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Properties of β^*-Homeomorphisms in Topological Spaces

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Abstract

The concept of β^*-homeomorphisms is introduced and investigated by Palanimani [13] earlier. In the present paper we investigate some more properties of β^*-homeomorphisms and also investigate contra β^*-homeomorphisms.

Keywords: *β^*-closed, β^*-continuous, β^*-closed map, β^*-homeomorphism, contra β^*-irresolute and contra β^*-homeomorphism.*

1 Introduction

N. Levine [7] introduced semi-continuous function using semiopen sets. Balachandran, Sundram and Maki [2] introduced the concept of generalized continuous maps and gc-irresolute maps on topological spaces. Several authors ([1], [3], [4], [5], [9] and [10]) studying the concepts of generalizations of continuous maps. Maki [8] introduced g-homeomorphism and gc-homeomorphism in topological spaces. Recently Palanimani [13] defined β^*-closed map and β^*-homeomorphism and studied some of their properties.

Throughout the present paper, (X, τ), (Y, σ) and (Z, η) (or X, Y and Z) represent nonempty topological spaces on which no separation axioms are assumed unless otherwise mentioned. The closure and interior of a subset $A \subseteq X$ will be denoted by $Cl(A)$ and $Int(A)$, respectively. The present paper is a continuatuion of [13] due to one of the present authors, we investigate more properties of functions preserving β^*-closed sets. In section 2, we recall some definitions on functions and we need some properties on functions (c.f. Lemma 2.5 and Theorem 2.6). In section 3, for a topological space (X, τ), we introduce and invetigate groups of functions, say $\beta^*h(X;\ \tau)$ preserving β^*-closed sets respectively, they contain the homeomorphism group $h(X, \tau)$ as a subgroup (cf. Theorem 3.3). Moreover, these groups have an important property that they are one of topological invarients (Theorem 3.3). Using the concept of contra β^*-irresoluteness, In section 4, we construct more groups of functions, say $\beta^*h(X; \tau) \cup con\text{-}\beta^*h(X; \tau)$ for a topological space (X, τ); they contain the homeomorphism group $h(X, \tau)$ as a subgroup (cf. Theorem 4.4).

2 Preliminaries

We need the following definition, lemma and Theorem.

Definition 2.1 *A subset A of a topological space $(X,\ \tau)$ is called a*

(i) generalized closed (briefly, g-closed) [6] if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in $(X,\ \tau)$.

(ii) β^*-closed [11] if $Cl(Int(A)) \subseteq U$ whenever $A \subseteq U$ and U is g-open in $(X,\ \tau)$.

The complements of the above mentioned closed sets are their respective open sets.

Definition 2.2 *[12] Let $f : X \to Y$ from a topological space X into a topological space Y is called β^*-continuous if the inverse image of every closed set in Y is β^*-closed in X.*

Definition 2.3 *[13] Let $f : X \to Y$ from a topological space X into a topological space Y is called β^*-closed map if for each closed set F of X, $f(F)$ is β^*-closed in Y.*

Definition 2.4 *[12] A map $f : X \to Y$ from a topological space X into a topological space Y is called β^*-irreslute if the inverse image of every β^*-closed set in Y is β^*-closed in X.*

Lemma 2.5 *Let $f : (X,\ \tau) \to (Y,\ \sigma)$ and $g : (Y,\ \sigma) \to (Z,\ \zeta)$ be two functions between topological spaces.*

(i) If f and g are β^*-irreslote, then the composition $g \circ f$ is also β^*-irresolute.

(ii) The identity function $1_X : (X,\ \tau) \to (Y,\ \sigma)$ is β^*-irresolutre.

Proof: The proofs are obvious from definitions.

Theorem 2.6 *Let $f : (X,\ \tau) \to (Y,\ \sigma)$ be a function then every homeomorphism is β^*-continuous.*

Proof: Let f be a homeomorphism. Then, $f^{-1} : (Y,\ \sigma) \to (X,\ \tau)$ is also a homeomorphism. By definition, it is shown that $f = (f^{-1})^{-1}$ is continuous. By Theorem 3.2 in [12], it is shown that f is β^*-continuous.

3 More on Functions Preserving β^*-Closed Sets

Definition 3.1 *A bijection $f : (X,\ \tau) \to (Y,\ \sigma)$ is called β^*-homeomorphism [13] if f is both β^*-continuous and β^*-closed (or f^{-1} is β^*-continuous).*

For a topological space $(X,\ \tau)$, we introduce the following:
(1) $h(X;\ \tau) = \{f | f : (X,\ \tau) \to (X,\ \tau)$ is a homeomorphism $\}$.
(2) $\beta^* h(X;\ \tau) = \{f | f : (X,\ \tau) \to (X,\ \tau)$ is a β^*-homeomorphism $\}$.

Theorem 3.2 *For a topological spaces $(X,\ \tau)$ then $h(X;\ \tau) \subseteq \beta^* h(X;\ \tau)$.*

Proof: Let $f \in h(X;\ \tau)$. Then by definition f and f^{-1} are continuous. By Theorem 3.2 in [12], it is shown that f and f^{-1} are β^*-continuous and so f is β^*-homeomorphism, i.e., $f \in \beta^* h(X;\ \tau)$.

Theorem 3.3 *Let $(X,\ \tau)$ be a topological space. Then, we have the following properties.*

(i) The collection $\beta^* h(X;\ \tau)$ forms a group under the composition of functions.

(ii) The homeomorphism group $h(X;\ \tau)$ is a subgroup of the group $\beta^* h(X;\ \tau)$.

Proof: (i) A binary operation $\eta_X : \beta^* h(X;\ \tau) \times \beta^* h(X;\ \tau) \to \beta^* h(X;\ \tau)$ is well defined by $\eta_X(a,\ b) = b \circ a$, where $b \circ a : X \to X$ is the composite function of the functions a and b such that $(b \circ a)(x) = b(a(x))$ for every point $x \in X$. Indeed, by Lemma 2.5 (i), it is shown that, for every β^*-homeomorphisms a and b, the composition $b \circ a$ is also β^*-homeomorphism. Namely, for every pair $(a,\ b) \in \beta^* h(X;\ \tau)$, $\eta_X(a,\ b) = b \circ a \in \beta^* h(X;\ \tau)$. Then, it is claimed that the binary operation $\eta_X : \beta^*h(X;\ \tau) \times \beta^*h(X;\ \tau) \to \beta^*h(X;\ \tau)$ satisfies the axiom of group. Namely, putting $a.b = \eta_X(a,\ b)$, the following properties hold $\beta^*h(X;\ \tau)$.

(1) $((a.b).c) = (a.(b.c))$ holds forevery elements $a,\ b,\ c \in \beta^*h(X;\ \tau)$;

(2) for all element $a \in \beta^*h(X;\ \tau)$, there exists an element $e \in \beta^*h(X;\ \tau)$ such that $a.e = e.a = a$ hold in $\beta^*h(X;\ \tau)$;

(3) for each element $a \in \beta^*h(X;\ \tau)$, there exists an element $a_1 \in \beta^*h(X;\ \tau)$ such that $a.a_1 = a_1.a = e$ hold in $\beta^*h(X;\ \tau)$.

Indeed, the proof of (1) is obvious; the proof of (2) is obtained by taking $e = 1_X$, where 1_X is the identity function on X and using Lemma 2.5 (ii); the proof of (3) is obtained by taking $a_1 = a^{-1}$ for each $a \in \beta^*h(X;\ \tau)$ and Definition 3.1, where a^{-1} is the inverse function of a. Therefore, by definition of groups, the pair $(\beta^*h(X;\ \tau),\ \eta_X)$ forms a group under the composition of functions. i.e., $\beta^*h(X;\ \tau)$ is a group.

(ii) It is obvious that $1_X : (X,\ \tau) \to (X,\ \tau)$ is a homeomorphism and so $h(X;\ \tau) \neq \phi$. It follows from Theorem 3.2 that $h(X;\ \tau) \subseteq \beta^*h(X;\ \tau)$. Let $a,\ b \in h(X;\ \tau)$. Then we have that $\eta_X(a,\ b^{-1}) = b^{-1} \circ a \in h(X;\ \tau)$, here $\eta_X : \beta^*h(X;\ \tau) \times \beta^*h(X;\ \tau) \to \beta^*h(X;\ \tau)$ is the binary operation (cf. Proof of Theorem 3.3 (i)). Therefore, the group $h(X;\ \tau)$ is a subgroup of $\beta^*h(X;\ \tau)$.

Theorem 3.4 *Let $(X,\ \tau)$ and $(Y,\ \sigma)$ be topological spaces. If $(X,\ \tau)$ and $(Y,\ \sigma)$ are homeomorphism, then there exist isomorphisms: i.e., $\beta^*h(X;\ \tau) \cong \beta^*h(Y;\ \sigma)$.*

Proof: It follows from assumption that there exists a homeomorphism, say $f : (X,\ \tau) \to (Y,\ \sigma)$. We define a function $f_* : \beta^*h(X,\ \tau) \to \beta^*h(Y,\ \sigma)$ by $f_*(a) = f \circ a \circ f^{-1}$ for every element $a \in \beta^*h(X,\ \tau)$; by Theorem 2.6 (or Theorem 3.2) and Lemma 2.5(i), the bijections $f \circ a \circ f^{-1}$ and $(f \circ a \circ f^{-1})^{-1}$ are β^*-closed and so f_* is well defined. The induced function f_* is a homeomoephism. Indeed, $f_*(\eta_X(a,\ b)) = f \circ b \circ f^{-1} \circ f \circ a \circ f^{-1} = (f_*(b)) \circ (f_*(a)) = \eta_X(f_*(a),\ f_*(a))$ hold. Obviously, f_* is bijective. Thus, we have (i), i.e., f_* is an isomorphism.

4 More on the Groups including the Homeomorphism Group $h(X;\ \tau)$ as Subgroup

Definition 4.1 *For a topological spaces $(X,\ \tau)$ and $(Y,\ \sigma)$, we define the following functions. A function $f : (X,\ \tau) \to (Y,\ \sigma)$ is said to be contra β^*-irresolute if $f^{-1}(V)$ is β^*-closed in $(X,\ \tau)$ for every β^*-open set V of $(Y,\ \sigma)$.*

For these we can immediately see the following lemma.

Lemma 4.2 *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two functions between topological spaces.*

(i) If f and g are contra β^*-irresolute, then the composition $g \circ f$ is also β^*-irresolute.

(ii) If f is β^*-irresolute (resp. contra β^*-irresolute) and g are contra β^*-irresolute (resp. β^*-irresolute), then the composition $g \circ f$ is contra β^*-irresolute.

Definition 4.3 *For a topological space $(X,\ \tau)$, we define the collection of functions con-$\beta^*h(X,\ \tau) = \{f | f : (X,\ \tau) \to (X,\ \tau)$ is a contra β^*-irresolute bijection and f^{-1} is contra β^*-irresolute $\}$.*

For a topological space (X, τ), we construct alternative groups, say $\beta^*h(X;\ \tau)\cup$ con-$\beta^*h(X;\ \tau)$.

Theorem 4.4 *Let $(X,\ \tau)$ be a topological space. Then, we have the following properties.*

(i) The union of two collections, $\beta^*h(X;\ \tau)\cup$ con-$\beta^*h(X;\ \tau)$, forms a group under the composition of functions.

(ii) The homeomorphism group $h(X;\ \tau)$ is a subgroup of $\beta^*h(X;\ \tau)\cup$ con-$\beta^*h(X;\ \tau)$.

Proof: (i) Let $B_X = \beta^*h(X;\ \tau)\cup$ con-$\beta^*h(X;\ \tau)$. A binary operation $w_X : B_X \times B_X \to B_X$ is well defined by $w_X(a,\ b) = b \circ a$, where $b \circ a : X \to X$ is the composite function of the functions a and b. Indeed, let $(a,\ b) \in B_X$; if $a \in \beta^*h(X;\ \tau)$ and $b \in$ con -$\beta^*h(X;\ \tau)$, then $b \circ a : (X,\ \tau) \to (X,\ \tau)$ a contra β^*-irresolute bijection and $(b \circ a)^{-1}$ is also contra β^*-irresolute and so $w_X(a,\ b) = b \circ a \in$ con-$\beta^*h(X;\ \tau) \subset B_X$ (cf. Lemma 4.2 (ii)) if $a \in \beta^*h(X;\ \tau)$ and $b \in \beta^*h(X;\ \tau)$ then $b \circ a : (X,\ \tau) \to (X,\ \tau)$ is a β^*-irresolute bijection and so $w_X(a,\ b) = b \circ a \in \beta^*h(X;\ \tau) \subseteq B_X$ (cf. Lemma 2.5 (i)), if $a \in$ con-$\beta^*h(X;\ \tau)$ and $b \in$ con-$\beta^*h(X;\ \tau)$, then $b \circ a : (X,\ \tau) \to (X,\ \tau)$ is a β^*-irresolute bijection and $(b \circ a)^{-1}$ is also β^*-irresolute and so $w_X(a,\ b) = b \circ a \in \beta^*h(X;\ \tau) \subset B_X$ (cf. Lemma 4.2(i)) if $a \in con\beta^*h(X;\ \tau)$ and $b \in \beta^*h(X;\ \tau)$ then $b \circ a : (X,\ \tau) \to (X,\ \tau)$ is a contra β^*-irresolute bijection and $(b \circ a)^{-1}$ is also β^*-irresolute and so $w_X(a,\ b) = b \circ a \in$ *con-*$\beta^*h(X;\ \tau) \subseteq B_X$ (cf. Lemma 4.2 (ii)). By the similar arguments of Theorem 3.3, it is claimed that the binary operation $w_X : B_X \times B_X \to B_X$ satifies the axiom of group; for the identy element e of B_X, $e = 1_X : (X,\ \tau) \to (X,\ \tau)$ (the identity function). Thus the pair $(B_X,\ w_X)$ forms a group under the composition of functions, i.e., $\beta^*h(X;\ \tau)\cup$ con-$\beta^*h(X;\ \tau)$ is a group.

(ii) By Theorem 3.3 (ii) above, it is shown that $h(X;\ \tau)$ is a subgroup of $\beta^*h(X;\ \tau)\cup$ con-$\beta^*h(X;\ \tau)$.

The groups of Theorem 4.4 are also invarient concepts under homeomorphisms between topological spaces (c.f. Theorem 3.4).

Theorem 4.5 *Let $(X,\ \tau)$ and $(Y,\ \sigma)$ be topological spaces. If $(X,\ \tau)$ and $(Y,\ \sigma)$ are homeomorphic, then there exist isomorphisms i.e., $\beta^*h(X;\ \tau)\cup$ con-$\beta^*h(X;\ \tau) \cong \beta^*h(Y;\ \sigma)\cup$ con-$\beta^*h(Y;\ \sigma)$.*

Proof: Let $f : (X,\tau) \to (Y,\sigma)$ be a homeomorphism. We put $B_X = \beta^*h(X;\ \tau)\cup$ con-$\beta^*h(X;\ \tau)$ (resp. $B_Y = \beta^*h(Y;\ \sigma)\cup$ con-$\beta^*h(Y;\ \sigma)$) for a topological space (X,τ) (resp. (Y,σ)). First we have a well defined function $f_* : B_X \to B_Y$ by $f_*(a) = f \circ a \circ f^{-1}$ for every element $a \in B_X$. Indeed by Theorem 2.6(ii) (or Theorem 3.2), f and f^{-1} are β^*-irresolute. By Lemma 2.5(i) and Lemma 4.2 (ii), the bijections $f \circ a \circ f^{-1}$ and $(f \circ a \circ f^{-1})^{-1}$ are β^*-irresolute or contra β^*-irresolute and so f_* is well defined. The induced function f_* is a homeomorphism. Indeed, $f_*(w_X(a,b)) = f \circ b \circ f^{-1} \circ f \circ a \circ f^{-1} = (f_*(b)) \circ (f_*(a)) = w_Y(f_*(a)) \circ (f_*(a))$ hold, $w_X : B_X \times B_X \to B_X$ and $w_Y : B_Y \times B_Y \to B_Y$ are the binary operations defined in Proof of Theorem 4.4(i). Obviously, f_* is bijective. Thus, we have the isomorphisms.

References

[1] I. Arockiarani, Studies on generalizations of generalized closed sets and maps in topological spaces, *Ph. D Thesis*, Bharathiar University, Coimbatore, (1997).

[2] K. Balachandran, P. Sundaram and H. Maki, On generalized continuous maps in topological spaces, *Mem. I ac Sci. Kochi Univ. Math.*, 12(1991), 5-13.

[3] R. Devi, K. Balachandran and H. Maki, Semi-generalized homeomorphisms and generalized semi-homeomorphisms in topological spaces, *Indian J. Pure Appl. Math.*, 20(1995), 271-284.

[4] R. Devi, K. Balachandran and H. Maki, On generalized α-continuous maps, *Far. East J. Math. Sci. Special*, 1(1997), 1-15.

[5] J. Dontchev and H. Maki, On θ-generalized closed sets, *Topology Atlass*, URL:http://www.Unipissing.ca/topology/p/a/b/a/08.htm.

[6] N. Levine, Generalized closed sets in topology, *Rend. Circ. Math. Palermo*, 19(2) (1970), 89-96.

[7] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, 70(1963), 36-41.

[8] H. Maki, R. Devi and K. Balachandran, Generalized α-closed sets in topology, *Bull. Fukuoka Univ. Ed. Part-III*, 42(1993), 13-21.

[9] A.S. Mashhour, I.A. Hasanein and S.N. El-Deep, α-continuous and α-open mappings, *Acta. Math. Hung*, 41(1983), 213-218.

[10] N. Palaniappan and K.C. Rao, Regular generalized closed sets, *Kyungpook Math. J.*, 33(1993), 211-219.

[11] P.G. Palanimani and R. Parimelazhagan, β^*-closed sets in topological spaces, *ROSR Journal of Mathematics*, 5(1) (2013), 47-50.

[12] ———-, β^*-continuous maps and pasting lemma in topological spaces, *International Journal of Scientific and Engineering Research*, 4(1) (2013), 1-4.

[13] ———-, β^*-homeomorphisms in topological spaces, *International Journal of Computer Applications*, 67(13) (2013), 1-3.