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C^1-Stable Ergodic Shadowable Invariant Sets and Hyperbolicity

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Abstract

Let f be a diffeomorphism of a closed C^∞ manifold. We define the notion of C^1-stable ergodic shadowing property for a closed f- invariant set Λ, and prove that Λ has C^1-stable ergodic shadowing property if and only if it is a hyperbolic elementary set.

Keywords: C^1*- stable ergodic shadowable, basic set, uniformly hyperbolic*

1 Introduction

The notion of pseudo orbits is very often appeared in studying dynamical systems theory. There are various type of (pseudo orbit) shadowing properties, all of them playing an important role in orbit study. Recently ergodic shadowing property is introduced, and its relation with shadowing property is argued [1]. Let (X, d) be a compact metric space and $f : X \longrightarrow X$ be a continuous map. For any two open subsets U and $V$of X, let $N(U, V, f) := \{m \in \mathbb{N}; f^m(U) \cap V \neq \emptyset\}$

f is called topologically transitive if for any two open subsets U and V of X, $N(U, V) \neq \emptyset$. *A mapping f is weakly mixing* if $f \times f$ is transitive on $X \times X$. *Topological mixing* means that for any two open subsets U and V , the set $N(U, V)$ contains any natural number $n \geq n_0$, for some fixed $n_0 \in \mathbb{N}$. *For $\delta > 0$, a sequence $\{x_i\}_{a \leq i \leq b}$ is called a $\delta-$pseudo orbit of f* if $d(f(x_i), x_{i+1}) < \delta$ for any $a \leq i \leq b$. If $a = 1, b < \infty$, the finite δ-pseudo orbit

$\{x_i\}_{1\leq i\leq b}$ of f is called a δ-chain of f from x_1 to x_b of length b. A point $x\in X$ is a chain recurrent point of f if for every $\epsilon>0$ there is an ϵ-chain from x to x. The set of all chain recurrent points of f is denoted by $CR(f)$. A sequence $\{x_i\}_{a\leq i\leq b}$ is said to be ϵ-shadowed by a point x in X if $d(f^i(x),x_i)<\epsilon$ for each $a\leq i\leq b$. *A mapping f is said to have POTP(resp. $POTP^+$)* if for any $\epsilon>0$ there is a $\delta>0$ such that every δ-pseudo orbit of f,$\{x_i\}_{i\in\mathbb{Z}}$, (resp. $\{x_i\}_{i\in\mathbb{N}}$) can be ϵ-shadowed by some point in X with $d(f^i(x),x_i)<\epsilon$ for $i\in\mathbb{Z}$,(resp. $d(f^i(x),x_i)<\epsilon$ for $i\in\mathbb{N}$). *A mapping f is called chain transitive* if for any two points $x,y\in X$ and every $\epsilon>0$ there exists an ϵ-chain from x to y. A mapping f is called chain mixing if for every two points $x,y\in X$ and any $\varepsilon>0$, there is a positive integer n_0 such that for any integer $n>n_0$ there is an ϵ-chain from x to y of length n.

Given a sequence $\xi=\{x_i\}_{i\in\mathbb{Z}}$ (resp. $\xi=\{x_i\}_{i\in\mathbb{N}}$), put $NPO(\xi,\delta):=\{i\in\mathbb{Z}:d(f(x_i),x_{i+1})\geq\delta\}$(resp.$NPO^+(\xi,\delta):=\{i\in\mathbb{N}:d(f(x_i),x_{i+1})\geq\delta\}$) and $NPO_n(\xi,\delta):=NPO(\xi,\delta)\bigcap\{-n,...,-2,-1,0,1,2,...,n\}$ (resp.$NPO_n^+(\xi,\delta):=NPO^+(\xi,\delta)\bigcap\{1,2,3,...,n\}$).

For a sequence $\xi=\{x_i\}_{i\in\mathbb{Z}}$ (resp. $\xi=\{x_i\}_{i\in\mathbb{N}}$) and a point x of X, put $NS(\xi,x,\delta):=\{i\in\mathbb{Z}:d(f^i(x),x_i)\geq\delta\}$ (resp.$NS^+(\xi,x,\delta):=\{i\in\mathbb{N}:d(f^i(x),x_i)\geq\delta\}$) and $NS_n(\xi,x,\delta):=NS(\xi,x,\delta)\bigcap\{-n,...,-2,-1,0,1,2,...,n\}$ (resp. $NS_n^+(\xi,x,\delta):=NS^+(\xi,x,\delta)\bigcap\{1,2,...,n\}$). $\xi=\{x_i\}_{i\in\mathbb{Z}}$ (resp. $\xi=\{x_i\}_{i\in\mathbb{N}}$) is called a *$\delta$-ergodic pseudo orbit* if $\lim_{|n|\to\infty}\frac{|NPO_n(\xi,\delta)|}{n}=0$ (resp. $\lim_{n\to\infty}\frac{|NPO_n^+(\xi,\delta)|}{n}=0$). A δ- ergodic pseudo orbit is said to be *ϵ-ergodic shadowed* by a point x in X, denote by EPOTP, (resp. EPOTP^+) if $\lim_{|n|\to\infty}\frac{|NS_n^+(\xi,x,\delta)|}{n}=0$

(resp. $\lim_{n\to\infty}\frac{|NS_n^+(\xi,x,\delta)|}{n}=0$). *For an f-invariant set Λ, we say that Λ has ergodic shadowing property (or Λ is ergodic shadowable for f)* if for every $\epsilon>0$ there exists $\delta>0$ such that every δ-ergodic pseudo orbit in Λ could be ϵ-ergodic shadowed by a point x in X.

2 Notations

Let M be a compact C^∞ manifold, and let $\mathrm{Diff}^1(M)$ be the space of diffeomorphisms of M endowed with C^1- topology. Denote by d the distance on M induced by a Riemanian metric $||.||$ on the tangent bundle TM. Let $f\in\mathrm{Diff}^1(M)$, and $P(f)$ be the set of periodic point of f.

It is well known that if $p\in P(f)$ is a hyperbolic saddle point with period k, then the sets $W^s(p)=\{x\in M:f^{kn}(x)\to p \text{ as } n\to\infty\}$ and $W^u(p)=\{x\in M:f^{-kn}(x)\to p \text{ as } n\to\infty\}$ are C^1-injectively immersed submanifolds of M. The dimension of the stable manifold $W^s(p)$ is called the index of p, denoted by index(p). For a closed f-invariant set $\Lambda\subset M$ and a compact neighborhood $U\subseteq M$ of Λ, let $\Lambda_f(U)=\bigcap_{n\in\mathbb{Z}}f^n(U)$. *A set Λ is called locally maximal in U*

if there is a compact neighborhood U of Λ such that $\Lambda = \Lambda_f(U)$.

Definition 2.1 *Let $f \in Diff^1(M)$. A closed f-invariant set $\Lambda \subset M$ has C^1- stable ergodic shadowing property if there exist a compact neighborhood U of Λ and a C^1-neighborhood $\mathcal{U}(f)$ of f such that*
(i) $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$, i.e. Λ is locally maximal in U;
(ii) Λ_g is ergodic shadowable for every $g \in \mathcal{U}(f)$, where $\Lambda_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$. Such a Λ is called C^1-stable ergodic shadowable with respect to U and $\mathcal{U}(f)$.

A closed f-invariant set $\Lambda \subset M$ is called hyperbolic if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$, $0 < \lambda < 1$ such that $||Df^n|E(x)|| \leq C\lambda^n$ and $||Df^{-n}|F(f^n(x))|| \leq C\lambda^n$.

3 Results

A set Λ is called a basic set (resp. elementary set) if Λ is locally maximal and $f|\Lambda$ is transitive (resp. topologically mixing). Thus every elementary set is a basic set. It is clear that if f has EPOTP then f has EPOTP$^+$. The following proposition is proved in [1]:

Proposition 3.1 *Suppose $f : X \to X$ be a continuous surjective mapping on a compact metric space X. Then f is ergodic shadowable, if and only if f is shadowable and topological mixing.*

Topological entropy of every transitive and continuous map $f : [0, 1] \to [0, 1]$ is at least $h_{topo}(f) \geq log\sqrt{2}$ (see [[4], Corollary 3.6]). We do not recall here the definition of topological entropy, since it is well known (see [5]).

Remark 3.2 *It is know that every connected compact one dimensional manifold is homeomorphic with $\mathbb{S}^1$ or $[0, 1]$, and it is easy to see that every homeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ has zero topological entropy and there exist no homeomorphism $f : [0, 1] \to [0, 1]$ with topologically mixing thus if homeomorphism $f : M \to M$ has ergodic shadowing property, we must suppose that manifold M is not connected compact one dimensional.*

Let Λ be a hyperbolic set which is not a basic set, then Λ does not have C^1-stable ergodic shadowing property.

In this paper, the following theorem is proved.

Theorem 3.3 *Let Λ contains a hyperbolic saddle point p. Then Λ has C^1-stable ergodic shadowing property if and only if Λ is a hyperbolic elementary set.*

Proof of the "if" part follows from the local stability of hyperbolic sets. For the"only if" we need some preliminaries.

Proposition 3.4 *Suppose that $f|\Lambda$ has ergodic shadowing property. If $p, q \in \Lambda$ are hyperbolic saddle points then $W^s(p) \bigcap W^u(q) \neq \emptyset \neq W^s(q) \bigcap W^u(p)$.*

Proof. Let $p, q \in \Lambda$ be hyperbolic saddle points. Choose $\epsilon_0 > 0$ such that both $W^{\sigma}_{\epsilon_0}(p)$, $W^{\sigma}_{\epsilon_0}(q)$ $(\sigma = s, u)$ are C^1-embedded disks. Since $f|\Lambda$ has ergodic shadowing property proposition 3.1 shows that $f|\Lambda$ has shadowing property. Let $0 < \delta = \delta(\epsilon_0)$ be the number in the definition of the shadowing property for $f|\Lambda$. Since $f|\Lambda$ is chain transitive for every $p, q \in \Lambda$ and $\delta > 0$, there exists a δ-pseudo orbit $\{x_i\}_{i=0}^{n_\delta}$ such that $x_0 = p$, $x_{n_\delta} = q$, and $\{x_i\}_{i=0}^{n_\delta} \subseteq \Lambda$.

Extend the δ- pseudo orbit by putting $x_{-i} = f^{-i}(p)$ and $x_{n_\delta + i} = f^i(q)$ for all $i \geq 0$. As $f|\Lambda$ has shadowing property, there is $y \in M$ such that $d(f^i(y), x_i) < \epsilon_0$ for all $i \in \mathbb{Z}$, and hence $y \in W^u_{\epsilon_0}(p)$ and $f^{n_\delta}(y) \in W^s_{\epsilon_0}(q)$. Thus $y \in W^s(q)$ and $y \in W^u(p)$ i.e. $W^s(q) \bigcap W^u(p) \neq \emptyset$. Proof of $W^s(p) \bigcap W^u(q) \neq \emptyset$ is similar.

If $p \in P(f)$ is hyperbolic, then for any $g \in$Diff$^1(M)$ C^1-near f, there exists a unique hyperbolic periodic point $p_g \in P(g)$ nearby p such that $\pi(p_g) = \pi(p)$ and index(p) =index(p_g). such a p_g is called the continuation of p.

Lemma 3.5 *Let Λ satisfies C^1- stable ergodic shadowing property and let $\mathcal{U}(f)$ be as in definition 2.1. Then for any pair of hyperbolic saddle point $p, q \in \Lambda_g(U) \bigcap P(g)$ $(g \in \mathcal{U}(f))$, index(p)=index(q).*

Proof. Let Λ satisfies C^1-stable ergodic shadowing property and let $\mathcal{U}(f)$ be as above. Fix any $g \in \mathcal{U}(f)$ and hyperbolic saddle points $p, q \in \Lambda_g(U) \bigcap P(f)$. Then there is a C^1- neighborhood $\mathcal{V}(f) \subseteq \mathcal{U}(f)$ containing g such that for any $\varphi \in \mathcal{V}(f)$ there are continuation p_φ and q_φ of p, q in $\Lambda_\varphi(U)$. Suppose that index(p)<index(q), and thus dim$W^s(p, g)$+ dim$W^u(q, g) <$ dim(M). Since Kupka-Smale diffeomorphisms are dense in Diff$^1(M)$ we choose a Kupka-Smale diffeomorphism $\varphi \in \mathcal{V}(f)$. Then $W^s(p_\varphi, \varphi) \bigcap W^u(q_\varphi, \varphi) = \emptyset$. On the other hand $\varphi \in \mathcal{U}(f)$, and $\varphi|\Lambda_\varphi(U)$ satisfies the ergodic shadowing property, thus proposition 3.4 shows that

$$W^s(p_\varphi, \varphi) \bigcap W^u(q_\varphi, \varphi) \neq \emptyset.$$

This is a contradiction.

Lemma 3.6 *(Frankś Lemma) Let $\mathcal{U}(f)$ be a given C^1-neighborhood of f. Then there exists $\epsilon > 0$ and a C^1-neighborhood $\mathcal{U}_0(f) \subseteq \mathcal{U}(f)$ of f such that for given $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, ..., x_N\}$, a neighborhood U of $\{x_1, x_2, ..., x_N\}$ and linear maps $L_i : T_{x_i}M \to T_{g_i(x)}M$ satisfying $||L_i - D_{x_i}g|| \leq \epsilon$, for all $1 \leq i \leq N$, there exists $\tilde{g} \in U(f)$ such that $\tilde{g}(x) = g(x)$ if $x \in \{x_1, x_2, ..., x_N\} \cup (M \setminus U)$ and $D_{x_i}\tilde{g} = L_i$ for all $1 \leq i \leq N$.*

Proof. see the proofs of lemma 1.1 in [2] or lemma II.2 in [6].

Let Λ be a closed f-invariant set, we have the following lemma:

Lemma 3.7 *Suppose that Λ is C^1-stable ergodic shadowable with respect to the neighborhood U of Λ and C^1-neighborhood $\mathcal{U}(f)$ of f. Then there exists a C^1-neighborhood $\mathcal{V}(f) \subseteq \mathcal{U}(f)$ of f such that every periodic point of $g \in \mathcal{V}(f)$ in Λ_g is hyperbolic, where $\Lambda_g = \cap_{n\in\mathbb{Z}} g^n(U)$.*

Proof. Let $\epsilon > 0$ and $\mathcal{U}_0(f) \subseteq \mathcal{U}(f)$ be as in $Frank\grave{s}$ Lemma. Suppose that there exists a non-hyperbolic periodic point $q \in \Lambda_g$ for some $g \in \mathcal{U}_0(f)$. Since Λ is locally maximal, reducing $\mathcal{U}(f)$ if necessary, we may assume that q is contained in the interior of U. To simplify the notations we let $g(q) = q$. Using $Frank\grave{s}$ Lemma we can construct $g_1 \in \mathcal{U}_0(f)$ C^1 -close to g possessing a g_1-invariant normally hyperbolic small arc $I_q \subseteq U$ central at q (resp. a g_1-invariant normally hyperbolic small circle $C_q \subseteq U$ with a small diameter centered at q) such that $g_1^k|I_q{=}id$ for some $k > 0$ (resp. $g_1|C_q$ is conjugate to an irrational rotation map) if λ is real (resp. complex) see [3] . We know that both I_q and C_q are contained in Λ_{g_1}. But g_1 does not have the ergodic shadowing property on I_q since $g_1^k|I_q$ is the identity. Moreover, if $g_1|C_q$ is conjugated to an irrational rotation map, then we can see that C_q is not ergodic shadowable for g_1.

Consider the locally maximal invariant set $\Lambda_f(U)$ of f in U, $\Lambda_f(U)$ is robustly transitive if there exist neighborhood $\mathcal{U}(f)$ of f such that $\Lambda_g(U)$ is transitive for all $g \in \mathcal{U}(f)$. Since $f|\Lambda_f(U)$ has ergodic shadowing property then $f|\Lambda_f(U)$ is topologically transitive. Thus if Λ has C^1- stable ergodic shadowing property then Λ is robustly transitive.

For proving theorem 3.3 We will use the following result due to Ma$\hat{n}$e [6].

Theorem 3.8 *Let $\Lambda_f(U)$ be robustly transitive, then the following conditions are equivalent:*

(1) there is a C^1- neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$ all periodic points of $\Lambda_g(U)$ are hyperbolic and has the same index;

(2) there is a C^1- neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $\Lambda_g(U)$ is hyperbolic.

Proof of theorem 3.3. Suppose that a close f-invariant set Λ has C^1-stable ergodic shadowing property, and let $\mathcal{U}(f)$ be as befor. To get the conclusion, using theorem 3.8 it is enough to show that for any $g \in \mathcal{U}(f)$ all periodic points of $\Lambda_g(U)$ are hyperbolic and has the same index. Which is clear from lemma 3.7 and lemma 3.5.

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