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SUBSETS OF THE PLANE WITH SMALL LINEAR SECTIONS AND INVARIANT EXTENSIONS OF THE TWO-DIMENSIONAL LEBESGUE MEASURE

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ABSTRACT. We consider some subsets of the Euclidean plane $\mathbf{R}^2$, having small linear sections (in all directions), and investigate those sets from the point of view of measurability with respect to certain invariant extensions of the classical Lebesgue measure on $\mathbf{R}^2$.

Many examples of paradoxical sets in a finite-dimensional Euclidean space, having small sections by hyperplanes in this space, are known in the literature. One of the earliest examples is due to Sierpiński who constructed a function $f : \mathbf{R} \to \mathbf{R}$ such that its graph is λ_2-thick in $\mathbf{R}^2$. Here λ_2 denotes the standard two-dimensional Lebesgue measure on the plane $\mathbf{R}^2$, and we say that a subset X of $\mathbf{R}^2$ is λ_2-thick in $\mathbf{R}^2$ if the inner λ_2-measure of the set $\mathbf{R}^2 \setminus X$ is equal to zero (cf. the corresponding definition in [1]). In particular, the λ_2-thickness of the graph of f implies that it is nonmeasurable with respect to λ_2 and, hence, f is not measurable in the Lebesgue sense. At the same time, any straight line in $\mathbf{R}^2$ parallel to the axis of ordinates meets the graph of f in exactly one point.

Further, Mazurkiewicz constructed a subset Y of $\mathbf{R}^2$ having the property that, for each straight line l in $\mathbf{R}^2$, the set $l \cap Y$ consists of exactly two points. Note that Y can also be chosen to be λ_2-thick and, consequently, nonmeasurable with respect to λ_2. Later on, various examples of sets with small sections, however large in some sense, were presented by other authors.

In this paper, we deal with similar sets in connection with the following natural question: how small are such sets from the point of view of invariant extensions of the Lebesgue measure λ_2? Namely, we are going to demonstrate in our further considerations that there are invariant extensions of λ_2, concentrated on sets with small linear sections (in all directions). Actually,

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a more stronger result will be obtained stating that the corresponding sets can even have small sections by all analytic curves lying in the plane.

Let λ_n denote the standard n-dimensional Lebesgue measure on the Euclidean space $\mathbf{R}^n$, where $n \geq 2$.

Below, we need the following auxiliary statement which is also rather useful in many other situations.

Lemma 1. *Let Z be a λ_n-measurable subset of the Euclidean space $\mathbf{R}^n$ with $\lambda_n(Z) > 0$ and let $\{M_i \ : \ i \in I\}$ be a family of analytic manifolds in $\mathbf{R}^n$, such that*

(1) $\mathrm{card}(I)$ *is strictly less than the cardinality continuum;*

(2) *for each index $i \in I$, the dimension of M_i is strictly less than n.*

Then the relation $Z \setminus \cup\{M_i \ : \ i \in I\} \neq \varnothing$ is satisfied.

The proof of this lemma is not difficult and can be carried out by induction on n (here the classical Fubini theorem plays an essential role). For details, see, e.g., [2]. Some applications of the lemma to certain questions of the geometry of Euclidean spaces may be found in [3]. Note also that the lemma does not hold true for topological manifolds in $\mathbf{R}^n$. For example, there are models of set theory in which the Euclidean plane can be covered by a family of Jordan curves, whose cardinality is strictly less than the cardinality continuum.

Starting with the above-mentioned lemma we are able to establish the following result.

Theorem 1. *Let G be the group of all analytic diffeomorphisms of $\mathbf{R}^n$. Then there exists a subset X of $\mathbf{R}^n$ such that*

(1) *X is almost G-invariant, i.e.,*

$$(\forall g \in G)(\mathrm{card}(g(X) \triangle X) < \mathbf{c})$$

where the symbol $\triangle$ denotes the operation of symmetric difference of sets and the symbol $\mathbf{c}$ denotes the cardinality continuum;

(2) *X is λ_n-thick in $\mathbf{R}^n$;*

(3) *for any analytic manifold M in $\mathbf{R}^n$ with $\dim(M) < n$, we have*

$$\mathrm{card}(M \cap X) < \mathbf{c}.$$

Proof. We use the method developed in [2], [3] and [4]. Let α denote the least ordinal number of cardinality continuum. Since the equality $\mathrm{card}(G) = \mathbf{c}$ holds, we may write

$$G = \cup\{G_\xi \ : \ \xi < \alpha\}$$

where $\{G_\xi \ : \ \xi < \alpha\}$ is some α-sequence of subgroups of G, satisfying these two conditions:

(a) for each ordinal $\xi < \alpha$, we have the inequality

$$\mathrm{card}(G_\xi) \le \mathrm{card}(\xi) + \omega,$$

where ω denotes the first infinite cardinal;

(b) the family $\{G_\xi \ : \ \xi < \alpha\}$ is increasing by inclusion.

Also, we may consider the family $\{Y_\xi \ : \ \xi < \alpha\}$ consisting of all Borel subsets of $\mathbf{R}^n$ with a strictly positive n-dimensional Lebesgue measure. Finally, we denote by $\{M_\xi \ : \ \xi < \alpha\}$ the family of all analytic manifolds in $\mathbf{R}^n$ whose dimensions are strictly less than n.

Let us now define, by the method of transfinite recursion, an injective α-sequence $\{x_\xi \ : \ \xi < \alpha\}$ of points in $\mathbf{R}^n$, satisfying the following relations:

(i) for any $\xi < \alpha$, the point x_ξ belongs to Y_ξ;

(ii) for any $\xi < \alpha$, we have

$$x_\xi \notin G_\xi(\cup\{M_\zeta \ : \ \zeta \le \xi\} \cup \{x_\zeta \ : \ \zeta < \xi\}).$$

Note that Lemma 1 guarantees, at each ξ-step of our recursion, the existence of a point x_ξ. So the recursion can be continued up to α. In this way, we will be able to construct the required α-sequence of points $\{x_\xi \ : \ \xi < \alpha\}$. Now, putting

$$X = \{G_\xi(x_\xi) \ : \ \xi < \alpha\},$$

we can easily check that the set X is the desired one. For example, let us verify that X is almost G-invariant.

Indeed, let $g \in G$. Then there exists $\xi < \alpha$ such that $g \in G_\zeta$ for all $\zeta \in [\xi, \alpha[$. Consequently,

$$g(G_\zeta(x_\zeta)) = G_\zeta(x_\zeta) \quad \forall \zeta \in [\xi, \alpha[.$$

From this we infer

$$g(X) \triangle X \subseteq g\Big(\bigcup_{\zeta<\xi} G_\zeta(x_\zeta)\Big) \bigcup \Big(\bigcup_{\zeta<\xi} G_\zeta(x_\zeta)\Big)$$

and since

$$\mathrm{card}\,\Big(\bigcup_{\zeta<\xi} G_\zeta(x_\zeta)\Big) \le (\mathrm{card}(\xi) + \omega)^2 < \mathbf{c},$$

we obtain finally

$$\mathrm{card}(g(X) \triangle X) < \mathbf{c},$$

which completes the argument. $\square$

The next statement is a trivial consequence of this theorem.

Theorem 2. *Let G denote the group of all isometric transformations of $\mathbf{R}^n$. Then there exists a subset X of $\mathbf{R}^n$ such that*

(1) *X is almost G-invariant;*

(2) *X is λ_n-thick in $\mathbf{R}^n$;*

(3) *for each analytic manifold M in $\mathbf{R}^n$ with $\dim(M) < n$, we have*

$$\mathrm{card}(X \cap M) < \mathbf{c}.$$

In particular, for any hyperplane L in $\mathbf{R}^n$, the inequality

$$\mathrm{card}(X \cap L) < \mathbf{c}$$

is true.

Moreover, if the Continuum Hypothesis holds, then, for each analytic manifold M in $\mathbf{R}^n$ with $\dim(M) < n$, we have

$$\mathrm{card}(X \cap M) \leq \omega.$$

Let us point out one application of Theorem 2. The following question arises naturally: does there exist a measure μ_n on $\mathbf{R}^n$ extending the Lebesgue measure λ_n, invariant under the group of all isometric transformations of $\mathbf{R}^n$ and concentrated on some subset X of $\mathbf{R}^n$ with small sections by hyperplanes?

Of course, here the smallness of sections means that, for any hyperplane L in $\mathbf{R}^n$, the cardinality of $L \cap X$ is strictly less than the cardinality continuum.

Theorem 2 yields immediately the positive answer to this question. Indeed, let G be the group of all isometric transformations of $\mathbf{R}^n$ and consider the G-invariant σ-ideal of subsets of $\mathbf{R}^n$, generated by the set $\mathbf{R}^n \setminus X$, where X is the set from Theorem 2. We denote this σ-ideal by J. Then, for any set $Z \in J$, the inner λ_n-measure of Z is equal to zero (since X is almost G-invariant and λ_n-thick in $\mathbf{R}^n$). Taking this fact into account and applying the standard methods of extending invariant measures (see, for instance, [2], [4], [5] and [6]), we infer that there exists a measure μ_n on $\mathbf{R}^n$ satisfying the relations:

(1) μ_n is complete and extends λ_n;

(2) μ_n is invariant under the group G;

(3) $J \subset \mathrm{dom}(\mu_n)$;

(4) for each set $Z \in J$, we have $\mu_n(Z) = 0$.

It follows from relation (4) that $\mu_n(\mathbf{R}^n \setminus X) = 0$, i.e., our measure μ_n is concentrated on X. At the same time, we know that X has small sections by all hyperplanes in $\mathbf{R}^n$ and, moreover, by all analytic manifolds in $\mathbf{R}^n$ whose dimensions are strictly less than n. In addition, the measure μ_n being complete and metrically transitive has the so-called uniqueness property (for the definition, see, e.g., [2]).

In particular, for $n = 2$, we obtain that there exists a complete measure μ_2 on the Euclidean plane $\mathbf{R}^2$, such that:

(1) μ_2 is an extension of the two-dimensional Lebesgue measure λ_2;

(2) μ_2 is invariant under the group of all isometric transformations of the plane $\mathbf{R}^2$;

(3) μ_2 is concentrated on some subset X of $\mathbf{R}^2$ having the property that all linear sections of X are small (i.e., of cardinality strictly less than the cardinality continuum).

Note that the question on the existence of a measure μ_2 on the plane, satisfying conditions (1)–(3), was formulated by R. D. Mabry (personal communication). Moreover, we see that the above-mentioned support X of μ_2 has a stronger property: for each analytic curve l in $\mathbf{R}^2$, the cardinality of $l \cap X$ is strictly less than $\mathbf{c}$. Let us point out that in [2] an analogous question was considered for extensions of λ_2 which are invariant under the group of all translations of $\mathbf{R}^2$. More precisely, it was demonstrated in [2] that there exists a translation invariant extension of λ_2 concentrated on a subset of $\mathbf{R}^2$ whose all linear sections are at most countable.

Remark 1. If the Continuum Hypothesis holds, then we also can conclude that the above-mentioned subset X of the plane has the property that all its linear sections are at most countable (obviously, the same is true for sections of X by analytic curves lying in the plane). Thus, in this case, the measure μ_2 is concentrated on a set with countable linear sections. In this connection, the following question seems to be natural: does there exist a measure ν_2 on the plane, extending λ_2, invariant under the group of all isometric transformations of $\mathbf{R}^2$ and concentrated on a set with finite linear sections? It turns out that the answer to this question is negative. For the proof of the corresponding result (and a more general statement), see paper [4] by the author.

Remark 2. By assuming some additional set-theoretical axioms, a much more stronger result than Theorem 1 can be established. Namely, let us suppose that, for any cardinal $\kappa < \mathbf{c}$, the space $\mathbf{R}^n$ cannot be covered by a κ-sequence of λ_n-measure zero sets. In fact, it suffices to assume this only for $n = 1$, i.e., that the real line $\mathbf{R}$ cannot be covered by a family of Lebesgue measure zero sets, whose cardinality is strictly less than $\mathbf{c}$. For instance, this hypothesis follows directly from the well-known Martin's Axiom (see, e.g., [7]). We say that a group G of transformations of $\mathbf{R}^n$ is admissible if each element g from G preserves the σ-ideal of all λ_n-measure zero subsets of $\mathbf{R}^n$. There are many natural examples of admissible groups. For instance, if G coincides with the group of all affine transformations of $\mathbf{R}^n$, then G is admissible. If G coincides with the group of all diffeomorphisms of $\mathbf{R}^n$, then G is admissible, too. Evidently, the cardinality of these two groups is equal to $\mathbf{c}$. It is not hard to check that there are also admissible groups G with card$(G) = 2^{\mathbf{c}}$. Let us fix an admissible group G with card$(G) = \mathbf{c}$. Then, applying an argument similar to the proof of Theorem 1, we easily obtain the following statement: there exists a subset X of $\mathbf{R}^n$ with card$(X) = \mathbf{c}$, such that:

(1) X is almost G-invariant;

(2) X is λ_n-thick in $\mathbf{R}^n$;

(3) for any λ_n-measure zero subset Z of $\mathbf{R}^n$, we have card$(Z \cap X) < \mathbf{c}$.

Condition (3) shows, in particular, that the above-mentioned set X is a generalized Sierpiński subset of $\mathbf{R}^n$ (for the definition and various properties of Sierpiński sets, see [8] or [9] where the dual objects to the Sierpiński sets – the so-called Luzin sets – are discussed as well).

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