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OSCILLATION AND NONOSCILLATION CRITERIA FOR A SECOND ORDER LINEAR EQUATION

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ABSTRACT. New oscillation and nonoscillation criteria are established for the equation

$$u'' + p(t)u = 0,$$

where $p :]1, +\infty[\to R$ is the locally integrable function. These criteria generalize and complement the well known criteria of E. Hille, Z. Nehari, A. Wintner, and P. Hartman.

We shall consider the equation

$$u'' + p(t)u = 0, \tag{0.1}$$

where the function $p :]1, +\infty[\to R$ is Lebesgue integrable on each finite segment from $[1, +\infty[$. By a solution of equation (0.1) is understood a function $u : [1, +\infty[\to R$ which is absolutely continuous together with its first derivative on each finite segment from $[1, +\infty[$ and which satisfies almost everywhere equation (0.1). Equation (0.1) is called *oscillatory* if there exists its solution with an infinite number of zeros and *nonoscillatory* otherwise.

Below we shall give some new oscillation and nonoscillation criteria for equation (0.1). The paper is organized as follows: the main results are formulated in Section 1; Section 2 contains remarks and comments; the auxiliary propositions are presented in Section 3, while the proofs of the main results can be found in Section 4.

Before we proceed to the formulation of the main results we want to introduce some notation.

Let

$$c(t) = \frac{1}{t} \int_1^t \int_1^s p(\xi)\, d\xi\, ds \quad \text{for} \quad t \geq 1.$$

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Below it will always be assumed that there exists a finite limit

$$c_0 \overset{\text{def}}{=} \lim_{t\to+\infty} c(t). \tag{0.2}$$

We set

$$Q(t) = t\bigg(c_0 - \int_1^t p(s)\,ds\bigg), \quad H(t) = \frac{1}{t}\int_1^t s^2 p(s)\,ds \ \text{ for } \ t \geq 1,$$

$$Q_* = \liminf_{t\to+\infty} Q(t), \quad Q^* = \limsup_{t\to+\infty} Q(t),$$

$$H_* = \liminf_{t\to+\infty} H(t), \quad H^* = \limsup_{t\to+\infty} H(t).$$

1. Formulation of the Main Results

Theorem 1.1. *Let*

$$\limsup_{t\to+\infty} \frac{t}{\ln t}(c_0 - c(t)) > \frac{1}{4}\,. \tag{1.1}$$

Then equation (0.1) *is oscillatory.*

Corollary 1.1. *Let* $Q_* > -\infty$ *and*

$$\limsup_{t\to+\infty} \frac{1}{\ln t}\int_1^t sp(s)\,ds > \frac{1}{4}\,.$$

Then equation (0.1) *is oscillatory.*

Corollary 1.2. *Let*

$$\liminf_{t\to+\infty} [Q(t) + H(t)] > 1/2\,. \tag{1.2}$$

Then equation (0.1) *is oscillatory.*

Theorem 1.2. *Let*

$$\limsup_{t\to+\infty} [Q(t) + H(t)] > 1. \tag{1.3}$$

Then equation (0.1) *is oscillatory.*

Corollary 1.2 readily implies (see equality (4.1) below) that if $Q_* > \frac{1}{4}$ then equation (0.1) is oscillatory, while from Theorem 1.1 it follows (see equalities (2.2) and (2.3) below) that the condition $H_* > \frac{1}{4}$ also guarantees the oscillation of equation (0.1) (see also [11]). Hence we shall limit our consideration to the case with $Q_* \leq \frac{1}{4}$ and $H_* \leq \frac{1}{4}$.

Theorem 1.3. *Let either*

$$0 \le Q_* \le 1/4 \quad \text{and} \quad H^* > 1/2\left(1+\sqrt{1-4Q_*}\right), \tag{1.4}$$

or

$$0 \le H_* \le 1/4 \quad \text{and} \quad Q^* > 1/2\left(1+\sqrt{1-4H_*}\right). \tag{1.5}$$

Then equation (0.1) *is oscillatory.*

Theorem 1.4. *Let*

$$0 \le Q_* \le 1/4 \quad \text{and} \quad 0 \le H_* \le 1/4\,. \tag{1.6}$$

Then each of the conditions

$$Q^* > Q_* + 1/2\left(\sqrt{1-4Q_*}+\sqrt{1-4H_*}\right) \tag{1.7}$$

and

$$H^* > H_* + 1/2\left(\sqrt{1-4Q_*}+\sqrt{1-4H_*}\right) \tag{1.8}$$

guarantees the oscillation of equation (0.1).

When condition (1.6) is fulfilled, Theorem 1.2 can be formulated in a more precise way.

Theorem 1.5. *Let condition* (1.6) *be fulfilled and*

$$\limsup_{t\to+\infty}\,[Q(t)+H(t)] > H_* + Q_* + 1/2\left(\sqrt{1-4Q_*}+\sqrt{1-4H_*}\right).$$

Then equation (0.1) *is oscillatory.*

To conclude the paragraph, we shall give two theorems on nonoscillation. In [3] and [12] it was respectively proved that if

$$-3/4 < Q_* \quad \text{and} \quad Q^* < 1/4\,, \tag{1.9}$$

or

$$-3/4 < H_* \quad \text{and} \quad H^* < 1/4\,, \tag{1.10}$$

then equation (0.1) is nonoscillatory. The theorem below complements these results.

Theorem 1.6. *Let either*

$$-\infty < Q_* \le -3/4 \quad \text{and} \quad Q^* < Q_* - 1 + \sqrt{1-4Q_*}, \tag{1.11}$$

or

$$-\infty < H_* \le -3/4 \quad \text{and} \quad H^* < H_* - 1 + \sqrt{1-4H_*}. \tag{1.12}$$

Then equation (0.1) *is nonoscillatory.*

Our next theorem is an attempt to reverse Corollary 1.1.

Theorem 1.7. *Let there exist a finite limit*

$$p_0 \stackrel{\text{def}}{=} \lim_{t\to+\infty} \frac{1}{\ln t} \int_1^t sp(s)\,ds < \frac{1}{4} \tag{1.13}$$

and

$$G^* < G_* + \sqrt{1-4p_0},$$

where

$$G_* = \liminf_{t\to+\infty} \ln t \left[\frac{1}{\ln t} \int_1^t sp(s)\,ds - p_0 \right],$$

$$G^* = \limsup_{t\to+\infty} \ln t \left[\frac{1}{\ln t} \int_1^t sp(s)\,ds - p_0 \right].$$

Then equation (0.1) *is nonoscillatory.*

Corollary 1.3. *Let*

$$-\infty < \limsup_{t\to+\infty} \int_1^t sp(s)\,ds < \liminf_{t\to+\infty} \int_1^t sp(s)\,ds + 1 < +\infty.$$

Then equation (0.1) *is nonoscillatory.*

2. Remarks

Among a great number of papers dealing with the oscillation of equation (0.1) we shall mention only those having a direct connection with the above-formulated theorems.

A. Wintner [2] and P. Hartman [4] proved respectively that if $\lim\limits_{t\to+\infty} c(t) = +\infty$ or

$$-\infty < \liminf_{t\to+\infty} c(t) < \limsup_{t\to+\infty} c(t) \le +\infty,$$

then equation (0.1) is oscillatory. Hence the case with the existence of the finite limit (0.2) seems to us the most interesting one to investigate.

Frequently, equation (0.1) is considered under the assumption that there exists a finite limit

$$\int_1^{+\infty} p(s)\,ds \stackrel{\text{def}}{=} \lim_{t\to+\infty} \int_1^t p(s)\,ds, \tag{2.1}$$

since in that case there also exist limit (0.2) (and $c_0 = \int\limits_1^{+\infty} p(s)\,ds$). However it is clear that (2.1) is not the necessary condition for (0.2). An example will be given below to show that Theorem 1.1 also covers the case with

$$p_* = \liminf_{t\to+\infty} \int\limits_1^t p(s)\,ds < c_0 < \limsup_{t\to+\infty} \int\limits_1^t p(s)\,ds = p^*.$$

In the particular case, where $p(t) \geq 0$ for $t > 1$, Corollary 1.1 was proved in [10], while Theorems 1.3 and 1.4 in [12]. In order that the function Q be bounded from below, it is necessary that $c_0 = p^*$. Since

$$c(t) = \int\limits_1^t p(s)\,ds - \frac{H(t)}{t} - \frac{1}{t}\int\limits_1^t \frac{1}{s}\,H(s)\,ds \quad \text{for} \quad t \geq 1,$$

for the function H to be bounded from below it is necessary that $c_0 = p_*$. Therefore condition (2.1) is necessary for Theorem 1.4, while the conditions $c_0 = p^*$ and $c_0 = p_*$ are necessary for the fulfilment of conditions (1.4) and (1.5) of Theorem 1.3, respectively. Note that the oscillation criterion $Q_* > \frac{1}{4}$ (originating from E. Hille [1]) is somewhat more general than in [5], since it does not demand that (2.1) be fulfilled.

Theorems 1.3 and 1.4 generalize and improve the well-known oscillation criterion from E. Hille [1]. In this paper, the case is considered when $p(t) \geq 0$ for $t > 1$ and an example is given, showing that the constant 1 in the oscillation criterion $Q^* > 1$ cannot be decreased. However, in this example $Q_* = 0$ and $H_* = 0$. If $Q_* > 0$ or $H_* > 0$, then, as follows from Theorems 1.3 and 1.4, the constant 1 can be decreased.

One can easily verify that for any pair of numbers (x_0, y_0), where $x_0 \leq y_0$, there is a function $p :]1, +\infty[\to R$ such that (0.2) holds and $Q_* = x_0$, $Q^* = y_0$ ($H_* = x_0$ and $H^* = y_0$). Therefore Theorems 1.3–1.6 are meaningful.

For (1.9) and (1.11) condition (2.1) is necessary.

By using the equality

$$c(t) = c(\tau) + \int\limits_\tau^t \frac{\ln s}{s^2}\left[\frac{1}{\ln s}\int\limits_1^s \xi p(\xi)\,d\xi\right] ds \quad \text{for} \quad t,\, \tau > 1 \tag{2.2}$$

one can readily find that if there exists a finite limit $\lim\limits_{t\to+\infty} \frac{1}{\ln t}\int_1^t sp(s)\,ds$, then (0.2) holds. Thus for Theorem 1.7 condition (0.2) is necessary.

As is known (see [8] and [9]), for equation (0.1) to be oscillatory when $p(t) = \frac{\mu}{t}\sin t$ for $t \geq 1$, it is necessary and sufficient that $|\mu| < \frac{1}{\sqrt{2}}$. This example shows that only condition (1.13) is not enough for the nonoscillation of equation (0.1).

It might seem at first glance that for conditions (1.10) and (1.12) it is not required that (0.2) be fulfilled. However, using equalities (2.2) and

$$\frac{1}{\ln t}\int_1^t sp(s)\,ds = \frac{1}{\ln t}\,H(t) + \frac{1}{\ln t}\int_1^t \frac{1}{s}\,H(s)\,ds \quad \text{for} \quad t > 1 \tag{2.3}$$

it is easy to show that if the function H is bounded (from both sides), then (0.2) holds. A similar situation arises for the oscillation criterion $H_* > \frac{1}{4}$ (originating from Z. Nehari [6]). By (2.2) and (2.3) one can easily verify that in that case either (0.2) holds or this limit is equal to $+\infty$. However in the latter case, equation (0.1) will be oscillatory by virtue of A. Wintner's above mentioned theorem [2]. Therefore for this criterion condition (0.2) is also necessary in a certain sense.

Finally, if we rewrite equation (0.1) as

$$v''(x) = -\frac{1}{\lambda^2}\,t^{2(1-\lambda)}\Big[p(t) - \frac{1-\lambda^2}{4t^2}\Big]\,v(x),$$

then, after transforming $u(t) = \frac{t^{\frac{1-\lambda}{2}}}{\sqrt{\lambda}}\,v(x)$, $x = t^\lambda$, $\lambda > 0$, and applying Theorems 1.3 and 1.4, we can generalize and improve the oscillation criteria of Z. Nehari [6].

Example. Let $\lambda \neq 0$ and γ be real numbers,

$$g(t) = -\gamma\,\frac{\ln t}{t} + \frac{\lambda}{1+\ln t}\,(\sin \ln^2 t - 1) \quad \text{for} \quad t \geq 1$$

and

$$p(t) = 2g'(t) + tg''(t) \quad \text{for} \quad t \geq 1.$$

It is easy to verify that

$$\int_1^t p(s)\,ds = g(t) + tg'(t) + \gamma, \quad c(t) = g(t) + \frac{t-1}{t}\,(\gamma - \lambda) + \lambda \quad \text{for} \quad t \geq 1,$$

$$\liminf_{t\to+\infty}\int_1^t p(s)\,ds = \gamma - 2|\lambda|, \quad \limsup_{t\to+\infty}\int_1^t p(s)\,ds = \gamma + 2|\lambda|, \quad c_0 = \gamma,$$

$$\frac{t}{\ln t}\,(c_0 - c(t)) = -\frac{t}{\ln t}\,g(t) + \frac{1}{\ln t}\,(\gamma - \lambda) \quad \text{for} \quad t \geq 1,$$

$$-\infty = \liminf_{t\to+\infty}\frac{t}{\ln t}\,(c_0 - c(t)), \quad \limsup_{t\to+\infty}\frac{t}{\ln t}\,(c_0 - c(t)) = \gamma \ \text{ for } \ \lambda < 0,$$

$$\gamma = \liminf_{t\to+\infty}\frac{t}{\ln t}\,(c_0 - c(t)), \quad \limsup_{t\to+\infty}\frac{t}{\ln t}\,(c_0 - c(t)) = +\infty \ \text{ for } \ \lambda > 0.$$

Thus, by Theorem 1.1, if $\lambda > 0$ or $\lambda < 0$ and $\gamma > \frac{1}{4}$ equation (0.1) is oscillatory.

3. Some Auxiliary Propositions

In this paragraph we establish some properties of solutions of equation (0.1). Throughout this paper it is assumed that the function $p :]1, +\infty[\to R$ is Lebesgue integrable on each finite segment from $[1, +\infty[$. For the convenience of reference we shall give one proposition without proving it (see, for example, [7], Lemma 7.1, p. 365).

Lemma 3.1. *Let equation* (0.1) *be nonoscillatory and* $u(t) \neq 0$ *for* $t \geq a$ *be its some solution. Then*

$$\int^{+\infty} \rho^2(s)\, ds < +\infty$$

and

$$\rho(t) = c_0 - \int_1^t p(s)\, ds + \int_t^{+\infty} \rho^2(s)\, ds \quad \textit{for} \quad t \geq a, \tag{3.1}$$

where

$$\rho(t) = \frac{u'(t)}{u(t)} \quad \textit{for} \quad t \geq a. \tag{3.2}$$

Lemma 3.2. *Let equation* (0.1) *be nonoscillatory and* $0 \leq Q_* \leq \frac{1}{4}$. *Then for each solution* u *of equation* (0.1) *the estimate*

$$\liminf_{t \to +\infty} \frac{tu'(t)}{u(t)} \geq \frac{1}{2}\left(1 - \sqrt{1 - 4Q_*}\right) \tag{3.3}$$

is valid.

Proof. Let $u(t) \neq 0$ for $t \geq a$ be some solution of equation (0.1). By Lemma 3.1 equality (3.1) is fulfilled, where the function ρ is defined by (3.2). We set

$$r = \liminf_{t \to +\infty} t\rho(t).$$

If $r = +\infty$, then there is nothing to prove. Therefore it will be assumed that $r < +\infty$. If $Q_* = 0$, then estimate (3.3) is trivial by virtue of (3.1). So it will assumed that $Q_* > 0$. For arbitrary $\varepsilon \in]0, Q_*[$ we choose $t_\varepsilon > a$ such that

$$Q(t) > Q_* - \varepsilon \quad \text{for} \quad t \geq t_\varepsilon. \tag{3.4}$$

Now from (3.1) we have $t\rho(t) > Q_* - \varepsilon$ for $t \ge t_\varepsilon$. Hence we readily conclude that $r \ge Q_*$. Choose $t_{1\varepsilon} > t_\varepsilon$ such that $t\rho(t) > r - \varepsilon$ for $t \ge t_{1\varepsilon}$. Taking this and inequality (3.4) into account, from (3.1) we find

$$t\rho(t) \ge Q_* - \varepsilon + (r-\varepsilon)^2 \quad \text{for} \quad t \ge t_{1\varepsilon}.$$

Therefore $r \ge Q_* - \varepsilon + (r-\varepsilon)^2$ and, since ε was arbitrary, we have $r^2 - r + Q_* \le 0$. Now by simple calculations we conclude that (3.3) is valid. $\square$

Lemma 3.3. *Let equation* (0.1) *be nonoscillatory and* $0 \le H_* \le \frac{1}{4}$. *Then for each solution* u *of equation* (0.1) *the estimate*

$$\limsup_{t\to+\infty} \frac{tu'(t)}{u(t)} \le \frac{1}{2}\left(1 + \sqrt{1-4H_*}\,\right) \tag{3.5}$$

holds.

Proof. Let $u(t) \ne 0$ for $t \ge a$ be some solution of equation (0.1). We introduce the function ρ by equality (3.2) and set

$$M = \limsup_{t\to+\infty} t\rho(t).$$

If $M \le 0$, then there is nothing to prove. Hence it will be assumed that $M > 0$.

Clearly, $\rho'(t) = -p(t) - \rho^2(t)$ for $t \ge a$. By multiplying both sides of this equality by t^2 and integrating from $\tau > a$ to t we obtain

$$t\rho(t) = -\,H(t) + \frac{1}{t}\int_\tau^t s\rho(s)(2 - s\rho(s))\,ds + \\ + \frac{1}{t}\int_1^\tau s^2 p(s)\,ds + \frac{\tau^2}{t}\,\rho(\tau) \quad \text{for} \quad t > \tau > a. \tag{3.6}$$

Since $s\rho(s)(2 - s\rho(s)) \le 1$ for $s \ge a$, (3.6) implies

$$t\rho(t) \le 1 - H(t) + \frac{1}{t}\int_1^\tau s^2 p(s)\,ds + \frac{\tau^2}{t}\,\rho(\tau) \quad \text{for} \quad t > \tau > a,$$

and therefore $M \le 1 - H_*$. Thus estimate (3.5) is valid for $H_* = 0$.

Now let us assume that $H_* > 0$. For arbitrary $0 < \varepsilon < \min\{H_*, 1 - M\}$ we choose $t_\varepsilon > a$ such that $t\rho(t) < M + \varepsilon$, $H(t) > H_* - \varepsilon$ for $t \ge t_\varepsilon$. Now (3.6) (for $\tau = t_\varepsilon$) implies

$$t\rho(t) \le -H_* + \varepsilon + (M+\varepsilon)(2 - M - \varepsilon) + \frac{1}{t}\int_1^{t_\varepsilon} s^2 p(s)\,ds + \frac{1}{t}\,t_\varepsilon^2 \rho(t_\varepsilon) \ \text{ for } \ t > t_\varepsilon.$$

Hence we easily find that $M \le -H_* + \varepsilon + (M+\varepsilon)(2-M-\varepsilon)$ and therefore $M^2 - M + H_* \le 0$. Now by simple calculations we conclude that (3.5) holds. $\square$

Lemma 3.4. *For equation* (0.1) *to be nonoscillatory it is necessary and sufficient that the equation*

$$\begin{gathered} v'' = -\frac{1}{t^2}\big(Q^2(t) + 2\alpha Q(t) + \alpha(\alpha-1)\big)\,v - \frac{2}{t}\,(\alpha + Q(t))\,v' \\ \Big(v'' = -\frac{1}{t^2}\big(H^2(t) + 2(1-\alpha)H(t) + \alpha(\alpha-1)\big)\,v + \frac{2}{t}\,(H(t)-\alpha)\,v'\Big), \end{gathered} \tag{3.7}$$

where α is some real number, be nonoscillatory.

Proof. It is easy to verify that if v is a solution of equation (3.7), then the function u defined by the equality

$$u(t) = t^\alpha v(t)\exp\Big[\int_1^t \frac{Q(s)}{s}\,ds\Big] \quad \text{for} \quad t \ge 1$$

$$\Big(u(t) = t^\alpha v(t)\exp\Big[-\int_1^t \frac{H(s)}{s}\,ds\Big] \quad \text{for} \quad t \ge 1\Big)$$

is a solution of equation (0.1). Therefore these equations are simultaneously either oscillatory or nonoscillatory. $\square$

Lemma 3.5. *For equation* (0.1) *to be nonoscillatory it is necessary and sufficient that the equation*

$$\begin{aligned} v'' = -\frac{1}{t^2}\big[G^2(t) - (2\alpha-1)G(t) + p_0 + \alpha(\alpha-1)\big]\,v \,+& \\ +\frac{2}{t}\,(\alpha - G(t))\,v', & \end{aligned} \tag{3.8}$$

where

$$G(t) = \int_1^t sp(s)\,ds - p_0 \ln t \quad \textit{for} \quad t \ge 1 \tag{3.9}$$

be nonoscillatory for some numbers α and p_0.

Proof. It is easy to verify that if v is a solution of equation (3.8), then the function u defined by the equality

$$u(t) = t^\alpha v(t)\exp\Big[-\int_1^t \frac{G(s)}{s}\,ds\Big] \quad \text{for} \quad t \ge 1$$

is a solution of equation (0.1). □

4. Proof of the Main Results

Proof of Theorem 1.1. Let us assume the opposite, i.e., that $u(t) \neq 0$ for $t \geq a$ is a solution of equation (0.1). Equality (3.1), where the function ρ is defined by (3.2), is fulfilled by virtue of Lemma 3.1. The integration of (3.1) from a to t gives

$$t(c_0 - c(t)) = \int_a^t \frac{s\rho(s)(1 - s\rho(s))}{s}\, ds - t \int_t^{+\infty} \rho^2(s)\, ds + a\rho(a) + \\ + \int_1^a sp(s)\, ds \quad \text{for} \quad t \geq a.$$

Since $s\rho(s)(1 - s\rho(s)) \leq \frac{1}{4}$ for $s \geq a$, the latter equality implies

$$\frac{t}{\ln t}\,(c_0 - c(t)) \leq \frac{1}{4} + \frac{1}{\ln t}\left[a\rho(a) + \int_1^a sp(s)\, ds\right] \quad \text{for} \quad t \geq a,$$

which contradicts condition (1.1). □

By virtue of the equality

$$\frac{t}{\ln t}\,(c_0 - c(t)) = \frac{1}{\ln t}\, Q(t) + \frac{1}{\ln t}\int_1^t sp(s)\, ds \quad \text{for} \quad t > 1,$$

one can easily show that Corollary 1.1 is valid.

Proof of Corollary 1.2. It is easy to find that

$$Q(t) + H(t) = \frac{2}{t}\int_1^t Q(s)\, ds + \frac{c_0}{t} \quad \text{for} \quad t \geq 1 \tag{4.1}$$

and

$$\frac{t}{\ln t}\,(c_0 - c(t)) = \frac{1}{\ln t}\int_1^t \frac{Q(s)}{s}\, ds + \frac{c_0}{\ln t} = \\ = \frac{1}{\ln t}\left[\frac{1}{t}\int_1^t Q(s)\, ds + \int_1^t \frac{1}{s^2}\Big(\int_1^s Q(\xi)\, d\xi\Big)\, ds\right] + \frac{c_0}{\ln t} \quad \text{for} \quad t > 1. \tag{4.2}$$

Using (1.2) and (4.1) we obtain

$$\liminf_{t\to+\infty} \frac{1}{t} \int_1^t Q(s)\,ds > \frac{1}{4}\,.$$

Hence by (4.2) we conclude that (1.1) is fulfilled. Therefore equation (0.1) will be oscillatory by virtue of Theorem 1.1. □

Proof of Theorem 1.2. Let us assume the opposite, i.e., that $u(t) \neq 0$, $t \geq a$, is a solution of equation (0.1). Equality (3.1), where the function ρ is defined by (3.2), holds by virtue of Lemma 3.1. Clearly,

$$\rho'(t) = -p(t) - \rho^2(t) \quad \text{for} \quad t \geq a.$$

By multiplying both sides of this equality by t^2 and integrating from $\tau \geq a$ to t we obtain equality (3.6).

Now, using (3.1), we find that

$$\begin{aligned} Q(t) + H(t) = \frac{1}{t} \int_\tau^t s\rho(s)(2 - s\rho(s))\,ds - t \int_t^{+\infty} \rho^2(s)\,ds \,+ \\ + \frac{1}{t} \int_1^\tau s^2 p(s)\,ds + \frac{1}{t}\,\tau^2\rho(\tau) \quad \text{for} \quad t \geq a. \end{aligned} \tag{4.3}$$

Hence we have

$$Q(t) + H(t) \leq 1 + \frac{1}{t} \int_1^\tau s^2 p(s)\,ds + \frac{1}{t}\,\tau^2\rho(\tau) \quad \text{for} \quad t \geq a,$$

which contradicts condition (1.3). □

Proof of Theorem 1.3. Let us assume the opposite, i.e., that $u(t) \neq 0$, $t \geq a$, is a solution of equation (0.1). Then (3.1) and (3.6), where the function ρ is defined by (3.2), are valid. By Lemma 3.2 when (1.4) is fulfilled and by Lemma 3.3 when (1.5) is fulfilled, for any sufficiently small $\varepsilon > 0$ there exists $t_\varepsilon > a$ such that,

$$t\rho(t) > r - \varepsilon \quad \text{and} \quad t\rho(t) < M + \varepsilon \;\; \text{for} \;\; t \geq t_\varepsilon, \tag{4.4}$$

respectively, where

$$r = \frac{1}{2}\big(1 - \sqrt{1 - 4Q_*}\big), \quad M = \frac{1}{2}\big(1 + \sqrt{1 - 4H_*}\,\big). \tag{4.5}$$

Hence, if (1.4) is fulfilled, (3.1) and (3.6) imply

$$H(t) \le -r + \varepsilon + 1 + \frac{1}{t}\, t_\varepsilon^2 \rho(t_\varepsilon) + \frac{1}{t} \int_1^{t_\varepsilon} s^2 p(s)\, ds \quad \text{for} \quad t \ge t_\varepsilon,$$

and if (1.5) is fulfilled, (3.1) and (3.6) imply $Q(t) \le M + \varepsilon$ for $t \ge t_\varepsilon$, which contradicts the conditions of the theorem. $\square$

Proof of Theorem 1.4. Let us assume that (1.8) is fulfilled (the case, where (1.7) is fulfilled, is proved similarly). Let $u(t) \ne 0$, $t \ge a$, be a solution of equation (0.1). Then (3.6), where the function ρ is defined by equality (3.2), is valid. We shall assume that $H_* > 0$, since for $H_* = 0$ condition (1.8) is equivalent to condition (1.4) of Theorem 1.3 which has been proved above. By Lemmas 3.2 and 3.3, for arbitrary $\varepsilon \in]0, 1-M[$, there exists $t_\varepsilon > a$ such that (4.4) holds, where r and M are the numbers defined by equalities (4.5). Since $M + \varepsilon < 1$, we have

$$s\rho(s)(2 - s\rho(s)) < (M+\varepsilon)(2 - M - \varepsilon) \quad \text{for} \quad s \ge t_\varepsilon. \tag{4.6}$$

Taking into account (4.4) and (4.6), from (3.6) we obtain

$$H(t) \le -r + \varepsilon + (M+\varepsilon)(2-M-\varepsilon) + \frac{1}{t} \int_1^{t_\varepsilon} s^2 p(s)\, ds + \frac{1}{t}\, t_\varepsilon^2 \rho(t_\varepsilon) \quad \text{for} \quad t \ge t_\varepsilon.$$

Hence we easily conclude that $H^* \le -r + M(2-M)$, which contradicts condition (1.8). $\square$

The proof of Theorem 1.5 repeats that of Theorem 1.2 with the only difference that one should use (4.4) and (4.6) in equality (4.3).

Proof of Theorem 1.6. Assume that (1.11) ((1.12)) is fulfilled. Choose $\varepsilon > 0$ such that

$$Q^* + \varepsilon < Q_* - \varepsilon - 1 + \sqrt{1 - 4(Q_* - \varepsilon)}$$
$$\Big(H^* + \varepsilon < H_* - \varepsilon - 1 + \sqrt{1 - 4(H_* - \varepsilon)} \Big).$$

We set

$$\alpha = \Big[\frac{1}{2}\big(1 + \sqrt{1 - 4(Q^* + \varepsilon)}\big)\Big]^2 \quad \Big(\alpha = 1 - \frac{1}{4}\big(1 + \sqrt{1 - 4(H^* + \varepsilon)}\big)^2\Big).$$

It is easy to verify that

$$Q^* + \varepsilon = \sqrt{\alpha} - \alpha \quad \big(H^* + \varepsilon = \sqrt{1-\alpha} - (1-\alpha)\big) \tag{4.7}$$

and

$$Q_* - \varepsilon > -\sqrt{\alpha} - \alpha \quad \big(H_* - \varepsilon > -\sqrt{1-\alpha} - (1-\alpha)\big). \tag{4.8}$$

Choose $t_\varepsilon > 1$ such that

$$Q_* - \frac{\varepsilon}{2} < Q(t) < Q^* + \frac{\varepsilon}{2} \quad \text{for} \quad t \geq t_\varepsilon$$

$$\Big(H_* - \frac{\varepsilon}{2} < H(t) < H^* + \frac{\varepsilon}{2} \quad \text{for} \quad t \geq t_\varepsilon \Big).$$

Now by (4.7) and (4.8) we have

$$-\sqrt{\alpha} - \alpha < Q(t) < \sqrt{\alpha} - \alpha \quad \text{for} \quad t \geq t_\varepsilon$$

$$\Big(-\sqrt{1-\alpha} - (1-\alpha) < H(t) < \sqrt{1-\alpha} - (1-\alpha) \quad \text{for} \quad t \geq t_\varepsilon \Big),$$

i.e.,

$$Q^2(t) + 2\alpha Q(t) + \alpha(\alpha - 1) < 0 \quad \text{for} \quad t \geq t_\varepsilon$$

$$\Big(H^2(t) + 2(1-\alpha)H(t) + \alpha(\alpha - 1) < 0 \quad \text{for} \quad t \geq t_\varepsilon \Big).$$

Thus equation (3.7) is nonoscillatory. Therefore, by Lemma 3.4, equation (0.1) is nonoscillatory. □

Proof of Theorem 1.7. Let the function G be defined by equality (3.9). Choose $\varepsilon > 0$ and $t_\varepsilon > 1$ such that $G^* + 2\varepsilon < G_* + \sqrt{1-4p_0}$ and $G_* - \varepsilon < G(t) < G^* + \varepsilon$ for $t \geq t_\varepsilon$.

We set

$$\alpha = G_* - \varepsilon + \frac{1}{2} + \frac{1}{2}\sqrt{1-4p_0}.$$

Clearly,

$$\alpha - \frac{1}{2} - \frac{1}{2}\sqrt{1-4p_0} < G(t) < \alpha - \frac{1}{2} + \frac{1}{2}\sqrt{1-4p_0} \quad \text{for} \quad t \geq t_\varepsilon,$$

i.e.,

$$G^2(t) - (2\alpha - 1)G(t) + p_0 + \alpha(\alpha - 1) < 0 \quad \text{for} \quad t \geq t_\varepsilon.$$

Thus equation (3.8) is nonoscillatory. Therefore by Lemma 3.5 equation (0.1) is nonoscillatory. □

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