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An Approximate Solution of Instability Phenomenon in Heterogeneous Porous Media with Mean Pressure

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Abstract

In important instability phenomenon arising in secondary oil recovery process, the flow of two immiscible fluids displacement in heterogeneous porous media with mean pressure effect has been discussed analytically. The phenomenon is formulated mathematically as a water-oil double phase flow problem. An approximate solution of non-linear partial differential equation of instability phenomenon has been obtained in term of ascending power series which represents saturation of injected water in instability phenomenon. The solution gives rise phase saturation distribution of injected water by using appropriate boundary condition and its graphical and numerical presentation given by using MATLAB.

Keywords: *Heterogeneous porous media, Instability phenomenon, Capillary pressure.*

1 Introduction

An oil reservoir is a porous medium, whose pores contain some hydrocarbon components, usually designated by the generic term "oil". The porous medium is often heterogeneous, which means that the rock properties like porosity and permeability may vary from one place to another. The most heterogeneous oil fields are the so-called "fractured oil fields", which consist of a collection of blocks of porous medium separated by a net of fractures. Oil recovery process includes (i) prime recovery process (ii) secondary oil recovery process (iii) enhanced oil recovery process. In primary recovery process the oil is recovered from oil basin without any external effect but remaining oil in oil formatted porous media can obtained by injecting different fluids like water, gas, steam and polymer injection in secondary oil recovery process [6].

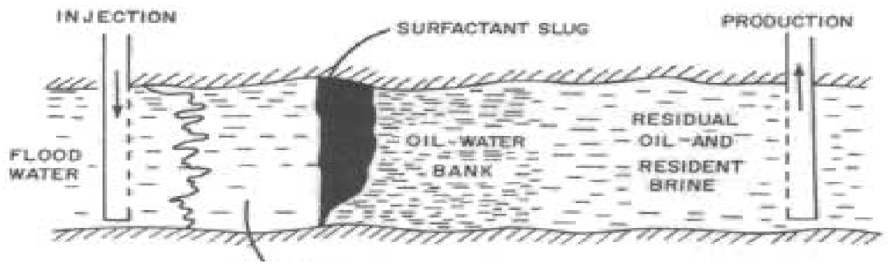

Figure 1: Actual formation in secondary oil recovery process

When the water is injected in oil formatted porous media then instead of regular displacement of common interface protuberance will occur due to the difference in viscosities of water and oil which gives arise to shape of fingers. This phenomenon is called instability (fingering) [3]. Many researchers have discussed this phenomenon with different points of view. The problem of the flow of two immiscible phases in homogeneous porous media without capillary pressure effect has been discussed by Buckely-Leverett [10]. Oroveanu [11] has formally extended this discussion for heterogeneous porous media. Verma [4] has discussed the statistical behaviour of fingering in a displacement process in heterogeneous porous medium with capillary pressure. Venkateswarlu[5] has discussed on the flow of immiscible liquids in a heterogeneous porous medium with capillary pressure and connate water saturation. Lyaghfouri [1] has discussed a free boundary problem for a fluid flow in a heterogeneous porous medium.

2 Statement of the Problem

During secondary oil recovery process, form oil formatted region, we considered that water is injected with velocity V_w into oil saturated heterogeneous cylindrical piece of porous medium of length L such that injected water shoots through oil formation at common interface $x = 0$ under the capillary pressure which gives rise to protuberance called instability (fingers) as per figure (2). It is assumed that the entire oil at the initial boundary $x = 0$ is displaced through a small distance due to the impact of injecting water. The schematic presentation of fingers at level x is expressed in fig. (3)

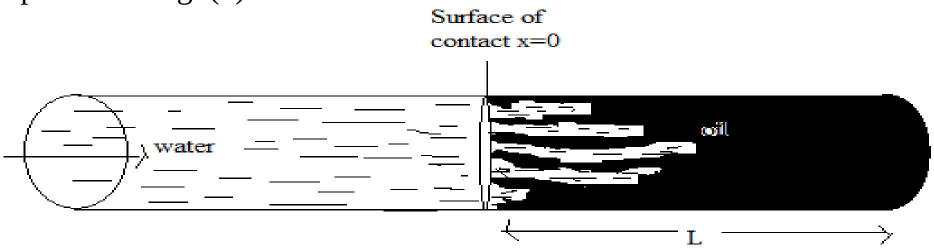

Figure 2: Formation of fingers in the cylindrical piece of porous media

Figure 3: Schematic representation of fingers at level "*x*"

Our particular interest in the present investigation is to determine the saturation of injecting water in well developed fingers due to water injection which push oil toward production well.

3 Fundamental Equations

During water injection in secondary oil recovery process, let injected water and native oil are two immiscible fluid governed by Darcy's law expressed [7] as

$$V_w = -\frac{K_w}{\mu_w} K \frac{\partial p_w}{\partial x} \tag{1}$$

$$V_o = -\frac{K_o}{\mu_o} K \frac{\partial p_o}{\partial x} \tag{2}$$

Where $K = K(x)$ is the variable permeability of the heterogeneous porous medium, K_w and K_o is relative permeabilities of displacing fluid, which are function of saturations S_w and S_o, p_w and p_o are pressures of displacing injected water and native oil, μ_w and μ_o are the constant kinematic viscosities of water and oil respectively.

The equation of continuity for flowing fluids are written respectively as

$$m\frac{\partial S_w}{\partial t}+\frac{\partial V_w}{\partial x}=0 \tag{3}$$

$$m\frac{\partial S_o}{\partial t}+\frac{\partial V_o}{\partial x}=0 \tag{4}$$

Where $m=m(x)$ is the variable porosity of heterogeneous porous media.
From the definition of phase saturation [2] gives

$$S_w+S_o=1 \tag{5}$$

4 Analytical Relationships

For definiteness we assume the following relationships:

4.1 Capillary Pressure

When water is injected then the flow takes place in interconnected capillary. Thus the capillary pressure p_c, defined as continuity of the flowing fluid across their common interface, is a function of the injected fluid saturation. It may be written as [3]

$$p_c(S_w)=p_o-p_w \tag{6}$$

Mehta [9] suggested that capillary pressure is proportional to saturation of displacing fluid and is in with opposite direction. He suggested the expression

$p_c=-\beta S_w$; β is proportionality constant (7)

4.2 Relative Permeability and Phase Saturation

The relationship between the relative permeability and phase saturation was given by Scheidegger and Johnson[2] is used here.
It is given as

$$K_w=S_w \tag{8}$$

$$K_o=S_o=1-S_w \tag{9}$$

4.3 Laws of Variation of the Characteristics of the Medium

Following Oroveanu [11], we take the laws of variation in the porosity and permeability of the uniform heterogeneous medium is defined as only function of *x* but for definiteness, we choose that porosity and permeability of heterogeneous porous media will varying with different distance *x* for time dependent constants. Hence,

$$m = m(x,t) = \frac{1}{a(t) - b(t)x} \qquad (10)$$

$$K = K(x,t) = K_c\left(1 + a_1(t)x\right) \qquad (11)$$

Where $a(t), b(t)$ and $a_1(t)$ are some time dependent constants. Since $m(x,t)$ cannot exceed unity, we assume further that $a(t) - b(t)x \geq 1$.

5 Equation of Motion for Saturation

The equation of motion for saturation is obtained by substituting the values of (V_w) and (V_o) from equation (1) and (2) to the equation (3) and (4) respectively,

$$m\frac{\partial S_w}{\partial t} = \frac{\partial}{\partial x}\left(\frac{K_w}{\mu_w}K\frac{\partial p_w}{\partial x}\right) \qquad (12)$$

$$m\frac{\partial S_o}{\partial t} = \frac{\partial}{\partial x}\left(\frac{K_o}{\mu_o}K\frac{\partial p_o}{\partial x}\right) \qquad (13)$$

Eliminating $\frac{\partial p_w}{\partial x}$ from equation (6), we have equation (12)

$$\frac{\partial}{\partial x}\left[\frac{K_w}{\mu_w}K\left\{\left(\frac{\partial p_o}{\partial x}\right) - \left(\frac{\partial p_c}{\partial x}\right)\right\}\right] = m\frac{\partial S_w}{\partial t} \qquad (14)$$

By combining equation (13) and (14), we get

$$\frac{\partial}{\partial x}\left\{\left(\frac{K_w}{\mu_w}K + \frac{K_o}{\mu_o}K\right)\frac{\partial p_o}{\partial x} - \frac{K_w}{\mu_w}K\frac{\partial p_c}{\partial x}\right\} = 0 \qquad (15)$$

On integrating equation (15) with respect to ‘*x*’, we get

$$\left\{\left(\frac{K_w}{\mu_w}K + \frac{K_o}{\mu_o}K\right)\frac{\partial p_o}{\partial x} - \frac{K_w}{\mu_w}K\frac{\partial p_c}{\partial x}\right\} = B(t) \qquad (16)$$

Where $B(t)$ is an arbitrary constant, dependent on time, and is evaluated by noting that

$K_w = 0$ at x=0, for all time . (17)

Further, since the striking oil (at x=0) maintains an uniform velocity V, we obtain from equation (2)

$$V_o\left(0,t\right) = -\left(\frac{K_o}{\mu_o} K \frac{\partial p_o}{\partial x}\right) = V \tag{18}$$

From equation (16), (17) and (18), we get

$$B\left(t\right) = -V \tag{19}$$

Substituting this value in equation (16), we obtain

$$\left\{\left(\frac{K_w}{\mu_w} K + \frac{K_o}{\mu_o} K\right)\frac{\partial p_o}{\partial x} - \frac{K_w}{\mu_w} K \frac{\partial p_c}{\partial x}\right\} = -V \tag{20}$$

Equation (20) gives,

$$\frac{\partial p_o}{\partial x} = -\frac{V}{K\left(\frac{K_w}{\mu_w} + \frac{K_o}{\mu_o}\right)} + \frac{\left(\frac{\partial p_c}{\partial x}\right)}{1 + \frac{k_0}{k_w}\frac{\mu_w}{\mu_o}} \tag{21}$$

Now, from equation (14) and (21), the following is obtained

$$m\frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x}\left[\frac{V}{1+\frac{K_o}{k_w}\frac{\mu_w}{\mu_o}} + \frac{\frac{K_o}{\mu_o} K\left(\frac{\partial p_c}{\partial x}\right)}{1+\frac{K_o}{K_w}\frac{\mu_w}{\mu_o}}\right] = 0 \tag{22}$$

The value of the pressure of oil (p_o) can be written [11] as,

$$p_o = \frac{p_o + p_w + p_o - p_w}{2}$$

$$p_o = \bar{p} + \left(1/2\right) p_c \ , \ \bar{p} = \frac{\left(p_o + p_w\right)}{2} \quad \text{which is the mean pressure and constant.} \tag{23}$$

It may be mentioned that the concept of mean pressure is justified in the statistical treatment of fingering [4].

On differentiating the above equation with respect to 'x' the following equation is obtained

$$\frac{\partial p_o}{\partial x} = \frac{1}{2}\frac{\partial p_c}{\partial x} \tag{24}$$

On substituting the value of $\frac{\partial p_o}{\partial x}$ from (24) to (20) we can obtained

$$V = \left[\frac{K}{2}\left(\frac{K_w}{\mu_w} - \frac{K_o}{\mu_o} \right)\frac{\partial p_c}{\partial x} \right] \tag{25}$$

On subtitling the value of V in equation (22) after simplification, we can obtained

$$m\frac{\partial S_w}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left[K\left(\frac{K_w}{\mu_w}\frac{dp_c}{dS_w}\frac{\partial S_w}{\partial x} \right) \right] = 0 \tag{26}$$

On substituting the value of p_c and k_w from equation (7) & (8) in the above equation, the following equation is obtained

$$m\frac{\partial S_w}{\partial t} = \frac{\beta}{2\mu_w}\frac{\partial}{\partial x}\left[\left(K\, S_w\frac{\partial S_w}{\partial x} \right) \right] \tag{27}$$

For simplification of equation (27), we consider, $K \propto m$ [13]

$$K = K_c\, m$$

Where we choose K_c as constant of proposnality.
Substituting this value in equation (27), so we get

$$m\frac{\partial S_w}{\partial t} = \frac{\beta K_c}{2\mu_w}\frac{\partial}{\partial x}\left[\left(m\, S_w\frac{\partial S_w}{\partial x} \right) \right] \tag{28}$$

Which is nonlinear partial differential equation of instability phenomenon in heterogeneous porous matrix during secondary recovery process which gives saturation of injected water for any distance x for $t \geq 0$.
The appropriate sets of condition to solve nonlinear equation (28) are
Let saturation of injected phase and variation in saturation of injected phase at common interface are respectively

$$S_w(0,t) = S_{w0} \quad \text{at } x = 0 \text{ for } t > 0 \tag{29}$$

$$\frac{\partial S_w}{\partial x}(0,t) = \omega \text{ very small near to zero} \tag{30}$$

6 Solution of the Equation

To convert equation (28) together with condition (29) and (30) in dimensionless variable,
Choose new dimensionless variable

$$X = \frac{x}{L} \quad and \quad T = \frac{\beta K_c}{2L^2 \mu_w} t$$

Substituting this value in equation (28), we have

$$\frac{\partial S_w}{\partial T} = \frac{\partial}{\partial X}\left[S_w \frac{\partial S_w}{\partial X}\right] + \frac{1}{m}\frac{\partial m}{\partial X} S_w \frac{\partial S_w}{\partial X} \tag{31}$$

For more simplification,

$\frac{1}{m}\frac{\partial m}{\partial X} = \frac{\partial}{\partial X}(\log m)$ and using value of m and neglecting higher order of x, we get

$$\frac{1}{m}\frac{\partial m}{\partial X} = \frac{1}{2\sqrt{T}} \tag{32}$$

;where choose $L\frac{b(T)}{a(T)} = \frac{L}{2\sqrt{T}}$ for any $0 < T \le 1$
Hence equation (31), will be

$$\frac{\partial S_w}{\partial T} = \frac{\partial}{\partial X}\left[S_w \frac{\partial S_w}{\partial X}\right] + \frac{L}{2\sqrt{T}} S_w \frac{\partial S_w}{\partial X} \tag{33}$$

The appropriate boundary conditions to solve (33) are

$$S_w(0,T) = S_{w0} \quad \text{at} \quad X = 0 \ \text{ for } T > 0 \tag{34}$$

$$\frac{\partial S_w}{\partial x}(0,T) = \omega \text{ for any } T > 0 \tag{35}$$

Choose similarity transformation,

$$S_w(X,T) = f(\eta), \text{ where } \eta = \frac{X}{2\sqrt{T}} \quad [9] \tag{36}$$

The governing equation (33) reduce to the ordinary differential equation

$$f^{'}(\eta)\left[f^{'}(\eta)+2\eta\right]+f(\eta)\,f^{''}(\eta)+Lf(\eta)\,f^{'}(\eta)=0 \tag{37}$$

and boundary conditions (34) and (35) will be

$$f(0)=S_{w0},\ X=0, T>0 \tag{38}$$

$$f^{'}(0)=\omega\neq 0\ \ for\ any\ T>0 \quad \text{(very small)} \tag{39}$$

To find successive coefficient's of Maclaurin's series at $\eta=0$ .Taking n[th] derivative of equation (40) and solving for $f^{(n+2)}(\eta)$ and evaluating at $\eta=0$, we have

$$f^{(n+2)}(0)=-\frac{1}{f(0)}\Big[\ f^{n+1}(0)\left\{f^{'}(0)+Lf(0)\right\}+\sum_{k=1}^{n}\binom{n}{k}\Big\{f^{k+1}(0)\,f^{(n-k+1)}(0)+$$

$$Lf^{(k)}(0)\,f^{(n-k+1)}(0)+f^{(n-k+2)}(0)\,f^{k}(0)\Big\}+2n\,f^{n}(0)\Big] \tag{40}$$

$; n\geq k$ n = 1, 2, 3, 4.............

For the solution, it is necessary to determine the derivatives $f^{(n)}(0)$ for all n = 1, 2, 3........ The derivative $f^{'}(0)$ can be determined by means of formula (39) and $f^{''}(0)$ from equation (37). Further all other higher derivatives can be determined from formula (40) by putting

n ≥1, 2, 3......

Thus the desired value of $f(\eta)$ can be computed by Maclaurin's series.

$$f(\eta)=\sum_{k=0}^{\infty}f^{(k)}(0)\,\frac{\eta^{k}}{k!} \tag{41}$$

$$f(\eta)=f(0)+\eta f^{'}(0)+\frac{\eta^{2}}{2!}f^{''}(0)+\frac{\eta^{3}}{3!}f^{'''}(0)+\frac{\eta^{4}}{4!}f^{iv}(0)+........ \tag{42}$$

Resubstituting value of $f(\eta)$ and η from equation (36), we get

$$S_{w}(X,T)=f(0)+\frac{X}{2\sqrt{T}}f^{'}(0)+\frac{X^{2}}{8\left(\sqrt{T}\right)^{2}}f^{''}(0)+\frac{X^{3}}{48\left(\sqrt{T}\right)^{3}}f^{'''}(0)+\frac{X^{4}}{384\left(\sqrt{T}\right)^{4}}f^{\mathbf{iv}}(0)+....$$

(43)

Where different coefficient of series (43) are calculated as bellow.

$$f(0) = S_{wo}$$
$$f^{'}(0) = \omega$$
$$f^{''}(0) = -\frac{1}{(S_{w0})}\left[\omega^2 + L\omega(S_{w0})\right]$$
$$f^{'''}(0) = \frac{1}{(S_{w0})^2}\left[3\omega^3 + 3L\omega^2(S_{w0}) + L\omega(S_{w0})^2 - 2\omega(S_{w0})\right]$$
$$f^{iv}(0) = -\frac{1}{(S_{w0})^3}\left[11\omega^4 + 14L\,\omega^3(S_{w0}) + 6L\omega^2(S_{w0})^2 - 10\omega^2(S_{w0}) + \omega\,L(S_{w0})^3 - 6L\omega(S_{w0})^2\right]$$

Equation (43) represents saturation of injected water in oil formation $X \geq 0$ during instability phenomenon in heterogeneous porous media when water is injected at common interface during secondary oil recovery process.

7 Result and Discussion

The solution (43) is saturation of injected water during water injection in secondary oil recovery process which is ascending power series of X with time $T > 0$. The solution (43) satisfies both conditions (34) and (35) which is also in term of convergent power series in X and T. We have considered only first fours term of power series hence it gives an approximate solution of instability phenomenon.

8 Numerical & Graphical Presentations

Numerical and graphical presentations of equation (43) have been obtained by using MATLAB coding. Figure 4 shows the graph of $S_w(X,T)$ vs. X for time $T = 0.1, 0.2, 0.3, 0.4, 0.5$, and Table 1 represent the numerical data. Figure 5, 6 and 7 shows that the graph of $S_w(X,T)$ vs. Distance X for T = 0.6, 0.7 and 0.8 respectively. All figures are denotes the graphical representations of the phenomenon showing the behaviour of the injected liquid.

Table 1: Numerical data for the saturation of injected water

Distance X	$S_i(X,T)$ T=0.1	$S_i(X,T)$ T=0.2	$S_i(X,T)$ T=0.3	$S_i(X,T)$ T=0.4	$S_i(X,T)$ T=0.5
0.0	0.1	0.1	0.1	0.1	0.1
0.1	0.1010	0.1008	0.1007	0.1006	0.1006
0.2	0.1017	0.1013	0.1011	0.1010	0.1010
0.3	0.1037	0.1018	0.1014	0.1013	0.1012
0.4	0.1102	0.1032	0.1020	0.1017	0.1015
0.5	0.1249	0.1064	0.1033	0.1024	0.1019
0.6	0.1530	0.1128	0.1060	0.1037	0.1028

0.7	0.2004	0.1239	0.1106	0.1062	0.1043
0.8	0.2745	0.1415	0.1180	0.1102	0.1067
0.9	0.3832	0.1676	0.1292	0.1162	0.1104
1	0.5359	0.2047	0.1451	0.1249	0.1158

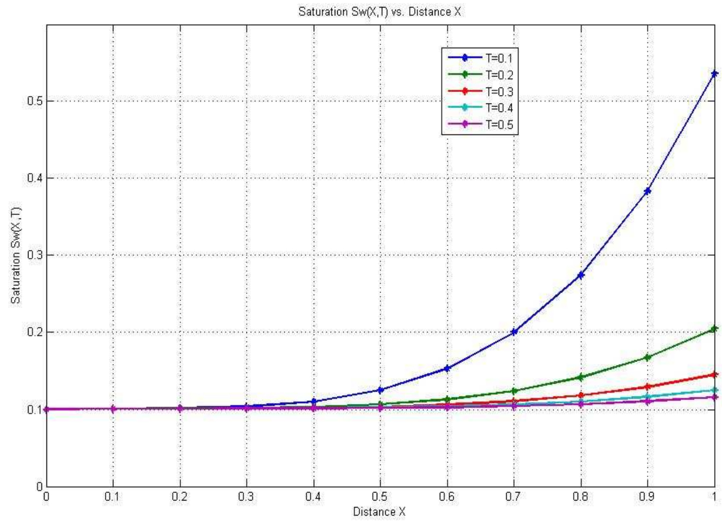

Figure 4: Saturation of injected water at different distance when T = 0.1, 0.2, 0.3, 0.4, 0.5 and ω = 0.01, S_{wo} = 0.1, L=5 fixed

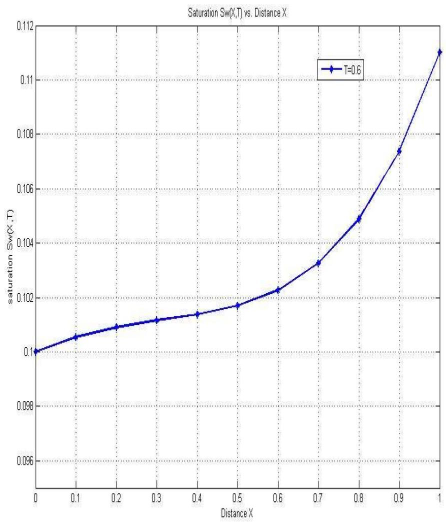

Figure 5: Saturation of injected water at different distance when T = 0.6 and ω = 0.01, S_{wo} = 0.1, L=5 fixed

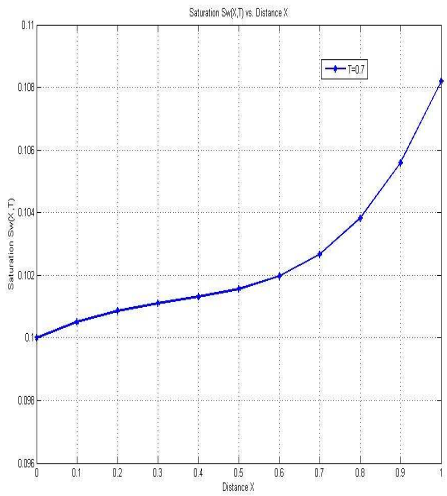

Figure 6: Saturation of injected water at different distance when T =0.7 and ω = 0.01, S_{wo} = 0.1, L=5 fixed

9 Conclusion

In instability phenomenon when water injection take place at common interface at $x = 0$ during secondary oil recovery process in heterogeneous porous media, there will be initial saturation $S_{w0} = 0.1$ which is shown in graph then protuberance will take place for any $X, T > 0$. Saturation steadily increasing up to the $X = 0.22$ then suddenly increasing fingers appears in phenomenon and saturation through interconnected capillaries of fingers will increases as distance X increases for different time $T > 0$ which is shown in graph but when T is increasing the saturation is very less decreasing than the previous time taken. From figure (5) to (7) it has been concluded that the saturation of injected water in instability phenomenon increases with respect to the distance X from common interface but effect of time is very less due to external force applied at time of water injection in instability phenomenon which is physically fact with phenomenon.

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