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About Dirichlet's Transformation and Theoretic-Arithmetic Functions

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Abstract

In this work, we are going to define a transformation from Dirichlet's series called discrete Dirichlet's transformation. We will obtain some classical results connected Riemann's zeta function and theoretic-arithmetic functions. Some probabilistic interpretations are maked explicit.

Keywords: *Dirichlet's transformation, Zeta function, Möbius transformation.*

1 Introduction

It is well-known (cf. [6]) that the Riemann zeta-function $\zeta(s)$ is holomorphic in the whole complex plane except for a simple pole at $s = 1$ with residue 1.

Riemann discovered the functional equation

$$\zeta(s)\Gamma(\frac{s}{2})\pi^{-s/2} = \zeta(1-s)\Gamma(\frac{1-s}{2})\pi^{-(1-s)/2}, \tag{1}$$

where $\Gamma(s)$ denotes Euler's Gamma-function.

This equation and the identity

$$\zeta(s) = \overline{\zeta(\overline{s})}, \;\; s \neq 1 \tag{2}$$

show some symmetries of $\zeta(s)$.

From (1) it follows that $\zeta(s)$ vanishes at the negative even integers, the so-called trivial zeros of $\zeta(s)$. It is also known that the other non-trivial zeros lie inside the so-called critical strip $0 \leq \Re e(s) \leq 1$, and they are non-real.

The famous, yet open Riemann hypothesis states that every non-trivial zero of $\zeta(s)$ satisfies $\Re e(s) = \frac{1}{2}$.

In this work, we are going from zeta function and a Dirichlet's series define one transformation called discrete Dirichlet's transformation. We obtain some classical results connected Riemann's zeta function and theoretic-arithmetic functions. Some applications stemed from number theory and theoretic-arithmetic function are given and probabilistic interpretations are maked explicit.

2 Dirichlet's Transformation

Let $f : \mathbb{N}^* \longrightarrow \mathbb{C}$ be a theoretic-arithmetic function. Associate to this last a Dirichlet series

$$\sum_{n\geq 1} \frac{f(n)}{n^s},\ s \in \mathbb{C}, \tag{3}$$

whose we will denote the abscissa of convergence $\lambda(f)$ and the abscissa of absolute convergence $\ell(f)$.

Introduce then $\mathbb{A}$ a class of theoretic-arithmetic functions such that $\lambda(f) < +\infty$. So for $f \in \mathbb{A}$, a Dirichlet series

$$\sum_{n\geq 1} \frac{f(n)}{n^s},\ s \in \mathbb{C}, \tag{4}$$

is convergent for $\Re e(s) > \lambda(f)$ and divergent for $\Re e(s) < \lambda(f)$. It represents a holomorphic function of a complex variable s in a half plane $\Re e(s) > \lambda(f)$ like that $\mathbb{A}$ equiped with addition process, multiplication by a scalar and a convolution product $*$, $(\mathbb{A}, +, \text{mult.by sca.}, *)$ is an algebra of theoretic-arithmetic functions and which is a sub-algebra of Dirichlet's algebra.

Next we introduce a class denoted $\mathfrak{C}$ of functions of complex variable s, defined on a half-plane $\Re e(s) > a$ where $a \in [-\infty, +\infty[$. $\mathfrak{C}$ equiped with operation $+$, multiplication by a scalar, ordinary product $\bullet$: $(\mathfrak{C}, +, \text{mult.by sca.}, \bullet)$ is an algebra, called functions algebra.

Definition 2.1 *We call discrete Dirichlet's transformation a mapping* $\wedge : \mathbb{A} \longrightarrow \mathfrak{C}$ *which to an element* $f \in \mathbb{A}$*, associates a function* $\widehat{f} \in \mathfrak{C}$ *defined by*

$$\widehat{f}(s) := \sum_{n\geq 1} \frac{f(n)}{n^s},\ \Re e(s) > \lambda(f). \tag{5}$$

A function $\widehat{f} \in \mathfrak{C}$ is said Dirichlet's transformation of f.

Proposition 2.2 *A mapping $\wedge$ is injective if and only if f and $g \in \mathbb{A}$ and $\widehat{f} = \widehat{g} \implies f = g$.*

Proof. It is a consequence of uniqueness theorem of a Dirichlet's series, see ([3], Theorem 3.3) and ([5]). □

3 Dirichlet's Transformation as a Homomorphism of Algebra

Theorem 3.1 *A Dirichlet's transformation of convolution product of two elements of $\mathbb{A}$ is equal to ordinary product of a Dirichlet's transformation of these two elements. More precisely, let f and $g \in \mathbb{A}$. Put $h = f * g$. Then we have*

a) $\ell(h) \leq \max\{\ell(f),\, \ell(g)\} < +\infty$, *hence* $h \in \mathbb{A}$;

b) $\widehat{h}(s) = \widehat{f}(s)\widehat{g}(s)$ *for* $\Re e(s) > \max\{\ell(f),\, \ell(g)\}$.

In shortcut

$$\widehat{f * g} = \widehat{f}.\widehat{g}. \tag{6}$$

Proof. Formally, we have

$$\widehat{f}(s).\widehat{g}(s) = (\sum_{k\geq 1} \frac{f(k)}{k^s})(\sum_{m\geq 1} \frac{g(m)}{m^s}) = \sum_{k,\, m\geq 1} \frac{f(k)g(m)}{(km)^s}. \tag{7}$$

Hence looking terms of same denominator, namely, in fact summing at km constant we have .

$$\widehat{f}(s).\widehat{g}(s) = \sum_{n\geq 1} \frac{1}{n^s}(\sum_{km=n} f(k)g(m)) = \sum_{n\geq 1} \frac{h(n)}{n^s} = \widehat{h}(s), \tag{8}$$

if everyone of series is convergent (absolutely convergent), namely $h < +\infty$, hence the theorem results. □

Theorem 3.2 *Dirichlet's transformation $\wedge : \mathbb{A} \longrightarrow \mathfrak{C}$ is an homomorphism from algebra ($\mathbb{A}$, $+$, mult.by scal., $*$) into algebra ($\mathfrak{C}$, $+$, mult.by scal., $\bullet$) :*

a) $f \overset{\wedge}{+} g = \hat{f} + \hat{g}$;

b) $\overset{\wedge}{\alpha f} = \alpha \hat{f}$, $\alpha \in \mathbb{N}$;

c) $f \overset{\wedge}{*} g = \hat{f}.\hat{g}$.

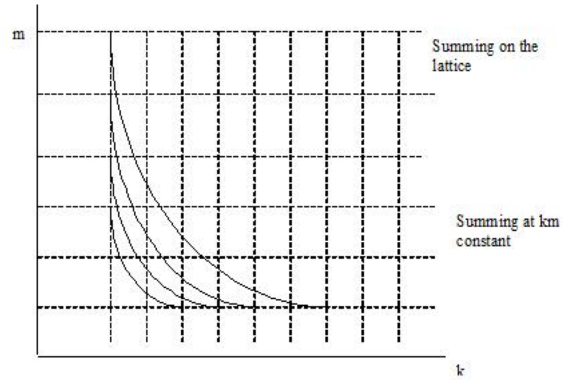

Figure 1:

4 Dirichlet's Transformation of Möbius Transformation

Theorem 4.1 *Let $f \in \mathbb{A}$ and F be its Möbius transformation : $F = 1 * f$. Then*

$$\hat{F}(s) = \zeta(s).\hat{f}(s), \quad \Re e(s) > Max\{\ell(f), 1\}. \tag{9}$$

Proof. Apply Theorem 3.1, then we have

$$\overset{\wedge}{1}(s) = \sum \frac{1}{n^s} = \zeta(s)\,;\ \ell(1) = 1$$

and $\hat{F} = \overset{\wedge}{1}.\hat{f}$. □

Probabilistic Interpretation

Interpret the expression (9) above in probabilistic meaning.

Suppose s real $> Max\{\ell(f), 1\}$, we have

$$\frac{1}{\zeta(s)}\hat{F}(s) = \hat{f}(s),$$

namely

$$\frac{1}{\zeta(s)}\sum_{n\geq 1}\frac{F(n)}{n^s} = \hat{f}(s),$$

and the mathematical expectation $E_s(F)$ of F is

$$E_s(F) = \hat{f}(s).$$

For the remainder of interpretation see ([3]).

5 Calculus of Dirichlet Transformations

5.1 Direct Calculus

Consider the following cases :

a) $f(n) = 1$, then

$$\widehat{f}(s) = \sum_{n\geq 1} \frac{1}{n^s} = \zeta(s), \ \ s > 1$$

b) $f(n) = n$, then

$$\widehat{f}(s) = \sum_{n\geq 1} \frac{1}{n^{s-1}} = \zeta(s-1), \ \ s > 2.$$

c) $f(n) = n^\alpha$, $\alpha \in \mathbb{R}$, then

$$\widehat{f}(s) = \sum_{n\geq 1} \frac{1}{n^{s-\alpha}} = \zeta(s-\alpha), \ \ s > \alpha + 1.$$

d) $f(n) =$ indicator function of the set of numbers of perfect squares, then

$$\widehat{f}(s) = \sum_{n\geq 1} \frac{1}{(n^2)^s} = \zeta(2s), \ \ s > \frac{1}{2}.$$

e) $f(n) =$ indicator function of powers k^{th} $(k \in \mathbb{N}^*)$, then

$$\widehat{f}(s) = \sum_{n\geq 1} \frac{1}{(n^k)^s} = \zeta(ks), \ \ s > \frac{1}{k}.$$

f) $f(n) = u(n) = \begin{cases} 1 & \text{if } \ n = 1, \\ 0 & \text{if } \ n > 1, \end{cases}$ then

$$\widehat{u}(s) = \sum_{n\geq 1} \frac{u(n)}{n^s} = 1.$$

g) Generally, $f(n) = \delta_a(n)$, where

$$\delta_a(n) = \begin{cases} 1 & \text{if } \ n = a, \\ 0 & \text{if } \ n \neq a, \end{cases} \quad a \in \mathbb{N}^* ,$$

then

$$\widehat{\delta}_{(a)}(s) = \sum_{n\geq 1} \frac{\delta_a(n)}{n^s} = \frac{1}{a^s}.$$

h) $f(n) = (-1)^{n-1}$, then

$$\widehat{f}(s) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s) = \eta(s).$$

Proof. We have

$$\eta(s) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + ...$$

$$= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + ... - 2(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + ...)$$

$$= \sum_{n\geq 1} \frac{1}{n^s} - 2(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + ...)$$

$$= \zeta(s) - 2\frac{1}{2^s}(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + ...) = (1 - 2^{1-s})\zeta(s).$$

□

Remark 5.1 *We have*

$$\eta(s) = (1 - 2^{1-s})\zeta(s).$$

a) $\zeta(s)$ *is definite for* $s > 1$ *;*
b) a function

$$\eta(s) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^s}$$

is definite for $s > 0$*, (it is an alternate series of abscissa of convergence* 0*) and*

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}.$$

We have

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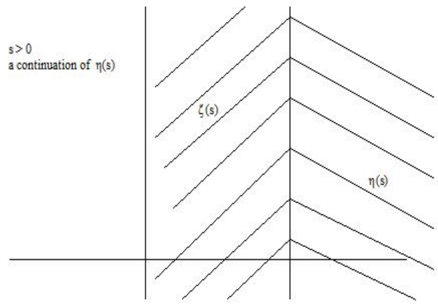

Figure 2:

6 Calculus of Dirichlet Transformations of Multiplicative Arithmetic Function

A calculus of Dirichlet transformations of multiplicative arithmetic function will be doing in pleasant way using generalized Euler's identity. Apply generalized Euler identity to the following multiplicative function

$$\frac{f(n)}{n^s}.$$

Theorem 6.1 *Let $f : \mathbb{N}^* \longrightarrow \mathbb{C}$ be a multiplicative function (no identically zero) and s be a real number such that series of general term*

$$\sum_{n\geq 1} \frac{f(n)}{n^s}$$

converges, namely, $\lambda(f) < +\infty$ and $\Re e(s) > \lambda(f)$. Then one has

$$\widehat{f}(s) = \sum_{n\geq 1} \frac{f(n)}{n^s} = \prod_p (1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + ...). \qquad (10)$$

Corollary 6.2 *A necessary and sufficient condition for an arithmetic function f to be multiplicative is that its Dirichlet's transformation can be written in the form*

$$h(s) = \prod_p (1 + \frac{c_p}{p^s} + \frac{c_p^2}{p^{2s}} + ...),$$

where c_p are complex numbers.

We have the following particular cases :

Proposition 6.3 *Suppose f strongly multiplicative, namely, for all $p \in \mathbb{P}$, for any $\alpha \in \mathbb{N}^*$, one has $f(p^\alpha) = f(p)$. Then*

$$\widehat{f}(s) = \prod_p (1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + ...) = \prod_p (1 + \frac{f(p)}{p^s - 1}).$$

Proposition 6.4 *If f is completely multiplicative, namely, for all $p \in \mathbb{P}$, for any $\alpha \in \mathbb{N}^*$, one has $f(p^\alpha) = (f(p))^\alpha$. Then*

$$\widehat{f}(s) = \prod_p (1 + \frac{f(p)}{p^s} + (\frac{f(p)}{p^s})^2 + ...).$$

Moreover, if for any p such that

$$\left| \frac{f(p)}{p^s} \right| < 1,$$

then

$$\widehat{f}(s) = \prod_p (\frac{1}{1 - \frac{f(p)}{p^s}}).$$

Example 6.5 *If $f = 1$ then*

$$\widehat{f}(s) = \prod_p (\frac{1}{1 - \frac{1}{p^s}}) = \zeta(s), \textit{ for } s > 1.$$

Indeed, $f = 1$ is an arithmetic function and completely multiplicative, $f(p) = 1$, for any p. Hence

$$\overset{\wedge}{1}(s) = \prod_p (\frac{1}{1 - \frac{1}{p^s}}) = \prod_p (1 - \frac{1}{p^s})^{-1} = \zeta(s), \textit{ for } s > 1.$$

Example 6.6 *Put $f = \mu$, where*

$$\mu(p^\alpha) = \begin{cases} -1 & \textit{if } \alpha = 1 \\ 0 & \textit{if } \alpha > 1. \end{cases}$$

Then

$$\widehat{\mu}(s) = \prod_p (1 - \frac{1}{p^s}) = \frac{1}{\zeta(s)}, \textit{ for } s > 1.$$

Indeed, $f = \mu$ is a multiplicative arithmetic function. According to previous example and a definition of μ, one has

$$\widehat{\mu}(s) = \prod_p (1 - \frac{1}{p^s}) = \frac{1}{\zeta(s)}, \textit{ for } s > 1.$$

Remark 6.7 *Functions* 1 *and* μ *are inverse one of the other in Dirichlet's algebra.*

Example 6.8 *Let* $f = |\mu| = \mu^2$ *be an indicator function of square free numbers. Then*

$$\left|\widehat{\mu}\right|(s) = \prod_p (1 + \frac{1}{p^s}) = \frac{\zeta(s)}{\zeta(2s)}, \ \textit{for } s > 1.$$

Proof. Indeed, one has $f = |\mu| = \mu^2$ the indicator function of square free numbers. It is a multiplicative arithmetic function and

$$|\mu|(p^\alpha) = \begin{cases} 1 & \text{if} \ \ \alpha = 1, \\ \\ 0 & \text{if} \ \ \alpha > 1. \end{cases}$$

Then according to previous examples and a definition of f we have

$$\left|\widehat{\mu}\right|(s) = \prod_p (1 + \frac{1}{p^s}).$$

But

$$1 + t = \frac{1 - t^2}{1 - t},$$

then

$$\left|\widehat{\mu}\right|(s) = \prod_p (\frac{1 - \frac{1}{p^{2s}}}{1 - \frac{1}{p^s}}) = \frac{\zeta(s)}{\zeta(2s)}, \ \text{for } s > 1.$$

□

Example 6.9 *Let* $f = \lambda$ *be Liouville's function, where*

$$\lambda(n) = \begin{cases} 1 & \textit{if} \ \ n = 1, \\ \\ (-1)^{\Omega(n)} & \textit{if} \ \ n > 1, \end{cases}$$

with $\lambda(p) = -1$*. Then*

$$\widehat{\lambda}(s) = \prod_p (\frac{1}{1 + \frac{1}{p^s}}) = \frac{\zeta(2s)}{\zeta(s)}, \ \textit{for } s > 1.$$

Proof. Indeed, let $f = \lambda$ be the Liouville's arithmetic function. It is a completely multiplicative arithmetic function.

According to previous examples and a definition of λ on prime numbers, we have

$$\widehat{\lambda}(s) = \prod_p \left(\frac{1}{1+\frac{1}{p^s}}\right).$$

But

$$\frac{1}{1+t} = \frac{1-t}{1-t^2},$$

then

$$\widehat{\lambda}(s) = \prod_p \left(\frac{1-\frac{1}{p^s}}{1-\frac{1}{p^{2s}}}\right) = \frac{\zeta(2s)}{\zeta(s)}, \text{ for } s > 1.$$

□

Remark 6.10 *We have*

$$\left.\begin{array}{l} \widehat{\lambda}(s) = \frac{\zeta(2s)}{\zeta(s)}, \\ \left|\widehat{\mu}\right|(s) = \frac{\zeta(s)}{\zeta(2s)} \end{array}\right\}$$

imply $|\mu|$ *and* λ *are inverse one of the other in Dirichlet's algebra.*

Example 6.11 *Put* $f = d$*, where* $d(n) =$ *number of divisors of* n*. Then*

$$\widehat{d}(s) = \prod_p \left(1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + ... + \frac{\alpha+1}{p^{\alpha s}}\right) = (\zeta(s))^2.$$

Proof. Let $f = d$ be the arithmetic function number of divisors of n. It is a multiplicative arithmetic function.

According to previous examples and a definition of d, we have

$$\widehat{d}(s) = \prod_p \left(1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + ... + \frac{\alpha+1}{p^{\alpha s}}\right).$$

But

$$1 + 2t + 3t^2 + 4t^3 + ... = \frac{1}{(1-t)^2},$$

then

$$\widehat{d}(s) = \prod_p \left(1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + ... + \frac{\alpha+1}{p^{\alpha s}}\right) = \prod_p \frac{1}{(1-\frac{1}{p^s})^2} = (\zeta(s))^2.$$

□

Example 6.12 *Let $f = d^*$ be a strongly multiplicative projection of d (see [1])*
$d^*(n) = 2^{\omega(n)}$ *number of square free divisors of n. Then*

$$\widehat{d^*}(s) = \prod_p \frac{1-\frac{1}{p^{2s}}}{(1-\frac{1}{p^s})^2} = \frac{(\zeta(s))^2}{\zeta(2s)}.$$

Proof. Indeed, let $f = d^*$ be a strongly multiplicative projection of d. Then f is a strongly multiplicative arithmetic function. According to previous examples and a definition of d^*, we have

$$\widehat{d^*}(s) = \prod_p (1 + \frac{1+\frac{d^*(p)-1}{p^s}}{1-\frac{1}{p^s}}) = \prod_p (\frac{1+\frac{1}{p^s}}{1-\frac{1}{p^s}}).$$

But

$$1 + t = \frac{1-t^2}{1+t},$$

hence

$$\widehat{d^*}(s) = \prod_p \frac{1-\frac{1}{p^{2s}}}{(1-\frac{1}{p^s})^2} = \frac{(\zeta(s))^2}{\zeta(2s)}.$$

□

Example 6.13 *Let $f = \varphi$ be the totient Euler's arithmetic function. Then*

$$\widehat{\varphi}(s) = \prod_p (\frac{1-\frac{1}{p^s}}{1-\frac{1}{p^{s-1}}}) = \frac{\zeta(s-1)}{\zeta(s)}, \text{ for } s > 2.$$

Proof. Indeed, let $f = \varphi$ be the totient Euler's arithmetic function. According to ([2]), Theorem 5.1, we have

$$\varphi(p^\alpha) = p^{\alpha-1}(p-1).$$

Hence

$$\widehat{\varphi}(s) = \prod_p (1 + \tfrac{\varphi(p)}{p^s} + \tfrac{\varphi(p^2)}{p^{2s}} + ...) = \prod_p (1 + \tfrac{p-1}{p^s} + \tfrac{p(p-1)}{p^{2s}} + \tfrac{p^2(p-1)}{p^{3s}} + ...)$$

$$= \prod_p (1 + \tfrac{p-1}{p^s}(1 + \tfrac{1}{p^{s-1}} + \tfrac{1}{p^{2s-1}} + ...)), \quad s > 2$$

$$= \prod_p (1 + \tfrac{p-1}{p^s} . \tfrac{1}{1-\frac{1}{p^{s-1}}}) = \prod_p (\tfrac{1-\frac{1}{p^s}}{1-\frac{1}{p^{s-1}}}) = \tfrac{\zeta(s-1)}{\zeta(s)}, \text{ for } s > 2.$$

□

7 Calculus of Dirichlet Transformations of Möbius Transformations

We have, if $F = 1 * f$, then

$$\widehat{F}(s) = \zeta(s).\widehat{f}(s).$$

We obtain the following results as propositions :

Proposition 7.1 *Put $f = 1$, then*

$$\widehat{f}(s) = \zeta(s), \text{ for } s > 1.$$

Put

$$d(n) = \sum_{d|n} 1.$$

We have $d = 1 * 1$. Then

$$\widehat{d}(s) = (\zeta(s))^2, \text{ for } s > 1.$$

Proposition 7.2 *Let $f(n) = n$, then*

$$\widehat{f}(s) = \sum_{n\geq 1} \frac{n}{n^s} = \zeta(s-1), \text{ for } s > 2 \text{ and } F(n) = \sum_{k|n} k.$$

We have

$$\sigma(n) = \sum_{k|n} k\,; \quad \sigma = 1 * f$$

and

$$\widehat{\sigma}(s) = \zeta(s)\zeta(s-1), \text{ for } s > 2.$$

Proposition 7.3 *Generally, let $f(n) = n^{\alpha}$, $\alpha \in \mathbb{R}$, then*

$$\widehat{f}(s) = \zeta(s-\alpha), \text{ for } s > \alpha + 1.$$

We have

$$\sigma_{\alpha}(n) = \sum_{k|n} k^{\alpha}\,; \quad \sigma_{\alpha} = 1 * f$$

and

$$\widehat{\sigma}_{\alpha}(s) = \zeta(s)\zeta(s-\alpha), \text{ for } s > \max(\alpha+1, 1).$$

We obtain some particular cases in the following corollary :

Corollary 7.4 *a) if* $\alpha = 0$ *then* $\sigma_0 = \alpha$ *;*
b) if $\alpha = 1$ *then* $\sigma_1 = \sigma$ *;*
c) if $\alpha = -1$ *then*

$$\sigma_{-1}(n) = \sum_{k|n} \frac{1}{k},$$

and $\widehat{\sigma}_{-1}(s) = \zeta(s)\zeta(s+1)$, *for* $s > 1$.

Proposition 7.5 *Let* $f = \varphi$*, where* φ *is a totient Euler's arithmetic function. Then*

$$\widehat{\varphi}(s) = \frac{\zeta(s-1)}{\zeta(s)}, \text{ for } s > 2. \tag{11}$$

Proof. Indeed, we have

$$\widehat{f}(s) = \sum_{n\geq 1} \frac{\varphi(n)}{n^s} \leq \sum_{n\geq 1} \frac{1}{n^{s-1}} = \zeta(s-1) \tag{12}$$

and

$$n = \sum_{k|n} \varphi(k) \text{ (Möbius's transformation)} \tag{13}$$

implies $\zeta(s-1) = \zeta(s)\widehat{\varphi}(s)$, for $s > 2$, hence

$$\widehat{\varphi}(s) = \frac{\zeta(s-1)}{\zeta(s)}, \text{ for } s > 2.$$

□

Proposition 7.6 *Let* $f = \Lambda$ *be von Mangoldt's arithmetic function introduced in ([6]). Then we have*

$$\widehat{\Lambda}(s) = -\frac{\zeta'(s)}{\zeta(s)}, \text{ for } s > 1.$$

Proof. Indeed, put $f = \Lambda$, then we have

$$\widehat{f}(s) = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s}, \text{ for } s > 1$$

with

$$\sum_{n\geq 1} \frac{\Lambda(n)}{n^s} \leq \sum_{n\geq 1} \frac{Logn}{n^s} < +\infty, \text{ for } s > 1$$

and

$$Logn = \sum_{k|n} \lambda(k),$$

hence

$$\sum_{n\geq 1}\frac{Logn}{n^s}=\zeta(s)\widehat{\Lambda}(s),\ \text{for } s>1.$$

But

$$\sum_{n\geq 1}\frac{Logn}{n^s}=-\frac{d}{ds}(\sum_{n\geq 1}\frac{1}{n^s})=-\zeta^{'}(s),\ \text{for } s>1,$$

hence

$$\widehat{\Lambda}(s)=-\frac{\zeta^{'}(s)}{\zeta(s)},\ \text{for } s>1.$$

□

Proposition 7.7 *Let $f=\mu$, then*

$$\widehat{\mu}(s)=\frac{1}{\zeta(s)},\ \textit{for } s>1.$$

Proof. We have

$$\widehat{f}(s)=\sum_{n\geq 1}\frac{\mu(n)}{n^s},$$

with

$$\sum_{n\geq 1}\frac{|\mu(n)|}{n^s}\leq\sum_{n\geq 1}\frac{1}{n^s}=\zeta(s),\ \text{for } s>1.$$

But

$$\sum_{k|n}\mu(k)=u(n)=\begin{cases}1 \text{ if } \ n=1,\\ \\ 0 \text{ if } \ n>1,\end{cases}$$

hence

$$\zeta(s)\widehat{\mu}(s)=1,\ \text{for } s>1.$$

□

We find the same results in ([6]) but in analytical way using real and complex analysis in its proofs.

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