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Stability of Quadratic Functional Equations in 2-Banach Space

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Abstract

In this paper, we investigate the Hyers-Ulam stability of the functional equation $f(2x+y) - f(x+2y) = 3f(x) - 3f(y)$ *in 2-Banach space.*

Keywords: *Hyers-Ulam stability, 2-Banach space, Quadratic functional equation.*

1 Introduction

Stability of for a function from a normed space to a Banach space has been studied by Hyers [4]. Skof [12] has proved Hyers-Ulam stability of the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1}$$

He has proved that for a function $f : X \longrightarrow Y$, a function between normed space X to Banach space Y satisfying

$$\|f(x+y) + f(x-y) = 2f(x) + 2f(y)\| \leq \delta$$

for each $x, y \in X$ and $\delta > 0$, there exists a unique quadratic function
$Q : X \longrightarrow Y$ such that

$$\|f(x) - Q(x)\| < \frac{\delta}{2}$$

The quadratic function $f(x) = cx^2$ satisfies the functional equation (1) and therefore Equation (1) is called the quadratic functional equation. Every solution of Equation (1) is said to be a quadratic mapping.

In fact several authors have studied the stability of different types of functional equations for functions from normed space to Banach space. (see [1, 2, 5, 6, 7, 8, 9, 10]).

Our aim is to study the Hyers-Ulam stability of the functional equation

$$f(2x+y) - f(x+2y) = 3f(x) - 3f(y) \tag{2}$$

introduced by [15], for a function from 2-normed space (normed space) to 2-Banach space.

Theorem 1.1 *[15] Let X and Y be real vector spaces, and let $f : X \longrightarrow Y$ be a function satisfies (2) if and only if $f(x) = B(x,x) + C$, for some symmetric bi-additive function $B : X \times X \longrightarrow Y$, for some C in Y. Therefore every solution f of functional equation (2) with $f(0) = 0$ is also a quadratic function.*

In the 1960s, S. Gähler [3] introduced the concept of 2-normed spaces. We first introduce 2-normed space and topology on it.

Definition 1.2 *Let X be a linear space over $\mathbb{R}$ with $\dim X > 1$ and let $\|\cdot,\cdot\| : X \times X \longrightarrow \mathbb{R}$ be a function satisfying the following properties:*

1. $\|x, y\| = 0$ *if and only if x and y are linearly dependent,*
2. $\|x, y\| = \|y, x\|$,
3. $\|ax, y\| = |a|\|x, y\|$,
4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$

for each $x, y, z \in X$ and $a \in \mathbb{R}$. Then the function $\|\cdot,\cdot\|$ is called a 2-norm on X and $(X, \|\cdot,\cdot\|)$ is called a 2-normed space.

We introduce a basic property of 2-normed spaces as follows. Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space, $x \in X$ and $\|x, y\| = 0$ for each $y \in X$. Suppose $x \neq 0$, since $\dim X > 1$, choose $y \in X$ such that $\{x, y\}$ is linearly independent so we have $\|x, y\| \neq 0$, which is a contradiction. Therefore, we have the following lemma.

Lemma 1.3 *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. If $x \in X$ and $\|x, y\| = 0$, for each $y \in X$, then $x = 0$.*

Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space. For $x, z \in X$, let $p_z(x) = \|x, z\|$, $x \in X$. Then for each $z \in X$, p_z is a real-valued function on X such that $p_z(x) = \|x, z\| \geq 0$, $p_z(\alpha x) = |\alpha|\|x, z\| = |\alpha| p_z(x)$ and $p_z(x + y) = \|x + y, z\| = \|z, x + y\| \leq \|z, x\| + \|z, y\| = \|x, z\| + \|y, z\| = p_z(x) + p_z(y)$, for each $\alpha \in \mathbb{R}$ and all $x, y \in X$. Thus p_z is a a semi-norm for each $z \in X$.

For $x \in X$, let $\|x, z\| = 0$, for each $z \in X$. By Lemma 1.3, $x = 0$. Thus for $0 \neq x \in X$, there is $z \in X$ such that $p_z(x) = \|x, z\| \neq 0$. Hence the family $\{p_z(x) : z \in X\}$ is a separating family of semi-norms.

Let $x_0 \in X$, for $\varepsilon > 0$, $z \in X$, let $U_{z,\varepsilon}(x_0) := \{x \in X : p_z(x - x_0) < \varepsilon\} = \{x \in X : \|x - x_0, z\| < \varepsilon\}$. Let $S(x_0) := \{U_{z,\varepsilon}(x_0) : \varepsilon > 0, z \in X\}$ and $\beta(x_0) := \{\cap\mathcal{F} : \mathcal{F}$ is a finite subcollection of $S(x_0)\}$. Define a topology τ on X by saying that a set U is open if for every $x \in U$, there is some $N \in \beta(x)$ such that $N \subset U$. That is, τ is the topology on X that has subbase $\{U_{z,\varepsilon}(x_0) : \varepsilon > 0, x_0 \in X, z \in X\}$. The topology τ on X makes X a topological vector space. Since for $x \in X$ collection $\beta(x)$ is a local base whose members are convex, X is locally convex.

In the 1960s, S. Gähler and A. White [14] introduced the concept of 2-Banach spaces.

Definition 1.4 *A sequence $\{x_n\}$ in a 2-normed space X is called a 2-Cauchy sequence if*

$$\lim_{m,n\to\infty} \|x_n - x_m, x\| = 0$$

for each $x \in X$.

Definition 1.5 *A sequence $\{x_n\}$ in a 2-normed space X is called a 2-convergent sequence if there is an $x \in X$ such that*

$$\lim_{n\to\infty} \|x_n - x, y\| = 0$$

for each $y \in X$. If $\{x_n\}$ converges to x, we write $\lim_{n\to\infty} x_n = x$.

Definition 1.6 *We say that a 2-normed space $(X, \|\cdot,\cdot\|)$ is a 2-Banach space if every 2-Cauchy sequence in X is 2-convergent in X.*

By using (2) and (4) of Definition 1.2 one can see that $\|\cdot,\cdot\|$ is continuous in each component. More precisely for a convergent sequence $\{x_n\}$ in a 2-normed space X,

$$\lim_{n\to\infty} \|x_n, y\| = \left\| \lim_{n\to\infty} x_n, y \right\|$$

for each $y \in X$.

2 Stability of a Functional Equation for Functions $f : (X, \| \cdot \|) \longrightarrow (X, \|\cdot, \cdot\|)$

Throughout this section, consider X a real normed linear space. We also consider that there is a 2-norm on X which makes $(X, \|\cdot, \cdot\|)$ a 2-Banach space. For a function $f : (X, \| \cdot \|) \longrightarrow (X, \|\cdot, \cdot\|)$, define $D_f : X \times X \longrightarrow X$ by

$$D_f(x, y) = f(2x + y) - f(x + 2y) - 3f(x) + 3f(y)$$

for each $x, y, \in X$.

Theorem 2.1 *Let $\varepsilon \geq 0, 0 < p, q < 2, r > 0$. If $f : X \longrightarrow X$ is a function such that*

$$\|D_f(x, y), z\| \leq \varepsilon(\|x\|^p + \|y\|^q)\|z\|^r \tag{3}$$

for each $x, y, z \in X$. Then there exists a unique quadratic function $Q : X \longrightarrow X$ satisfying (2) *and*

$$\|f(x) - Q(x) - f(0), z\| \leq \frac{\varepsilon\|x\|^p\|z\|^r}{4 - 2^p} \tag{4}$$

for each $x, z \in X$.

Proof 2.1 *Let $g : X \longrightarrow X$ be a function defined by $g(x) = f(x) - f(0)$, for each $x \in X$. Then $g(0) = 0$. Also*

$$\begin{aligned} \|D_g(x, y), z\| &= \|g(2x + y) - g(x + 2y) - 3g(x) + 3g(y), z\| \\ &\leq \varepsilon(\|x\|^p + \|y\|^q)\|z\|^r \end{aligned} \tag{5}$$

for each $x, z \in X$. Putting $y = 0$ in (5)*, we get*

$$\|g(2x) - 4g(x), z\| \leq \varepsilon\|x\|^p\|z\|^r \tag{6}$$

for each $x, z \in X$. Therefore

$$\left\|g(x) - \frac{1}{4}g(2x), z\right\| \leq \frac{\varepsilon}{4}\|x\|^p\|z\|^r \tag{7}$$

for each $x, z \in X$. Replacing x by $2x$ in (7)*, we get*

$$\left\|g(2x) - \frac{1}{4}g(4x), z\right\| \leq \frac{\varepsilon 2^p}{4}\|x\|^p\|z\|^r \tag{8}$$

for each $x, z \in X$. *By* (7) *and* (8), *we get*

$$\begin{aligned}\left\|g(x)-\frac{1}{16}g(4x),z\right\| &\le \left\|g(x)-\frac{1}{4}g(2x),z\right\|+\left\|\frac{1}{4}g(2x)-\frac{1}{16}g(4x),z\right\| \\ &\le \frac{\varepsilon}{4}\|x\|^p\|z\|^r+\frac{\varepsilon}{4}\frac{2^p}{4}\|x\|^p\|z\|^r \\ &= \frac{\varepsilon\|x\|^p\|z\|^r}{4}\left[1+\frac{2^p}{4}\right]\end{aligned}$$

for each $x, z \in X$. *By using induction on* n, *we get*

$$\begin{aligned}\left\|g(x)-\frac{1}{4^n}g(2^nx),z\right\| &\le \frac{\varepsilon\|x\|^p\|z\|^r}{4}\sum_{j=0}^{n-1}\frac{2^{pj}}{4^j} \\ &= \frac{\varepsilon\|x\|^p\|z\|^r}{4}\left[\frac{1-2^{(p-2)n}}{1-2^{p-2}}\right] \qquad (9)\end{aligned}$$

for each $x, z \in X$. *For* $m, n \in \mathbb{N}$, *for* $x \in X$

$$\begin{aligned}\left\|\frac{1}{4^m}g(2^mx)-\frac{1}{4^n}g(2^nx),z\right\| &= \left\|\frac{1}{4^{m+n-n}}g(2^{m+n-n}x)-\frac{1}{4^n}g(2^nx),z\right\| \\ &= \frac{1}{4^n}\left\|\frac{1}{4^{m-n}}g(2^{m-n}\cdot 2^nx)-g(2^nx),z\right\| \\ &\le \frac{\varepsilon\|2^nx\|^p\|z\|^r}{4\cdot 4^n}\sum_{j=0}^{m-n-1}2^{(p-2)j} \\ &= \frac{\varepsilon\|x\|^p\|z\|^r}{4}\sum_{j=0}^{m-n-1}2^{(p-2)(n+j)} \\ &= \frac{\varepsilon\|x\|^p\|z\|^r}{4}\frac{2^{(p-2)n}\left(1-2^{(p-2)(m-n)}\right)}{1-2^{p-2}} \\ &\longrightarrow 0 \ \text{as } m,n\to\infty\end{aligned}$$

for each $z \in X$. *Therefore,* $\{\frac{1}{4^n}g(2^nx)\}$ *is a* 2*-Cauchy sequence in* X, *for each* $x \in X$. *Since* X *is a* 2*-Banach space,* $\{\frac{1}{4^n}g(2^nx)\}$ 2*-converges, for each* $x \in X$. *Define the function* $Q : X \longrightarrow X$ *as*

$$Q(x) = \lim_{n\to\infty}\frac{1}{4^n}g(2^nx)$$

for each $x \in X$. *Now, from* (9)

$$\lim_{n\to\infty}\left\|g(x)-\frac{1}{4^n}g(2^nx),z\right\| \le \frac{\varepsilon\|x\|^p\|z\|^r}{4}\frac{1}{1-2^{p-2}}$$

for each $x, z \in X$*. Therefore*

$$\|f(x) - Q(x) - f(0), z\| \leq \frac{\varepsilon\|x\|^p\|z\|^r}{4-2^p}$$

for each $x, z \in X$*. Next we show that* Q *satisfies* (2)*. For* $x \in X$

$$\begin{aligned}
\|D_Q(x,y), z\| &= \lim_{n\to\infty} \frac{1}{4^n}\|D_g(2^n x, 2^n y), z\| \\
&= \lim_{n\to\infty} \frac{\varepsilon}{4^n}(\|2^n x\|^p + \|2^n y\|^q)\|z\|^r \\
&= \lim_{n\to\infty} \varepsilon\left[2^{(p-2)n}\|x\|^p + 2^{(q-2)n}\|y\|^q\right]\|z\|^r \\
&= 0
\end{aligned}$$

for each $z \in X$*. Therefore* $\|D_Q(x,y), z\| = 0$*, for each* $z \in X$*. So we get* $D_Q(x,y) = 0$*. Next we prove the uniqueness of* Q*. Let* Q' *be another quadratic function satisfying* (2) *and* (4)*. Since* Q *and* Q' *are quadratic,* $Q(2^n x) = 4^n Q(x)$*,* $Q'(2^n x) = 4^n Q'(x)$*, for each* $x \in X$*. Now for* $x \in X$

$$\begin{aligned}
\|Q(x) - Q'(x), z\| &= \frac{1}{4^n}\|Q(2^n x) - Q'(2^n x), z\| \\
&\leq \frac{1}{4^n}[\|Q(2^n x) - g(2^n x), z\| + \|g(2^n x) - Q'(2^n x), z\|] \\
&\leq \frac{1}{4^n}\frac{2\varepsilon\|2^n x\|^p\|z\|^r}{4-2^p} \\
&= 2^{(p-2)n}\frac{2\varepsilon}{4-2^p}\|x\|^p\|z\|^r \\
&\longrightarrow 0 \text{ as } n \to \infty
\end{aligned}$$

for each $z \in X$*. Therefore* $\|Q(x) - Q'(x), z\| = 0$*, for each* $z \in X$*. Therefore* $Q(x) = Q'(x)$*, for each* $x \in X$*.*

Theorem 2.2 *Let* $\varepsilon \geq 0, p, q > 2, r > 0$*. If* $f : X \longrightarrow X$ *is a function such that*

$$\|D_f(x,y), z\| \leq \varepsilon(\|x\|^p + \|y\|^q)\|z\|^r \tag{10}$$

for each $x, y, z \in X$*. Then there exists a unique quadratic function* $Q : X \longrightarrow X$ *satisfying* (2) *and*

$$\|f(x) - Q(x) - f(0), z\| \leq \frac{\varepsilon\|x\|^p\|z\|^r}{2^p-4} \tag{11}$$

for each $x, z \in X$*.*

Proof 2.2 *By* (6) *of Theorem* 2.1*, we have*

$$\|g(2x) - 4g(x), z\| \leq \varepsilon \|x\|^p \|z\|^r \tag{12}$$

for each $x, z \in X$*. Replacing* x *by* $\frac{x}{2}$ *in* (12)*, we get*

$$\left\|g(x) - 4g\left(\frac{x}{2}\right), z\right\| \leq \varepsilon 2^{-p} \|x\|^p \|z\|^r \tag{13}$$

for each $x, z \in X$*. Replacing* x *by* $\frac{x}{2}$ *in* (13)*, we get*

$$\left\|g\left(\frac{x}{2}\right) - 4g\left(\frac{x}{4}\right), z\right\| \leq \varepsilon 2^{-2p} \|x\|^p \|z\|^r \tag{14}$$

for each $x, z \in X$*. Combining* (13) *and* (14)*, we get*

$$\begin{aligned}\|g(x) - 16g(\frac{x}{4}), z\| &\leq \left\|g(x) - 4g\left(\frac{x}{2}\right), z\right\| + \left\|4g\left(\frac{x}{2}\right) - 16g\left(\frac{x}{4}\right), z\right\| \\ &\leq \varepsilon 2^{-p} \|x\|^p \|z\|^r + 4\varepsilon 2^{-2p} \|x\|^p \|z\|^r \\ &= \varepsilon \|x\|^p \|z\|^r [2^{-p} + 2^{-p} \cdot 4]\end{aligned}$$

for each $x, z \in X$*. By using induction on* n*, we have*

$$\begin{aligned}\left\|g(x) - 4^n g\left(\frac{x}{2^n}\right), z\right\| &\leq \varepsilon \|x\|^p \|z\|^r \sum_{j=0}^{n-1} 4^j 2^{p(-j-1)} \\ &= \varepsilon \|x\|^p \|z\|^r \sum_{j=0}^{n-1} 2^{(-p+2)j-p} \\ &= \varepsilon \|x\|^p \|z\|^r \left(\frac{2^{-p}(1 - 2^{(2-p)n})}{1 - 2^{2-p}}\right)\end{aligned} \tag{15}$$

for each $x, z \in X$*. For* $m, n \in \mathbb{N}$ *and for* $x \in X$

$$\begin{aligned}\left\|4^m g\left(\frac{x}{2^m}\right) - 4^n g\left(\frac{x}{2^n}\right), z\right\| &= \left\|4^{m+n-n} g\left(\frac{x}{2^{m+n-n}}\right) - 4^n g\left(\frac{x}{2^n}\right), z\right\| \\ &= 4^n \left\|4^{m-n} g\left(\frac{x}{2^{m-n} \cdot 2^n}\right) - g\left(\frac{x}{2^n}\right), z\right\| \\ &\leq 4^n \cdot \varepsilon \left\|\frac{x}{2^n}\right\|^p \|z\|^r \sum_{j=0}^{m-n-1} 2^{(-p+2)j-p} \\ &= \varepsilon \|x\|^p \|z\|^r \sum_{j=0}^{m-n-1} 2^{(2-p)(n+j)-p} \\ &= \varepsilon \|x\|^p \|z\|^r \left[\frac{2^{(-p+2)n-p}\left(1 - 2^{(-p+2)n}\right)}{1 - 2^{-p+2}}\right] \\ &\longrightarrow 0 \text{ as } n \to \infty\end{aligned}$$

for each $z \in X$. Therefore $\{4^n f(\frac{x}{2^n})\}$ is a 2-Cauchy sequence in X, for each $x \in X$. Since X is a 2-Banach space, the sequence $\{4^n f(\frac{x}{2^n})\}$ 2-converges, for each $x \in X$. Define $Q : X \longrightarrow X$ as

$$Q(x) := \lim_{n\to\infty} 4^n f\Big(\frac{x}{2^n}\Big)$$

for each $x \in X$. Now from (15),

$$\lim_{n\to\infty} \Big\| g(x) - 4^n g\Big(\frac{x}{2^n}\Big), z\Big\| \leq \varepsilon \|x\|^p \|z\|^r \frac{2^{-p}}{1 - 2^{2-p}}$$

for each $x, z \in X$. Therefore

$$\|f(x) - Q(x) - f(0), z\| \leq \frac{\varepsilon \|x\|^p \|z\|^r}{2^p - 4}$$

for each $x, z \in X$. The further part of the proof is similar to that of the proof of Theorem 2.1.

3 Stability of a Functional Equation for Function $f : (X, \|\cdot,\cdot\|) \longrightarrow (X, \|\cdot,\cdot\|)$

In this section we study similar problems which we have studied in section 2 for functions $f : X \longrightarrow X$, where $(X, \|\cdot,\cdot\|)$ is a 2-Banach space.

Theorem 3.1 *Let $\varepsilon \geq 0, 0 < p, q < 2$. If $f : X \longrightarrow X$ is a function such that*

$$\|D_f(x, y), z\| \leq \varepsilon(\|x, z\|^p + \|y, z\|^q) \tag{16}$$

for each $x, y, z \in X$. Then there exists a unique quadratic function $Q : X \longrightarrow X$ satisfying (2) *and*

$$\|f(x) - Q(x) - f(0), z\| \leq \frac{\varepsilon \|x, z\|^p}{4 - 2^p} \tag{17}$$

for each $x, z \in X$.

Proof 3.1 *Let $g : X \longrightarrow X$ be a function defined by $g(x) = f(x) - f(0)$, for each $x \in X$. Then $g(0) = 0$. Also*

$$\begin{aligned}\|D_g(x, y), z\| &= \|g(2x + y) - g(x + 2y) - 3g(x) + 3g(y), z\| \\ &\leq \varepsilon(\|x, z\|^p + \|y, z\|^q)\end{aligned} \tag{18}$$

for each $x, z \in X$. Putting $y = 0$ in (18)*, we get*

$$\|g(2x) - 4g(x), z\| \leq \varepsilon \|x, z\|^p \tag{19}$$

for each $x, z \in X$. *Therefore*

$$\left\| g(x) - \frac{1}{4}g(2x), z \right\| \leq \frac{\varepsilon}{4}\|x, z\|^p \tag{20}$$

for each $x, z \in X$. *Replacing* x *by* $2x$ *in* (20), *we get*

$$\left\| g(2x) - \frac{1}{4}g(4x), z \right\| \leq \frac{\varepsilon 2^p}{4}\|x, z\|^p \tag{21}$$

for each $x, z \in X$. *By* (20) *and* (21), *we get*

$$\begin{aligned}
\left\| g(x) - \frac{1}{16}g(4x), z \right\| &\leq \left\| g(x) - \frac{1}{4}g(2x), z \right\| + \left\| \frac{1}{4}g(2x) - \frac{1}{16}g(4x), z \right\| \\
&\leq \frac{\varepsilon}{4}\|x\|^p\|z\|^r + \frac{\varepsilon}{4}\frac{2^p}{4}\|x, z\|^p \\
&= \frac{\varepsilon\|x, z\|^p}{4}\left[1 + \frac{2^p}{4}\right]
\end{aligned}$$

for each $x, z \in X$. *By using induction on* n, *we get*

$$\begin{aligned}
\left\| g(x) - \frac{1}{4^n}g(2^n x), z \right\| &\leq \frac{\varepsilon\|x, z\|^p}{4}\sum_{j=0}^{n-1}\frac{2^{pj}}{4^j} \\
&= \frac{\varepsilon\|x, z\|^p}{4}\sum_{j=0}^{n-1}2^{(p-2)j} \\
&= \frac{\varepsilon\|x, z\|^p}{4}\left[\frac{1 - 2^{(p-2)n}}{1 - 2^{p-2}}\right]
\end{aligned} \tag{22}$$

for each $x, z \in X$. *For* $m, n \in \mathbb{N}$ *for* $x \in X$

$$\begin{aligned}
\left\| \frac{1}{4^m}g(2^m x) - \frac{1}{4^n}g(2^n x), z \right\| &= \left\| \frac{1}{4^{m+n-n}}g(2^{m+n-n}x) - \frac{1}{4^n}g(2^n x), z \right\| \\
&= \frac{1}{4^n}\left\| \frac{1}{4^{m-n}}g(2^{m-n} \cdot 2^n x) - g(2^n x), z \right\| \\
&\leq \frac{\varepsilon\|2^n x, z\|^p}{4 \cdot 4^n}\sum_{j=0}^{m-n-1}2^{(p-2)j} \\
&= \frac{\varepsilon\|x, z\|^p}{4}\sum_{j=0}^{m-n-1}2^{(p-2)(n+j)} \\
&= \frac{\varepsilon\|x, z\|^p}{4}\frac{2^{(p-2)n}\left(1 - 2^{(p-2)(m-n)}\right)}{1 - 2^{p-2}} \\
&\longrightarrow 0 \text{ as } m, n \to \infty
\end{aligned}$$

for each $z \in X$. *Therefore,* $\{\frac{1}{4^n} g(2^n x)\}$ *is a 2-Cauchy sequence in* X, *for each* $x \in X$. *Since* X *is a 2-Banach space,* $\{\frac{1}{4^n} g(2^n x)\}$ *2-converges, for each* $x \in X$. *Define the function* $Q : X \longrightarrow X$ *as*

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} g(2^n x)$$

for each $x \in X$. *Now, by* (22)

$$\lim_{n\to\infty} \left\| g(x) - \frac{1}{4^n} g(2^n x), z \right\| \leq \frac{\varepsilon \|x, z\|^p}{4} \frac{1}{1 - 2^{p-2}}$$

for each $x, z \in X$. *Therefore*

$$\|f(x) - Q(x) - f(0), z\| \leq \frac{\varepsilon \|x, z\|^p}{4 - 2^p}$$

for each $x, z \in X$. *Next we show that* Q *satisfies* (2). *For* $x \in X$

$$\begin{aligned}
\|D_Q(x, y), z\| &= \lim_{n\to\infty} \frac{1}{4^n} \|D_g(2^n x, 2^n y), z\| \\
&= \lim_{n\to\infty} \frac{\varepsilon}{4^n} (\|2^n x, z\|^p + \|2^n y, z\|^q) \\
&= \lim_{n\to\infty} \varepsilon \left[2^{(p-2)n} \|x, z\|^p + 2^{(q-2)n} \|y, z\|^q\right] \\
&= 0
\end{aligned}$$

for each $z \in X$. *Therefore* $\|D_Q(x, y), z\| = 0$, *for each* $z \in X$. *So we get* $D_Q(x, y) = 0$. *Next we prove the uniqueness of* Q. *Let* Q' *be another quadratic function satisfying* (2) *and* (17). *Since* Q *and* Q' *are quadratic,* $Q(2^n x) = 4^n Q(x)$, $Q'(2^n x) = 4^n Q'(x)$, *for each* $x \in X$. *Now for* $x \in X$

$$\begin{aligned}
\|Q(x) - Q'(x), z\| &= \frac{1}{4^n} \|Q(2^n x) - Q'(2^n x), z\| \\
&\leq \frac{1}{4^n} [\|Q(2^n x) - g(2^n x), z\| + \|g(2^n x) - Q'(2^n x), z\|] \\
&\leq \frac{1}{4^n} \frac{2\varepsilon \|2^n x, z\|^p}{4 - 2^p} \\
&= 2^{(p-2)n} \frac{2\varepsilon \ \|x, z\|^p}{4 - 2^p} \\
&\longrightarrow 0 \ \textit{as} \ n \to \infty
\end{aligned}$$

for each $z \in X$. *Therefore* $\|Q(x) - Q'(x), z\| = 0$, *for each* $z \in X$. *Therefore* $Q(x) = Q'(x)$, *for each* $x \in X$.

Theorem 3.2 *Let $\varepsilon \geq 0, p, q > 2, r > 0$. If $f : X \longrightarrow X$ is a function such that*

$$\|D_f(x, y), z\| \leq \varepsilon(\|x, z\|^p + \|y, z\|^q) \tag{23}$$

for each $x, y, z \in X$. Then there exists a unique quadratic function $Q : X \longrightarrow X$ satisfying (2) *and*

$$\|f(x) - Q(x) - f(0), z\| \leq \frac{\varepsilon\|x, z\|^p}{2^p - 4} \tag{24}$$

for each $x, z \in X$.

Proof 3.2 *By* (19) *of Theorem* 3.1*, we have*

$$\|g(2x) - 4g(x), z\| \leq \varepsilon\|x, z\|^p \tag{25}$$

for each $x, z \in X$. Replacing x by $\frac{x}{2}$ in (25)*, we get*

$$\left\|g(x) - 4g\left(\frac{x}{2}\right), z\right\| \leq \varepsilon 2^{-p}\|x, z\|^p \tag{26}$$

for each $x, z \in X$. Replacing x by $\frac{x}{2}$ in (26)*, we get*

$$\left\|g\left(\frac{x}{2}\right) - 4g\left(\frac{x}{4}\right), z\right\| \leq \varepsilon 2^{-2p}\|x, z\|^p \tag{27}$$

for each $x, z \in X$. Combining (26) *and* (27)*, we get*

$$\begin{aligned}\|g(x) - 16g\left(\frac{x}{4}\right), z\| &\leq \left\|g(x) - 4g\left(\frac{x}{2}\right), z\right\| + \left\|4g\left(\frac{x}{2}\right) - 16g\left(\frac{x}{4}\right), z\right\| \\ &\leq \varepsilon 2^{-p}\|x, z\|^p + 4\varepsilon 2^{-2p}\|x, z\|^p \\ &= \varepsilon\|x, z\|^p[2^{-p} + 2^{-p} \cdot 4]\end{aligned}$$

for each $x, z \in X$. By using induction on n, we have

$$\begin{aligned}\left\|g(x) - 4^n g\left(\frac{x}{2^n}\right), z\right\| &\leq \varepsilon\|x, z\|^p \sum_{j=0}^{n-1} 4^j 2^{p(-j-1)} \\ &= \varepsilon\|x, z\|^p \sum_{j=0}^{n-1} 2^{(-p+2)j-p} \\ &= \varepsilon\|x, z\|^p \left(\frac{2^{-p}(1 - 2^{(2-p)n})}{1 - 2^{2-p}}\right)\end{aligned} \tag{28}$$

for each $x, z \in X$*. For* $m, n \in \mathbb{N}$*, For* $x \in X$

$$\begin{aligned}\Big\|4^m g\Big(\frac{x}{2^m}\Big) - 4^n g\Big(\frac{x}{2^n}\Big), z\Big\| &= \Big\|4^{m+n-n} g\Big(\frac{x}{2^{m+n-n}}\Big) - 4^n g\Big(\frac{x}{2^n}\Big), z\Big\| \\ &= 4^n\Big\|4^{m-n} g\Big(\frac{x}{2^{m-n}\cdot 2^n}\Big) - g\Big(\frac{x}{2^n}\Big), z\Big\| \\ &\le 4^n \cdot \varepsilon \Big\|\frac{x}{2^n}\Big\|^p \|z\|^r \sum_{j=0}^{m-n-1} 2^{(-p+2)j-p} \\ &= \varepsilon \|x, z\|^p \sum_{j=0}^{m-n-1} 2^{(2-p)(n+j)-p} \\ &= \varepsilon \|x, z\|^p \Big[\frac{2^{(-p+2)n-p}\big(1 - 2^{(-p+2)n}\big)}{1 - 2^{-p+2}}\Big] \\ &\longrightarrow 0 \text{ as } n \to \infty\end{aligned}$$

for each $z \in X$*. Therefore* $\{4^n f(\frac{x}{2^n})\}$ *is a 2-Cauchy sequence in* X*, for each* $x \in X$*. Since* X *is a 2-Banach space, the sequence* $\{4^n f(\frac{x}{2^n})\}$ *2-converges, for each* $x \in X$*. Define* $Q : X \longrightarrow X$ *as*

$$Q(x) := \lim_{n\to\infty} 4^n f\Big(\frac{x}{2^n}\Big)$$

for each $x \in X$*. Now, by* (28)

$$\lim_{n\to\infty} \Big\|g(x) - 4^n g\Big(\frac{x}{2^n}\Big), z\Big\| \le \varepsilon \|x, z\|^p \frac{2^{-p}}{1 - 2^{2-p}}$$

for each $x, z \in X$*. Therefore*

$$\|f(x) - Q(x) - f(0), z\| \le \frac{\varepsilon \|x, z\|^p}{2^p - 4}$$

for each $x, z \in X$*. The further part of the proof is similar to that of the proof of Theorem* 3.1*.*

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