^{3} E_{(k)}\frac{\partial}{\partial s}\ln\sigma_k$ denotes a special fundamental matrix that reduces the first boundary value problem to a Fredholm integral equation of second order. It is assumed here that $\Gamma = \mathrm{Im}\sum\limits_{k=1}^{3}\|A_{pq}^{(k)}\|_{3\times 3}\ln\sigma_k$ is the basic fundamental matrix for equation (1), Γ is the symmetric matrix whose all elements are one-valued functions on the entire plane and possess a logarithmic singularity. It is easy to show that every element of the matrix Γ is a solution of equation (1) at any $x \neq y$. For the expression of the matrix Γ see [8]. The elements of the real symmetric matrix h are defined as follows:

$$h_{11} = c_{11}c_{44}\frac{1}{\sqrt{a_1a_2a_3}}, \quad \alpha_k = i\sqrt{a_k}, \quad k = 1,2,3,$$
$$h_{22} = c_{11}c_{44}C - \alpha B + c_{44}c_{33}A,$$
$$-h_{33} = c_{11}\varepsilon_{11}C + B[c_{11}\varepsilon_{33} + c_{44}\varepsilon_{11} + (e_{13}+e_{15})^2] + c_{44}\varepsilon_{33}A,$$
$$h_{23} = c_{11}e_{15}C - B[e_{13}(c_{13}+c_{44}) + c_{13}e_{15} - c_{11}e_{33}] + c_{44}e_{33}A,$$
$$\alpha = c_{13}^2 - c_{11}c_{33} + 2c_{13}c_{44}, \quad A_1 = \frac{b_0}{c_{11}}C + \alpha_0 B + \frac{b_3}{c_{44}}A,$$
$$b_0 = c_{11}(c_{44}\varepsilon_{11} + e_{15}^2), \quad b_3 = c_{44}(c_{33}\varepsilon_{33} + e_{33}^2), \quad h_{12} = h_{13} = 0$$

(the values of A, B, C are given in [8]).

A^{∞} and B^{∞} are the preassigned column vectors whose components are:

$$A_1^{\infty} = a_{11}\tau_{11}^{\infty} + a_{13}\tau_{33}^{\infty} + a_{14}\tau_{43}^{\infty},$$
$$A_3^{\infty} = \frac{1}{2}(b_{11}\tau_{13}^{\infty} + (b_{33}-a_{14})\tau_{41}^{\infty}) + \varepsilon^{\infty},$$
$$A_4^{\infty} = (b_{33}-a_{14})\tau_{13}^{\infty} - b_{44}\tau_{41}^{\infty},$$
$$B_1^{\infty} = A_3^{\infty} - 2\varepsilon^{\infty}, \quad B_3^{\infty} = a_{13}\tau_{11}^{\infty} + a_{33}\tau_{33}^{\infty} + a_{34}\tau_{43}^{\infty},$$
$$B_4^{\infty} = a_{14}\tau_{11}^{\infty} + a_{34}\tau_{33}^{\infty} - a_{44}\tau_{43}^{\infty}.$$

Here $\tau_{11}, \tau_{33}, \dots, \tau_{43}$ denote the components of the electromechanical stress tensor (see [8]); $a_{11}, \dots, a_{44}$ are the real constant values which are the combinations of the known piezoelastic, elastic and dielectric constants

$$a_{11} = \frac{b_3}{c_{44}\Delta}, \quad a_{44} = \frac{c_{11}c_{33} - c_{13}^2}{\Delta}, \quad a_{34} = \frac{c_{11}e_{33} - c_{13}e_{13}}{\Delta},$$
$$a_{33} = \frac{c_{11}\varepsilon_{33} + e_{13}^2}{\Delta}, \quad a_{13} = -\frac{\varepsilon_{33}c_{13} + e_{33}e_{13}}{\Delta}, \quad a_{14} = \frac{c_{33}e_{13} - e_{33}c_{13}}{\Delta},$$
$$\Delta = \frac{c_{11}c_{33} - c_{13}^2}{c_{33}c_{44}} b_3 + \frac{1}{c_{33}}(c_{33}e_{13} - e_{33}c_{13})^2,$$
$$b_{11} = \frac{c_{11}\varepsilon_{11}}{b_0}, \quad b_{44} = \frac{c_{11}c_{44}}{b_0}, \quad b_{33} - a_{14} = \frac{c_{11}e_{15}}{b_0};$$

α_k, $k = 1, \dots, 6$, are the roots of the characteristic equation

$$\begin{aligned} &b_3\alpha^6 + b_2\alpha^4 + b_1\alpha^2 + b_0 = 0,\\ &b_1 = c_{11}\alpha_0 - \varepsilon_{11}(c_{13} + c_{44})^2 + c_{44}(e_{13} + e_{15})^2 -\\ &\qquad - 2e_{15}(e_{13} + e_{15})(c_{13} + c_{44}) + c_{44}\frac{b_0}{c_{11}},\\ &b_2 = c_{11}\frac{b_3}{c_{44}} + c_{44}\alpha_0 - \varepsilon_{33}(c_{13} + c_{44})^2 + c_{33}(e_{13} + e_{15})^2 -\\ &\qquad - 2e_{33}(c_{13} + c_{44})(e_{13} + e_{15}). \end{aligned}$$

$M^{(j)}$, $j = 1, \dots, p+1$, are the unknown real constant vectors to be defined later on; g and h are the unknown real vectors from the Hölder class that have derivatives in the class H^*.

We write

$$V_j(z) = \frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{3} A_{(k)} \frac{(z_k - b_j^{(k)})\ln(z_k - b_j^{(k)}) - (z_k - a_j^{(k)})\ln(z_k - a_j^{(k)})}{b_j^{(k)} - a_j^{(k)}} M^j,$$
$$a_j^{(k)} = \operatorname{Re} a_j + \alpha_k \operatorname{Im} a_j, \quad b_j^{(k)} = \operatorname{Re} b_j + \alpha_k \operatorname{Im} b_j.$$

The vector $V_j(z)$ satisfies the following conditions:

1. V_j has the logarithmic singularity at infinity

$$V_j = \frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{3} A_{(k)}(-\ln z_k + 1)M^j + O(z_k^{-1}).$$

2. By V_j is meant a branch which is uniquely defined on the plane cut along l_j.

3. V_j is continuously extendable on l_j from the left and the right, the end points a_j and b_j inclusive, i.e., we have the equalities

$$V_j^+(a_j) = V_j^-(a_j), \quad V_j^+(b_j) = V_j^-(b_j),$$

$$V_j^+ - V_j^- = 2\operatorname{Re}\sum_{k=1}^{3} A_{(k)}\,\frac{\tau_k - a_j^{(k)}}{b_j^{(k)} - a_j^{(k)}}\,M^j, \quad j = 1,\dots,p.$$

To define the unknown density, we obtain, by virtue of (3)–(2), a system of singular integral equations of the normal type

$$\pm g(\tau) + \frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{3} E_{(k)} \int\limits_{l} \frac{\partial \ln(\tau_k - \zeta_k)}{\partial s}\,(g + ih)\,ds +$$
$$+ \sum_{j=1}^{p} V_j^{\pm} + A^{\infty}\operatorname{Re}\tau + B^{\infty}\operatorname{Im}\tau + M^{p+1} = f^{\pm}.$$

This formula implies

$$2g(\tau) = f^+ - f^- - \operatorname{Re}\sum_{j=1}^{p}\sum_{k=1}^{3} A_{(k)}\,\frac{\tau_k - a_j^{(k)}}{b_j^{k} - a_j^{(k)}}\,M^j, \tag{4}$$

$$\frac{1}{\pi}\int\limits_{l}\frac{h(\zeta)ds}{\zeta - \tau} + \frac{1}{\pi}\int\limits_{l} K(\tau,\zeta)h(\zeta)\,ds = \Omega(\tau), \tag{5}$$

where

$$K(\tau,\zeta) = -i\,\frac{\partial\Theta}{\partial s}\,E + \operatorname{Re}\sum_{k=1}^{3} E_{(k)}\,\frac{\partial}{\partial s}\ln\Big(1 + \lambda_k\,\frac{\overline{\tau} - \overline{\zeta}}{\tau - \zeta}\Big),$$

$$\lambda_k = \frac{1 + i\alpha_k}{1 - i\alpha_k}, \quad \Theta = \arg(\tau - \zeta), \quad \Omega(\tau) = \frac{1}{2}\,(f^+ + f^-) -$$
$$-\frac{1}{2}\sum_{j=1}^{p}(V_j^+ + V_j^-) - A^{\infty}\operatorname{Re}\tau - B^{\infty}\operatorname{Im}\tau - M^{p+1} -$$
$$-\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{3} E_{(k)}\int\limits_{l}\frac{\partial}{\partial s}\,ln(\tau_k - \zeta_k)g(\zeta)\,ds.$$

Thus we have defined the vector g on l. It is not difficult to verify that $g \in H$, $g' \in H^*$, $g(a_j) = g(b_j) = 0$, $\Omega \in H$, $\Omega' \in H^*$. Formula (5) is a system of singular integral equations of the normal type with respect to the vector h. The points a_j and b_j are nonsingular, while the total index of the class h_{2p} is equal to $-3p$ (for the definition of the class h_{2p} see [7]).

A solution of system (5), if it exists, will be expressed by a vector of the Hölder class that vanishes at the end points a_j, b_j, and has derivatives in the class H^*.

Next we shall prove that the homogeneous system of equations corresponding to (5) admits on a trivial solution in the class h_{2p}. Let the contrary be true. Assume $h^{(0)}$ to be a nontrivial solution of the homogeneous system corresponding to (5) in the class h_{2p} and construct the potential

$$U_0(z)=\frac{1}{\pi}\operatorname{Re}\sum_{k=1}^{3}E_{(k)}\int\limits_{l}\frac{\partial}{\partial s}\ln(z_k-\zeta_k)\,h^0(\zeta)ds.$$

Clearly, $U_0^+=U_0^-=0$ and by the uniqueness theorem we have $U_0(z)=0$, $z\in D^-$. Then $TU_0=0$, $z\in D^-$ and

$$(TU_0)^+-(TU_0)^-=2\|h_{ij}\|\frac{b_3}{A_1h_{11}}\,\frac{\partial h^{(0)}}{\partial s}=0.$$

Therefore, since $h^0(a_j)=0$, we obtain $h^0=0$, which completes the proof. Thus the homogeneous system adjoint to (5) will have $3p$ linearly independent solutions σ_j, $j=1,\dots,3p$, in the adjoint class and the condition for system (5) to be solvable will be written as

$$\int\limits_{l}\Omega\sigma_j\,ds=0,\quad j=1,\dots,3p. \tag{6}$$

Taking into account the latter condition and that

$$\int\limits_{l}\big[(TU)^+-(TU)^-\big]\,ds=-2\sum_{j=1}^{p}M^j, \tag{7}$$

we obtain a system of $3p+3$ algebraic equations with the same number of unknowns with respect to the components of the unknown vector M^j.

We shall show that system (6)–(7) is solvable. Assume that the homogeneous system obtained from (6)–(7) has a nontrivial solution $M_j^0=(M_{1j}^0,M_{3j}^0,M_{4j}^0)$, $j=1,\dots,p+1$, and construct the potential

$$U_0=\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{3}E_{(k)}\int\limits_{l}\frac{\partial}{\partial s}\ln(z_k-\zeta_k)\,(g_0+ih_0)\,ds+\sum_{j=1}^{p}V_j^0+M_{P+1}^0,$$

where

$$g_0=-\operatorname{Re}\sum_{k=1}^{3}A_{(k)}\,\frac{\tau_k-a_p^{(k)}}{b_p^{(k)}-a_p^{(k)}}\,M_p^{(0)},\quad \tau\in l_p,$$

$$V_j^{(0)}=\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{3}A_{(k)}\,\frac{(z_k-b_j^{(k)})\ln(z_k-b_j^{(k)})-(z_k-a_j^{(k)})\ln(z_k-a_j^{(k)})}{b_j^{(k)}-a_j^{(k)}}\,M_j^{(0)}.$$

It is obvious that $U_0^+=U_0^-=0$. Applying the formulas $\sum\limits_{j=1}^{P}M_j^0=0$ and $\int\limits_{l_j}((TU_0)^+-(TU_0)^-)ds=-2M_j^0=0$, $j=1,\dots,p$, $U_0(\infty)=M_0^{p+1}=0$, we obtain $M_j^0 = 0$, which contradicts the assumption. Therefore system (6)–(7) has a unique solution.

For $M_0^{(j)}$ system (5) is solvable in the class h_{2p}. The solution of the problem posed is given by potential (3) constructed using the solution h of system (5) and the vector g.

Let us consider a particular case with a rectilinear cut on the segment ab of the real axis. Assuming that the principal vector of external forces, displacement, electrostatic potential and induction vector vanish at infinity, we obtain

$$U(z) = \frac{1}{2\pi} \operatorname{Im} \sum_{k=1}^{3} E_{(k)} \int_a^b \frac{u^+(t) - u^-(t)}{t - z_k}\, dt + \\ + \frac{1}{2\pi} \operatorname{Im} \sum_{k=1}^{3} E_{(k)} X(z_k) \int_a^b \frac{u^+(t) - u^-(t)}{X^+(t)(t - z_k)}\, dt,$$

where $X(z_k) = \sqrt{(z_k - a)(b - z_k)}$.

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Author's address:
I. Vekua Institute of Applied Mathematics
Tbilisi State University
2, University St., Tbilisi 3800043
Georgia