*Gen. Math. Notes, Vol. 21, No. 2, April 2014, pp.114-124*
*ISSN 2219-7184; Copyright ©ICSRS Publication, 2014*
*www.i-csrs.org*
*Available free online at http://www.geman.in*

# Some Common Fixed Point Theorems on G-Metric Space

**Giniswamy[1] and Maheshwari P.G.[2]**

[1]Department of Mathematics
PES College of Science, Arts and Commerce
Mandya, Karnataka, India–571401
E-mail: gswamypes@gmail.com
[2]Department of Mathematics
Government First Grade College, Vijayanagar
Bangalore, Karnataka, India–560104
E-mail: maheshwari616@yahoo.com

(Received: 7-1-14 / Accepted: 15-2-14)

**Abstract**

*The purpose of this paper is to present some common fixed point theorems in G-Metric spaces, by employing the notion of Common limit in the range property and to demonstrate suitable examples. These results extend and generalizes several well known results in the literature.*

**Keywords:** *Coincidence point, Fixed point, G-Metric Space, CLRg Property, Weakly compatible maps.*

## 1 Introduction and Preliminaries

Inspired by the fact that metric fixed point theory has a wide application in almost all fields of quantitative sciences, many authors have directed their attention to generalize the notion of a metric space. In this respect, several generalized metric spaces have come through by many authors, in the last decade. Among all the generalized metric spaces, the notion of G-Metric space has attracted considerable attention from fixed point theorists. The concept of a G-Metric space was introduced by Mustafa and Sims in [19], wherein the authors discussed the topological properties of this space and proved the analog of the banach contraction principle in the context of G-metric spaces. Following these results, many authors have studied and developed several common fixed point theorems in this framework.

In 2002, M.Aamri and D.El Moutawakil[9] introduced the property (E.A), which is a true generalization of noncompatible maps in metric spaces. Under this notion many common fixed point theorems were studied in the literature (see [1,3,5,9,14,15] and the references therein). In 2011, the concept of Common limit in the range of $g$ ($CLRg$) property for a pair of self mappings in Fuzzy metric space was introduced by Sintunavarat et al.[17]. The importance of this property is, it ensures that one does not require the closeness of the range subspaces and hence, now a days, authors are giving much attention to this property for generalizing the results present in the literature(see [11]-[13] and the references therein). Very recently, this was extended to two pairs of self mappings as $CLR_{(S,T)}$ property by E.Karampur et al.[4].

In the present paper, by employing the notions of common limit in the range property for two as well as four self maps and weak compatibility , which is an efficient tool in providing the common fixed points, we derive some common fixed point theorems in the realm of G-metric space, which generalizes various comparable results in [2,7,9,10,14,16,18]and others. At the same time, we present suitable examples to exhibit the utility of the main results presented.

The following are the basic definitions needed in the main results.

**Definition 1.1** [19] *Let $X$ be a nonempty set and $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties:*

**G1** $G(x, y, z) = 0$ *if* $x = y = z$.

**G2** $0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$.

**G3** $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $z \neq y$.

**G4** $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$ *(symmetry in all three variables).*

**G5** $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$ *(rectangle inequality).*

*Then the function $G$ is called a generalized metric or, more specifically, a G-metric on $X$, and the pair $(X, G)$ is called a G-metric space.*

**Definition 1.2** [19] *A G-Metric space $(X, G)$ is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.*

**Example :** Let $(X, d)$ be the usual metric space Then the function $G : X \times X \times X \to [0, \infty)$ defined by $G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$ for all $x, y, z \in X$, is a G-Metric space.

In their initial paper Mustafa and Sims [19] also proved the following proposition.

**Proposition 1.3** *Let $(X, G)$ be a G-metric space. Then, for any $x, y, z, a \in X$, it follows that*

1. *if $G(x, y, z) = 0$, then $x = y = z$.*
2. $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$.
3. $G(x, y, y) \leq 2G(y, x, x)$.
4. $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$.
5. $G(x, y, z) \leq \frac{2}{3}[G(x, y, a) + G(x, a, z) + G(a, y, z)]$.
6. $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$

**Definition 1.4** [9] *Let $f, g$ be two self mappings of a metric space $(X, d)$. Then we say that $f$ and $g$ satisfy the property (E.A), if there exists a sequence $\{x_n\}$ in $X$ such that $\lim_n fx_n = \lim_n gx_n = t$ for some $t \in X$.*

**Definition 1.5** [4] *Let $f, g, S, T$ be four self mappings defined on a symmetric space $(X, d)$. Then the pairs $(f, S)$ and $(g, T)$ are said to have the common limit range property (with respect to S and T) often denoted by $CLR_{(S,T)}$ property, if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that*
*$\lim_n fx_n = \lim_n Sx_n = \lim_n gy_n = \lim_n Ty_n = t$ with $t = Su = Tw$ for some $t, u, w \in X$.*
*If $f = g$ and $S = T$, then the above definition implies $CLRg$ property due to Sintunavarat et al.[17].*

# 2 Main Result

The first result is a common fixed point theorem for a pair of self mappings using a generalized strict contractive condition, which extend Theorem 1 of [9].

**Theorem 2.1** *Let (X,G) be a symmetric G-Metric space and f,g be two weakly compatible self mappings on X satisfying*

1. *CLRg property.*
2. $G(fx, fy, fz) < max\{G(gx, gy, gz), \frac{G(fx,gx,gx)+G(fy,gy,gy)+G(fz,gz,gz)}{3}, \frac{G(fy,gx,gx)+G(fz,gy,gy)+G(fx,gz,gz)}{3}\}$ $\forall x, y, z \in X$ *with* $x \neq y$.

*Then $f$ and $g$ have a unique common fixed point.*

**Proof:** Since $f$ and $g$ satisfies CLRg property, there exists a sequence $\{x_n\}$ in $X$ such that $\lim_n fx_n = \lim_n gx_n = gx$ for some $x \in X$. Consider
$$G(fx_n, fx, fx) < max\{G(gx_n, gx, gx), \frac{G(fx_n,gx_n,gx_n)+G(fx,gx,gx)+G(fx,gx,gx)}{3}, \frac{G(fx,gx_n,gx_n)+G(fx,gx,gx)+G(fx_n,gx,gx)}{3}\}$$
Letting $n \to \infty$, we obtain $G(gx, fx, fx) \le \frac{2}{3}G(gx, fx, fx)$ which implies $gx = fx$. Thus $x$ is the coincidence point of $f$ and $g$. Let $z = fx = gx$.
Since $(f, g)$ are weakly compatible, we have $fz = fgz = gfz = gz$.
Now we will prove that $fz = z$. Suppose $fz \ne z$, then
$$G(fz, z, z) = G(fz, fx, fx) < max\{G(gz, gx, gx), \frac{G(fz,gz,gz)+G(fx,gx,gx)+G(fx,gx,gx)}{3}, \frac{G(fx,gz,gz)+G(fx,gx,gx)+G(fz,gx,gx)}{3}\}$$
$$< G(fz, z, z), \text{ which is a contradiction.}$$
Hence $fz = z = gz$. Thus $z$ is the common fixed point of $f$ and $g$. The uniqueness of the fixed point can be proved easily.

We now illustrate this theorem by giving an example.

**Example 1:** Let $X = [2, 20]$ and $G : X \times X \times X \to [0, \infty)$ defined by $G(x, y, z) = 0$ if $x = y = z$ and $G(x, y, z) = \max\{x, y, z\}$ in all other cases. Then $(X, G)$ is a symmetric G-Metric space. Let $f, g$ be two self maps on $X$ defined by
$fx = 5$ if $x \le 5$ , $fx = 3$ if $x > 5$ and $gx = \frac{x+5}{2}$ if $x \le 5$ , $gx = 10$ if $x > 5$.
Here $f$ and $g$ satisfies the CLRg property. To see this, consider a sequence $\{x_n\} = \{5-\frac{1}{n}\}\ \forall n$. Then $fx_n = f(5-\frac{1}{n}) \to 5$ and $gx_n = g(5-\frac{1}{n}) = \frac{5-\frac{1}{n}+5}{2} \to 5$. Therefore $\lim_n fx_n = \lim_n gx_n = 5 = g5$ .
Further, $(f, g)$ are weakly compatible and
$$G(fx, fy, fz) < max\{G(gx, gy, gz), \frac{G(fx,gx,gx)+G(fy,gy,gy)+G(fz,gz,gz)}{3}, \frac{G(fy,gx,gx)+G(fz,gy,gy)+G(fx,gz,gz)}{3}\}\ \forall x, y, z \in X \text{ with } x \ne y.$$
Thus $f$ and $g$ satisfy all the conditions of Theorem 2.1 and have a unique common fixed point at $x = 5$.

In 1977, Mathkowski[8] introduced the $\Phi$-map as the following:
Let $\Phi$ be the set of auxiliary functions $\phi$ such that $\phi : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $\lim_n \phi^n(t) = 0$ for all $t \in (0, \infty)$. If $\phi \in \Phi$, then $\phi$ is called a $\Phi$-map. Further $\phi(t) < t$ for all $t \in (0, \infty)$ and $\phi(0) = 0$.

In the next result, we extracted a unique common fixed point for two pairs of self mappings which involves a $\phi$-map under the lipschitz type of contractive condition. This result extend and generalize Theorem 2 of M.Aamri and El Moutawakil [9] and Theorem 2.1 of S Manro et al.[14].

**Theorem 2.2** *Let $(X,G)$ be a symmetric G-Metric space and $f,g,S,T$ be four self mappings on $X$ such that*

1. *(f,S) and (g,T) satisfies* $CLR_{(S,T)}$ *property.*

2. $G(fx, gy, gz) \leq \phi(max\{G(Sx, Ty, Tz), G(Sx, gy, gz), G(Ty, gy, gz), G(gy, Ty, Tz)\})\ \forall x, y, z \in X.$

3. *(f,S) and (g,T) are weakly compatible.*

*Then f ,g,S and T have a unique common fixed point.*

**Proof:** Since $(f, S)$ and $(g, T)$ satisfies $CLR_{(S,T)}$ property, there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_n fx_n = \lim_n Sx_n = \lim_n gy_n = \lim_n Ty_n = t$ with $t = Sx = Ty$ for some $t, x, y \in X$. Consider,

$$G(fx, gy_n, gy_n) \leq \phi(max\{G(Sx, Ty_n, Ty_n), G(Sx, gy_n, gy_n), G(Ty_n, gy_n, gy_n), G(gy_n, Ty_n, Ty_n)\})$$

on letting $n \to \infty$, we obtain $G(fx, t, t) \leq \phi(0) = 0$ which implies $fx = t = Sx$. Hence $x$ is the coincidence point of $f$ and $S$. Since $(f, S)$ are weakly compatible, we have $ffx = fSx = Sfx = SSx$.

Now we prove that $Ty = gy$. Consider,

$$G(fx_n, gy, gy) \leq \phi(max\{G(Sx_n, Ty, Ty), G(Sx_n, gy, gy), G(Ty, gy, gy), G(gy, Ty, Ty)\})$$

As $n \to \infty$, we have $G(t, gy, gy) \leq \phi(G(t, gy, gy))$ which implies $G(t, gy, gy) = 0$. Therefore $gy = t = Ty$. i.e. $y$ is the coincidence point of $g$ and $T$.

Since $(g, T)$ are weakly compatible we have $ggy = gTy = Tgy = TTy$. Also note that $fx = Sx = gy = Ty = t$.

Now we prove that $ffx = fx$. Suppose $fx \neq ffx$, then

$$G(ffx, fx, fx) = G(ffx, gy, gy)$$
$$\leq \phi(max\{G(Sfx, Ty, Ty), G(Sfx, gy, gy), G(Ty, gy, gy), G(gy, Ty, Ty)\})$$
$$< G(ffx, fx, fx), \text{a contradiction.}$$

Hence $ffx = fx = Sfx$, which implies $fx$ is the common fixed point of $f$ and $S$. Similarly one can prove $gy$ is the common fixed point of $g$ and $T$. Since $fx = gy$, $z = fx$ is the common fixed point of $f, g, S$ and $T$. The uniqueness of the fixed point follows easily.

As a corollary of Theorem 2.2, we derive the following sharpened version of Theorem 2.1 contained in S.Manro[14], as conditions on the ranges of involved mappings are relatively lightened.

**Corollary 2.3** *Let (X,G) be a symmetric G-Metric space and f,g,S,T be four self mappings on X such that*

1. *(f,S) and (g,T) satisfies* $CLR_{(S,T)}$ *property.*

2. $G(fx, gy, gy) \leq \phi(max\{G(Sx, Ty, Ty), G(Sx, gy, gy), G(Ty, gy, gy), G(gy, Ty, Ty)\})\ \forall x, y \in X.$

3. *(f,S) and (g,T) are weakly compatible.*

*Then f ,g,S and T have a unique common fixed point.*

**Proof:** Put $z = y$ in Theorem 2.2.

By restricting $f, g, S, T$ suitably, one can derive the corollaries involving two as well as three self mappings as follows:

**Corollary 2.4** *Let (X,G) be a symmetric G-Metric space and f,g,S be three self mappings on X such that*

1. *(f,S) and (g,S) satisfies $CLR_S$ property.*
2. $G(fx, gy, gz) \leq \phi(max\{G(Sx, Sy, Sz), G(Sx, gy, gz), G(Sy, gy, gz),$
$G(gy, Sy, Sz)\}) \; \forall x, y, z \in X.$
3. *(f,S) and (g,S) are weakly compatible.*

*Then f ,g and S have a unique common fixed point.*

**Proof:** Follows from Theorem 2.2 by setting $S = T$.

We now demonstrate this theorem by the following example.

**Example 2:** Let $X = [0, 6]$ and $G : X \times X \times X \to [0, \infty)$ defined by $G(x, y, z) = 0$ if $x = y = z$ and $G(x, y, z) = \max\{x, y, z\}$ in all other cases. Then clearly $(X, G)$ is a symmetric G-Metric space. Let $f, g, S, T$ be four self maps on $X$ defined by
$fx = 3$ if $x \leq 3$ , $fx = 4$ if $x > 3$ and $Sx = 6 - x$ if $x \leq 3$ , $Sx = 5$ if $x > 3$
$gx = 4$ if $x < 3$ , $gx = \frac{x+3}{2}$ if $x \geq 3$, and $Tx = 6$ if $x < 3$ , $Tx = \frac{2x+3}{3}$ if $x \geq 3$.
Now $(f, S)$ and $(g, T)$ satisfy the $CLR_{(S,T)}$ property. To see this, choose two sequences $\{x_n\} = \{3 - \frac{1}{n}\}$ and $\{y_n\} = \{3 + \frac{1}{n}\}$ $\forall n$. Then $fx_n = f(3 - \frac{1}{n}) \to 3$, $Sx_n = S(3 - \frac{1}{n}) = 6 - (3 - \frac{1}{n}) \to 3$, $gy_n = g(3 + \frac{1}{n}) = \frac{3+\frac{1}{n}+3}{2} \to 3$ and $Ty_n = T(3 + \frac{1}{n}) = \frac{2(3+\frac{1}{n})+3}{3} \to 3$. Therefore $\lim_n fx_n = \lim_n Sx_n = \lim_n gy_n = \lim_n Ty_n = t = 3$ with $3 = S3 = T3$. Also $(f, S)$ and $(g, T)$ are weakly compatible. Let $\phi : [0, \infty) \to [0, \infty)$ be a function defined by $\phi(t) = \frac{t}{2}$, $\forall t \in [0, \infty)$. Then $\phi(0) = 0$ and $0 < \phi(t) < t$, $\forall t \in (0, \infty)$.
Further, $G(fx, gy, gz) \leq \phi(max\{G(Sx, Ty, Tz), G(Sx, gy, gz), G(Ty, gy, gz),$
$G(gy, Ty, Tz)\}) \; \forall x, y, z \in X.$
Thus all the conditions of Theorem 2.2 are satisfied and $x = 3$ is the unique common fixed point of $f, g, S$ and $T$.

The next two theorems involved with Hardy Roger's type of contractive condition for two pairs of self mappings, which extend the results contained in Theorem 2.8 of [10], Theorem 3.1 of [7] and Theorem 2.2 of [10], Theorem 3.11 of [18].

**Theorem 2.5** *Let (X,G) be a G-Metric space and f,g,S,T be four self mappings on X such that*

1. *(f,S) and (g,T) satisfies $CLR_{(S,T)}$ property.*
2. $G(fx, gy, gz) \leq pG(fx, Sx, Sx) + qG(Sx, Ty, Ty) + rG(Ty, gz, gz) + t[G(fx, Ty, Tz) + G(Sx, gy, gz)]$, $\forall x, y, z \in X$ *where* $p, q, r, t \in [0, 1)$ *satisfying* $p + q + r + 2t < 1$.

*Then (f,S) and (g,T) have a unique point of coincidence in X. Moreover, if (f,S) and (g,T) are weakly compatible, then f ,g,S and T have a unique common fixed point.*

**Proof:** Since $(f, S)$ and $(g, T)$ satisfies $CLR_{(S,T)}$ property, there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_n fx_n = \lim_n Sx_n = \lim_n gy_n = \lim_n Ty_n = u$ with $u = Sx = Ty$ for some $u, x, y \in X$. Consider

$$G(fx, gy_n, gy_n) \leq pG(fx, Sx, Sx) + qG(Sx, Ty_n, Ty_n) + rG(Ty_n, gy_n, gy_n) + t[G(fx, Ty_n, Ty_n) + G(Sx, gy_n, gy_n)]$$

On letting $n \to \infty$, we obtain $[1 - (p + t)]G(fx, u, u) \leq 0$ which gives $fx = u = Sx$, since $p + q + r + 2t < 1$.

Hence $x$ is the coincidence point of $f$ and $S$. Similarly $y$ is the coincidence point of $g$ and $T$. Thus $u = fx = Sx = gy = Ty$.

Uniqueness of coincidence point:

Let $u_1$ and $u_2$ be two points of coincidence of $(f, S)$ and $(g, T)$.

$\Rightarrow u_1 = fx_1 = Sx_1 = gy_1 = Ty_1$ and $u_2 = fx_2 = Sx_2 = gy_2 = Ty_2$. Consider

$$G(u_1, u_1, u_2) = G(fx_1, gy_2, gy_2) \leq pG(fx_1, Sx_1, Sx_1) + qG(Sx_1, Ty_2, Ty_2) + rG(Ty_2, gy_2, gy_2) + t[G(fx_1, Ty_2, Ty_2) + G(Sx_1, gy_2, gy_2)]$$

which implies $[1 - (q + 2t)]G(u_1, u_1, u_2) \leq 0$ i.e.$u_1 = u_2$.

Since $(f, S)$ and $(g, T)$ are weakly compatible, we have

$$ffx = fSx = Sfx = SSx \text{ and } ggy = gTy = Tgy = TTy.$$

Now we prove that $ffx = fx$. Consider

$$G(ffx, fx, fx) = G(ffx, gy, gy) \leq pG(ffx, Sfx, Sfx) + qG(Sfx, Ty, Ty) + rG(Ty, gy, gy) + t[G(ffx, Ty, Ty) + G(Sfx, gy, gy)]$$

which gives $[1 - (q + 2t)]G(ffx, fx, fx) \leq 0$. Hence $fx = ffx = Sfx$.

i.e.$fx$ is the common fixed point of $f$ and $S$. Similarly we can prove $gy$ is the common fixed point of $g$ and $T$. Hence $z = fx$ is the common fixed point of $f, g, S$ and $T$. The uniqueness of the fixed point can be proved easily.

We now furnish an example to illustrate Theorem 2.5.

**Example 3:** Let $X = [0, 4]$ and $G : X \times X \times X \to [0, \infty)$ defined by $G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$, where $d$ is the usual metric on $X$ . Then clearly $(X, G)$ is a G-Metric space. Let $f, g, S, T$ be four self maps on $X$ defined by

$$fx = 2 \ , \ Sx = 4 - x \ , \ gx = \frac{5x+12}{11} \text{ and } Tx = \frac{x+2}{2} \ \forall x \in X.$$

Now $(f, S)$ and $(g, T)$ satisfy the $CLR_{(S,T)}$ property. To see this, choose two sequences $\{x_n\} = \{2 + \frac{1}{n}\}$ and $\{y_n\} = \{2 - \frac{1}{n}\}$ $\forall n$. Then $fx_n = f(2 + \frac{1}{n}) \to 2$, $Sx_n = S(2 + \frac{1}{n}) = 4 - (2 + \frac{1}{n}) \to 2$, $gy_n = g(2 - \frac{1}{n}) = \frac{5(2-\frac{1}{n})+12}{11} \to 2$ and $Ty_n = T(2 - \frac{1}{n}) = \frac{2-\frac{1}{n}+2}{2} \to 2$. Therefore $\lim_n fx_n = \lim_n Sx_n = \lim_n gy_n = \lim_n Ty_n = t = 2$ with $2 = S2 = T2$. Further, $(f, S)$ and $(g, T)$ are weakly compatible and $f, g, S$ and $T$ satisfy the contractive condition (2) for $p = \frac{1}{8}, q = \frac{1}{3}, r = \frac{1}{8}, t = \frac{1}{7}$. Thus all the conditions of Theorem 2.5 are satisfied and $x = 2$ is the unique common fixed point of $f, g, S$ and $T$.

**Theorem 2.6** *Let (X,G) be a G-Metric space and f,g,S,T be four self mappings on X such that*

1. *(f,S) and (g,T) satisfies $CLR_{(S,T)}$ property.*
2. *$G(fx, gy, gz) \leq hu(x, y, z)$ where $h \in (0, 1)$ and $\forall x, y, z \in X$*
   $u(x, y, z) \in \{G(fx, Sx, Sx), G(Sx, Ty, Ty), G(Ty, gz, gz), \frac{G(fx,Ty,Tz)+G(Sx,gy,gz)}{2}\}$.
3. *(f,S) and (g,T) are weakly compatible*

*Then f ,g,S and T have a unique common fixed point.*

**Proof:** Since $(f, S)$ and $(g, T)$ satisfies $CLR_{(S,T)}$ property, there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_n fx_n = \lim_n Sx_n = \lim_n gy_n = \lim_n Ty_n = t$ with $t = Sx = Ty$ for some $t, x, y \in X$.
Consider $G(fx, gy_n, gy_n) \leq hu(x, y_n, y_n)$
where $u(x, y_n, y_n) \in \{G(fx, Sx, Sx), G(Sx, Ty_n, Ty_n), G(Ty_n, gy_n, gy_n), \frac{G(fx,Ty_n,Ty_n)+G(Sx,gy_n,gy_n)}{2}\}$
On letting $n \to \infty$, we obtain
$G(fx, t, t) \leq hu(x, y_n, y_n)$, where $u(x, y_n, y_n) \in \{G(fx, t, t), \frac{G(fx,t,t)}{2}\}$.
If $u(x, y_n, y_n) = G(fx, t, t)$, then $G(fx, t, t) \leq hG(fx, t, t)$ which gives $fx = t = Sx$. If $u(x, y_n, y_n) = \frac{G(fx,t,t)}{2}$, then $G(fx, t, t) \leq h\frac{G(fx,t,t)}{2}$ which also gives $fx = t = Sx$.
Therefore in both the cases $fx = Sx = t$. Hence $x$ is the coincidence point of $f$ and $S$. Similarly $y$ is the coincidence point of $g$ and $T$. Thus $t = fx = Sx = gy = Ty$.

Since $(f, S)$ and $(g, T)$ are weakly compatible, we have

$$ffx = fSx = Sfx = SSx \text{ and } ggy = gTy = Tgy = TTy.$$

Now we prove that $ffx = fx$.
Consider $G(ffx, fx, fx) = G(ffx, gy, gy) \leq hu(fx, y, y)$,
where $u(fx, y, y) \in \{G(ffx, Sfx, Sfx), G(Sfx, Ty, Ty), G(Ty, gy, gy), \frac{G(ffx,Ty,Ty)+G(Sfx,gy,gy)}{2}\}$
i.e. $u(fx, y, y) = G(ffx, fx, fx)$.
Therefore $G(ffx, fx, fx) \leq hG(ffx, fx, fx)$ which implies $fx = ffx = Sfx$. Hence $fx$ is the common fixed point of $f$ and $S$. Similarly we can prove $gy$ is the common fixed point of $g$ and $T$. Since $fx = gy$, $z = fx$ is the common fixed point of $f, g, S$ and $T$. The uniqueness of the fixed point can be proved easily.

**Example 4:** Let $X = [0, 4]$ and $G : X \times X \times X \to [0, \infty)$ defined by $G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$, where $d$ is the usual metric on $X$. Then clearly $(X, G)$ is a G-Metric space. Let $f, g, S, T$ be four self maps on $X$ defined by

$$fx = 3 \ , \ Sx = x \ , \ gx = \frac{x+24}{9} \text{ and } Tx = \frac{x+3}{2} \ \forall x \in X.$$

Now $(f, S)$ and $(g, T)$ satisfy the $CLR_{(S,T)}$ property. To see this, choose two sequences $\{x_n\} = \{3 + \frac{1}{n}\}$ and $\{y_n\} = \{3 - \frac{1}{n}\}$ $\forall n$. Then $fx_n = f(3 + \frac{1}{n}) \to 3$, $Sx_n = S(3 + \frac{1}{n}) = 3 + \frac{1}{n} \to 3$, $gy_n = g(3 - \frac{1}{n}) = \frac{3-\frac{1}{n}+24}{9} \to 3$ and $Ty_n = T(3 - \frac{1}{n}) = \frac{3-\frac{1}{n}+3}{2} \to 3$. Therefore $\lim_n fx_n = \lim_n Sx_n = \lim_n gy_n = \lim_n Ty_n = t = 3$ with $3 = S3 = T3$. Further, $(f, S)$ and $(g, T)$ are weakly compatible and $f, g, S$ and $T$ satisfies the contractive condition (2) for $h = \frac{1}{2}$. Thus all the conditions of Theorem 2.6 are satisfied and $x = 3$ is the unique common fixed point of $f, g, S$ and $T$.

Our last result is also a common fixed point theorem but for expansive type of contractive condition, which extend and improve the results of Theorem 2.1 of [16] and Theorem 3.1 of [18].

**Theorem 2.7** *Let f,g be two self maps of a G-Metric space (X,G) satisfying CLRg property and*
$G(gx, gy, gz) \geq aG(fx, fy, fz)+bG(fx, gx, gx)+cG(fy, gy, gy)+eG(fz, gz, gz)$ $\forall x, y, z \in X$ *where* $a, b, c, e > 0$. *Then f and g have a coincidence point. If* $a > 1$, *then the coincidence point is unique. Moreover, if f and g are weakly compatible, then they have a unique common fixed point.*

**Proof:** Since $f$ and $g$ satisfies CLRg property, there exist a sequence $\{x_n\}$ in $X$ such that $\lim_n fx_n = \lim_n gx_n = gx$ for some $x \in X$.

$$\text{Consider } G(gx_n, gx, gx) \geq aG(fx_n, fx, fx) + bG(fx_n, gx_n, gx_n) + cG(fx, gx, gx) + eG(fx, gx, gx).$$

On letting $n \to \infty$, we obtain $(\frac{a}{2} + c + e)G(fx, gx, gx) \leq 0$ which implies $fx = gx$. Thus $x$ is the coincidence point of $f$ and $g$.
Uniqueness of the coincidence point:
Let $z = fx = gx$ and $w = fy = gy$ be the two points of coincidence of $f$ and $g$. Then, $G(z, w, w) = G(gx, gy, gy)$

$$\geq aG(fx, fy, fy) + bG(fx, gx, gx) + cG(fy, gy, gy) + eG(fy, gy, gy)$$

i.e.$(a-1)G(z, w, w) \leq 0$. Since $a > 1$, we get $z = w$.
Now $(f, g)$ are weakly compatible implies $ffx = fgx = gfx = ggx$.
To prove that $fx = ffx$. Consider
$G(ffx, fx, fx) = G(gfx, gx, gx)$
$\geq aG(ffx, fx, fx) + bG(ffx, gfx, gfx) + cG(fx, gx, gx) + eG(fx, gx, gx)$
which implies $ffx = fx = gfx$. Hence $fx$ is the common fixed point of $f$ and $g$.The uniqueness of the fixed point can be proved easily.

# References


[1] A.H. Soliman, M. Imdad and M. Hasan, Proving unified common fixed point theorems via common property (E.A) in symmetric spaces, *Commun. Korean Math. Sci.*, 25(4) (2010), 629-645.

[2] A. Kaewcharoen, Common fixed point theorems for contractive mappings satisfying Φ-maps in G-metric spaces, *Banach J. Math. Anal.*, 6(1) (2012), 101-111.

[3] D. Gopal, M. Imdad and C. Vetro, Impact of common property (E.A) on fixed point theorems in fuzzy metric spaces, *Fixed Point Theory and Applications*, Article ID 297360 (2011), 14 pages.

[4] E. Karampur, D.K. Patel, M. Imdad and D. Gopal, Some nonunique fixed point theorems in symmetric spaces through $CLR_{(S,T)}$ property, *Int. J. Math. Math. Sci.*, Article ID 753965 (2013), 8 pages.

[5] F. Gu and W. Shantanawi, Common fixed points for generalized weakly G-contraction mappings satisfying common (E.A) property in G-metric space, *Fixed Point Theory and Applications*, Article ID: 2013:309(2013), 15 pages.

[6] G. Jungck, Common fixed points for noncontinuous nonself maps on non metric spaces, *Far East J. Math. Sci.*, 4(1996), 199-215.

[7] J.G. Mehta and M.L. Joshi, On complete cone metric space and fixed point theorems, *J. Sci. Res.*, 3(2) (2011), 303-309.

[8] J. Mathkowski, Fixed point theorems for mappings with a contractive iterate at a point, *Proc. Amer. Math. Soc.*, 62(1977), 344-348.

[9] M. Aamri and D.El. Moutawakil, Some new common fixed point theorems under strict contractive conditions, *J. Math. Anal. Appl.*, 270(2002), 181-188.

[10] M. Abbas, B.E. Rhoades and T. Nazir, Common fixed points for four maps in cone metric spaces, *Appl. Math. Comput.*, 216(1) (2010), 80-86.

[11] M. Kumar, P. Kumar and S. Kumar, Some common fixed point theorems using CLRg property in cone metric spaces, *Advances in Fixed Point Theory*, 2(3) (2012), 340-356.

[12] M. Imdaad and S. Chauhan, Employing common limit range property to prove unified metrical common fixed point theorems, *Int. J. Anal.*, Article ID: 763261 (2013), 10 pages.

[13] Giniswamy and P.G. Maheshwari, Fixed point theorems in cone metric spaces by using common limit range property, *Int. Res. J. Math. Sci.*, 3(1) (2014), 275-278.

[14] S. Manro, S.S. Bhatia, S. Kumar and C. Vetro, A common fixed point theorem for two weakly compatible pairs in G-metric spaces using the property E.A, *Fixed Point Theory and Applications*, Article ID: 2013:41(2013), 9 pages.

[15] V. Pant and R.P. Pant, Fixed point in fuzzy metric space for noncompatible maps, *Soochow J. Math.*, 33(4) (2007), 647-655.

[16] W. Shatanawi and F. Awawedh, Some fixed and coincidence point theorems for expansive maps in cone metric spaces, *Fixed Point Theory and Applications*, Article ID: 2012:19(2012), 10 pages.

[17] W. Sintunavarat and P. Kumam, Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, *Journal of Applied Mathematics*, Article ID 637958 (2011), 14 pages.

[18] Z. Kadelburg, P.P. Murthy and S. Radenovic, Common fixed points for expansive mappings in cone metric spaces, *Int. J. Math. Anal.*, 5(27) (2011), 1309-1319.

[19] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, *J. Nonlinear Convex Anal.*, 7(2006), 289-297.