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Approximate Solution of Convection-Diffusion Equation by the Homotopy Perturbation Method

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Abstract

In recent years, a new difference scheme with high accuracy has been applied for solving convection-diffusion equation [1]. In this letter, we solve this equation by homotopy perturbation method (HPM) [2-4]. To illustrate the ability and reliability of the method some examples are provided. The results reveal that the method is very effective and simple

Keywords: *Homotopy perturbation method; Convection-diffusion equation*

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1 Introduction

Consider the convection-diffusion equation [1]

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$$\frac{\partial u}{\partial t}+\varepsilon\frac{\partial u}{\partial x}=\gamma\frac{\partial^2 u}{\partial x^2} \qquad 0\le x\le 1,\ t\ge 0. \qquad (1)$$

Subject to the initial condition, $u(x,0)=g(x)$, $0\le x\le 1$ and boundary conditions $u(0,t)=0$, $t\ge 0$. $u(1,t)=0$, $t\ge 0$. where the parameter γ is the viscosity coefficient and ε is the phase speed and both are assumed to be positive. g is a given function of sufficient smoothness.

To illustrate the basic concepts of homotopy perturbation method, consider the following non-linear functional equation:

$$A(u)=f(r), \quad r\in\Omega, \qquad (2)$$

With the following boundary conditions:

$$B\left(u,\partial u/\partial n\right)=0,\ r\in\Gamma.$$

Where A is a functional operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω. Generally speaking, the operator A can be decomposed into two parts L and N, where L is a linear and N is a non-linear operator. Therefore Eq. (2) can be rewritten as the following:

$$L(u)+N(u)-f(r)=0. \qquad (3)$$

We construct a homotopy $v(r,p):\Omega\times[0,1]\to R$, which satisfies:

$$H(v,p)=(1-p)\left[L(v)-L(u_0)\right]+p\left[A(v)-f(r)\right]=0,\ p\in[0,1],\ r\in\Omega.$$

Or

$$H(v,p)=L(v)-L(u_0)+pL(u_0)+p\left[N(v)-f(r)\right]=0,\ p\in[0,1],\ r\in\Omega.,$$

Where u_0 is an initial approximation to the solution of Eq. (2). In this method, homotopy perturbation parameter p is used to expand the solution, as a power series, say;

$$v=v_0+pv_1+p^2v_2+\cdots,$$

Usually an approximation to the solution, will be obtained by taking the limit, as p tends to 1,

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 \cdots ,$$

For solving Eq. (1), by homotopy perturbation method, we construct the following homotopy:

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} + \varepsilon \frac{\partial v}{\partial x} - \gamma \frac{\partial^2 v}{\partial x^2}\right) = 0,$$

Or

$$\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} + p\left(\varepsilon \frac{\partial v}{\partial x} - \gamma \frac{\partial^2 v}{\partial x^2} + \frac{\partial u_0}{\partial t}\right) = 0, \qquad (4)$$

Suppose that the solution of Eq. (4) to be in the following form

$$v = v_0 + p v_1 + p^2 v_2 + \dots \qquad (5)$$

Substituting Eq. (5) into Eq. (4), and equating the coefficients of the terms with the identical powers of p,

$$p^0 : \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0,$$

$$p^1 : \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} + \varepsilon \frac{\partial v_0}{\partial x} - \gamma \frac{\partial^2 v_0}{\partial x^2} = 0, \qquad v_1(x,0) = 0$$

$$p^2 : \frac{\partial v_2}{\partial t} + \varepsilon \frac{\partial v_1}{\partial x} - \gamma \frac{\partial^2 v_1}{\partial x^2} = 0, \qquad v_2(x,0) = 0$$

$$p^3 : \frac{\partial v_3}{\partial t} + \varepsilon \frac{\partial v_2}{\partial x} - \gamma \frac{\partial^2 v_2}{\partial x^2} = 0, \qquad v_3(x,0) = 0$$

$$\vdots$$

$$p^j : \frac{\partial v_j}{\partial t} + \varepsilon \frac{\partial v_{j-1}}{\partial x} - \gamma \frac{\partial^2 v_{j-1}}{\partial x^2} = 0, \qquad v_j(x,0) = 0$$

$$\vdots$$

For simplicity we take

$$v_0(x,t) = u_0(x,t) = u(x,0)$$

Having this assumption we get the following iterative equation

$$v_j = \int_0^t \left(\gamma \frac{\partial^2 v_{j-1}}{\partial x^2} - \varepsilon \frac{\partial v_{j-1}}{\partial x}\right) dt, \qquad j = 1, 2, 3, \dots$$

Therefore, the approximated solutions of Eq. (1) can be obtained, by setting $p=1$

$$u=\lim_{p\to 1} v = v_0+v_1+v_2+v_3+...$$

2 numerical examples

In this section, we present examples of convection-diffusion equation and results will be compared with the exact solutions.

Example1. Let us consider the convection-diffusion equation

$$\frac{\partial u}{\partial t}+0.1\frac{\partial u}{\partial x}=0.01\frac{\partial^2 u}{\partial x^2} \qquad 0\le x\le 1,\ t\ge 0.$$

With the following initial condition $u(x,0)=e^{5x}\sin \pi x$.

The exact solution is $u(x,t)=e^{5x-\left(0.25-0.01\pi^2\right)t}\sin \pi x$.
Approximation to the solution of example 1 can be readily obtained by

$$u_{20}=\sum_{i=0}^{20} v_i$$

The results corresponding absolute errors are presented in Fig.1.

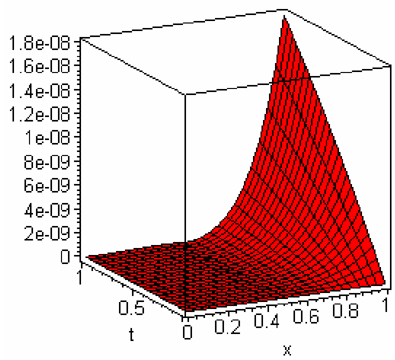

Fig.1. The absolute error between exact and numerical solutions in Example 1.

Example 2. Consider the following the convection-diffusion equation with boundary conditions $u(x,0)=e^{0.22x}\sin\pi x$.

The exact solution is $u(x,t)=e^{0.22x-\left(0.0242+0.5\pi^2\right)t}\sin\pi x$.

$$\frac{\partial u}{\partial t}+0.22\frac{\partial u}{\partial x}=0.5\frac{\partial^2 u}{\partial x^2}\qquad 0\le x\le 1,\ t\ge 0.$$

Approximation to the solution of example 2 can be readily obtained by

$$u_{20}=\sum_{i=0}^{20}v_i$$

The results corresponding absolute errors are presented in Fig.2.

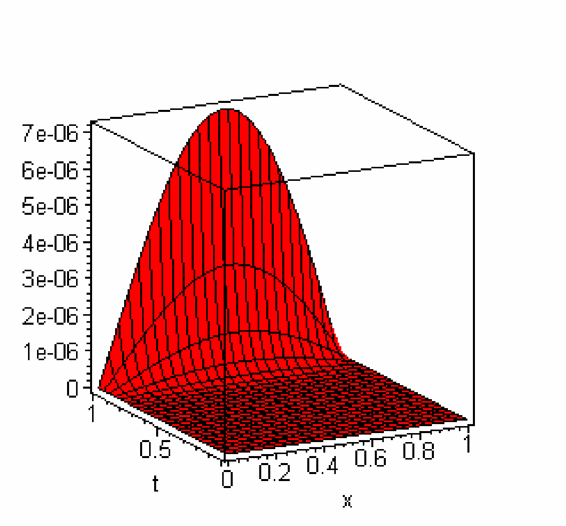

Fig.2. The absolute error between exact and numerical solutions in Example 2.

Example3. We consider the convection-diffusion equation with boundary conditions

$$u(x,0)=e^{0.25x}\sin\pi x.$$

The exact solution is $u(x,t)=e^{0.25x-\left(0.0125+0.2\pi^2\right)t}\sin\pi x$.

$$\frac{\partial u}{\partial t}+0.1\frac{\partial u}{\partial x}=0.2\frac{\partial^2 u}{\partial x^2}\qquad 0\le x\le 1,\ t\ge 0.$$

Approximation to the solution of example 3 can be readily obtained by

$$u_{20} = \sum_{i=0}^{20} v_i$$

The results corresponding absolute errors are presented in Fig.3.

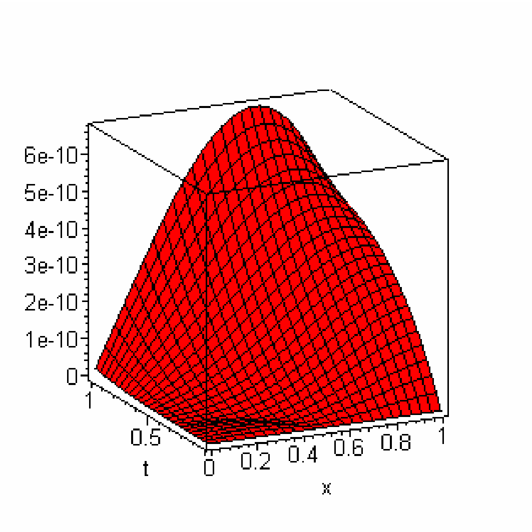

Fig.3. The absolute error between exact and numerical solutions in Example 3.

4 Conclusion

In this paper, we proposed the homotopy perturbation method for solving the convection-diffusion equations. The obtained solutions, in comparison with exact solutions admit a remarkable accuracy. The computations associated with the examples in this paper were performed using maple 10.

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