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Generalization of Titchmarsh's Theorem for the Jacobi-Dunkl Transform

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Abstract

In this paper, using a generalized Jacobi-Dunkl translation operator, we prove a generalization of Titchmarsh's theorem for functions in the k-Jacobi-Dunkl-Lipschitz class defined by the finite differences of order $k \in \mathbb{N}^*$ *and Sobolev spaces associated with the Jacobi-Dunkl operator.*

Keywords: *Generalized Jacobi-Dunkl translation, Jacobi-Dunkl Lipschitz class, Jacobi-Dunkl transform, Titchmarsh's theorem.*

1 Introduction

Titchmarsh's theorem characterizes the set of functions satisfying the Cauchy-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have:

Theorem 1.1. *[12] Let* $\alpha \in (0,1)$ *and* $f \in L^2(\mathbb{R})$ *. Then the following are equivalents:*

1. $\|f(t+h) - f(t)\| = O(h^\alpha)$ *, as* $h \to 0$ *;*

2. $\displaystyle\int_{|\lambda|\geq r} |\hat{f}(\lambda)|^2 d\lambda = O(r^{-2\alpha})$ *, as* $r \to +\infty$ *.*

where $\hat{f}$ is the Fourier transform of f .

A similar result of theorem 1.1 has been established for the Jacobi transform (see [8], theorem 2.2). Furthermore, a generalization of this result was proved in the Sobolev spaces associated with Jacobi transform (see [1], theorem 2.1).

In this paper, we prove a similar result for Jacobi-Dunkl transform, we consider functions in Sobolev spaces $W^{2,k}_{\alpha,\beta}$ (associated with Jacobi-Dunkl operator (see [5])) belonging to the k-Jacobi-Dunkl-Lipschitz class defined by the finite difference of order $k \in \mathbb{N}^*$. For this purpose we use the generalized translation and Jacobi-Dunkl operators.

The paper is organized as follows: in section 2 we recapitulate some results related to the harmonic analysis associated with the Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$ (see [2, 3, 4, 5, 7]). Section 3 is devoted to the main result (theorem 3.3). Before, we define the class $Lip(\delta, 2, \alpha, \beta)$ of functions in $W^{2,k}_{\alpha,\beta}$ satisfying a certain condition correspondent to the generalized Jacobi-Dunkl translation. Titchmarsh's theorem for Jacobi-Dunkl transform is given as a corollary of theorem 3.3.

2 Notations and Preliminaries

In the following, $\alpha\,,\beta$ *and* ρ denote 3 reals such that $\alpha \geq \beta \geq -\frac{1}{2}$, $\alpha \neq -\frac{1}{2}$ *and* $\rho = \alpha + \beta + 1$.

Notations:

- $A_{\alpha,\beta}(x) = 2^{\rho}(\sinh|x|)^{2\alpha+1}(\cosh|x|)^{2\beta+1}$.
- $$d\sigma_{\alpha,\beta}(\lambda) = \frac{|\lambda|}{8\pi\sqrt{\lambda^2-\rho^2}|C_{\alpha,\beta}(\sqrt{\lambda^2-\rho^2})|}\mathbb{I}_{\mathbb{R}\setminus]-\rho,\rho[}(\lambda)d\lambda$$
  where, $$C_{\alpha,\beta}(\mu) = \frac{2^{\rho-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma(\frac{1}{2}(\rho+i\mu))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\mu))}\ ,\ \mu \in \mathbb{C}\setminus(i\mathbb{N})\ .$$
  and $\mathbb{I}_{\Omega}$ is the characteristic function of Ω .
- $L^p(A_{\alpha,\beta})$ *(resp.* $L^p(\sigma_{\alpha,\beta})$, $p \in]0,+\infty[$, the space of measurable functions g on $\mathbb{R}$ such that
  $$||g||_{L^p(A_{\alpha,\beta})} = \left(\int_{\mathbb{R}}|g(t)|^p A_{\alpha,\beta}(t)dt\right)^{1/p} < +\infty\,.$$
  $$(resp.\ ||g||_{L^p(\sigma_{\alpha,\beta})} = \left(\int_{\mathbb{R}}|g(\lambda)|^p d\sigma_{\alpha,\beta}(\lambda)\right)^{1/p} < +\infty)\,.$$

- $\mathcal{D}(\mathbb{R})$ the space of C^∞-functions on $\mathbb{R}$ with compact support.
- $\mathcal{S}(\mathbb{R})$ the usual Schwartz space of C^∞-functions on $\mathbb{R}$ rapidly decreasing together with their derivatives, equipped with the topology of semi-norms $L_{m,n}$, $(m,n) \in \mathbb{N}^2$, where

$$L_{m,n}(f) = \sup_{x\in\mathbb{R}, 0\leq k\leq n}\left[(1+x^2)^m\left|\frac{d^k}{dx^k}f(x)\right|\right] < +\infty.$$

- $\mathcal{S}^1(\mathbb{R}) = \{(\cosh t)^{-2\rho}f;\ f \in \mathcal{S}(\mathbb{R})\}$.
  The topology of this space is given by the semi-norms $L^1_{m,n}$, $(m,n) \in \mathbb{N}^2$, where

$$L^1_{m,n}(f) = \sup_{x\in\mathbb{R}, 0\leq k\leq n}\left[(\cosh t)^{-2\rho}(1+x^2)^m\left|\frac{d^k}{dx^k}f(x)\right|\right] < +\infty.$$

- $(\mathcal{S}^1(\mathbb{R}))'$ the topological dual of $\mathcal{S}^1(\mathbb{R})$.

Now, we introduce the Jacobi-Dunkl Transform and its basic properties:

The Jacobi-Dunkl function with parameters (α,β) , $\alpha \geq \beta \geq -\frac{1}{2}, \alpha \neq -\frac{1}{2}$, is defined by :

$$\forall x \in \mathbb{R}, \quad \psi_\lambda^{(\alpha,\beta)}(x) = \begin{cases} \varphi_\mu^{(\alpha,\beta)}(x) - \dfrac{i}{\lambda}\dfrac{d}{dx}\varphi_\mu^{(\alpha,\beta)}(x) & , \text{ if } \lambda \in \mathbb{C}\setminus\{0\}; \\ 1 & , \text{ if } \lambda = 0. \end{cases} \tag{1}$$

with $\lambda^2 = \mu^2 + \rho^2,\ \rho = \alpha+\beta+1$ and $\varphi_\mu^{(\alpha,\beta)}$ is the Jacobi function given by:

$$\varphi_\mu^{(\alpha,\beta)}(x) = F\left(\frac{\rho+i\mu}{2}, \frac{\rho-i\mu}{2}; \alpha+1, -(\sinh x)^2\right), \tag{2}$$

where F is the Gaussian hypergeometric function given by

$$F(a,b,c,z) = \sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m m!}z^m \ , |z| < 1,$$

$a, b, z \in \mathbb{C}$ *and* $c \notin -\mathbb{N}$;
$(a)_0 = 1,\ (a)_m = a(a+1)...(a+m-1)$. (see [2, 9, 10]).
$\psi_\lambda^{(\alpha,\beta)}$ is the unique C^∞-solution on $\mathbb{R}$ of the differentiel-difference equation

$$\begin{cases} \Lambda_{\alpha,\beta}u = i\lambda u & , \ \lambda \in \mathbb{C}; \\ u(0) = 1. \end{cases} \tag{3}$$

where $\Lambda_{\alpha,\beta}$ is the Jacobi-Dunkl operator given by:

$$\Lambda_{\alpha,\beta}u(x) = \frac{du}{dx}(x) + \frac{A'_{\alpha,\beta}(x)}{A_{\alpha,\beta}(x)} \times \frac{u(x)-u(-x)}{2}\ ;\ i.e.$$

$$\Lambda_{\alpha,\beta}u(x) = \frac{du}{dx}(x) + [(2\alpha+1)\coth x + (2\beta+1)\tanh x] \times \frac{u(x)-u(-x)}{2}\,.$$

The function $\psi_\lambda^{(\alpha,\beta)}$ can be written in the form below (See [3]),

$$\psi_\lambda^{(\alpha,\beta)}(x) = \varphi_\mu^{(\alpha,\beta)}(x) + i\frac{\lambda}{4(\alpha+1)}\sinh(2x)\varphi_\mu^{(\alpha+1,\beta+1)}(x)\ ,\ \forall x \in \mathbb{R}\ , \tag{4}$$

where $\lambda^2 = \mu^2 + \rho^2$, $\rho = \alpha+\beta+1$.

The Jacobi-Dunkl transform of a function $f \in L^1(A_{\alpha,\beta})$ is defined by :

$$\mathcal{F}_{\alpha,\beta}(f)(\lambda) = \int_{\mathbb{R}} f(x)\psi_{-\lambda}^{(\alpha,\beta)}(x)A_{\alpha,\beta}(x)dx,\ \forall\lambda \in \mathbb{R}\ ; \tag{5}$$

The inverse Jacobi-Dunkl transform of a function $h \in L^1(\sigma_{\alpha,\beta})$ is:

$$\mathcal{F}_{\alpha,\beta}^{-1}(h)(t) = \int_{\mathbb{R}} h(\lambda)\psi_\lambda^{(\alpha,\beta)}(t)d\sigma_{\alpha,\beta}(\lambda)\,. \tag{6}$$

$\mathcal{F}_{\alpha,\beta}$ is a topological isomorphism from $\mathcal{S}^1(\mathbb{R})$ onto $\mathcal{S}(\mathbb{R})$, and extends uniquely to a unitary isomorphism from $L^2(A_{\alpha,\beta})$ onto $L^2(\sigma_{\alpha,\beta})$. The Plancherel formula is given by

$$\|f\|_{L^2(A_{\alpha,\beta})} = \|\mathcal{F}_{\alpha,\beta}(f)\|_{L^2(\sigma_{\alpha,\beta})}\ . \tag{7}$$

For $f \in \mathcal{S}^1(\mathbb{R})$ we have the following inversion formula

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_{\alpha,\beta}(f)(\lambda)\psi_\lambda^{(\alpha,\beta)}(x)d\sigma_{\alpha,\beta}(\lambda)\,,\ \forall x \in \mathbb{R}\,, \tag{8}$$

and the relation

$$\mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta}f)(\lambda) = i\lambda\mathcal{F}_{\alpha,\beta}(f)(\lambda)\,. \tag{9}$$

Let $f \in L^2(A_{\alpha,\beta})$. For all $x \in \mathbb{R}$ the operator of Jacobi-Dunkl translation τ_x is defined by:

$$\tau_x f(y) = \int_{\mathbb{R}} f(z)d\nu_{x,y}^{\alpha,\beta}(z)\ ,\ \forall\ y \in \mathbb{R}\ . \tag{10}$$

where $\nu_{x,y}^{\alpha,\beta}$, $x, y \in \mathbb{R}$ are the signed measures given by

$$d\nu_{x,y}^{\alpha,\beta}(z) = \begin{cases} K_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)dz & ,\ \text{if}\ \ x,y \in \mathbb{R}^*; \\ \delta_x & ,\ \text{if}\ \ y = 0; \\ \delta_y & ,\ \text{if}\ \ x = 0. \end{cases} \tag{11}$$

Here, δ_x is the Dirac measure at x. And

$$K_{\alpha,\beta}(x,y,z) = M_{\alpha,\beta}(\sinh(|x|)\sinh(|y|)\sinh(|z|))^{-2\alpha}\mathbb{I}_{I_{x,y}} \times \int_0^\pi \rho_\theta(x,y,z) \times (g_\theta(x,y,z))_+^{\alpha-\beta-1}\sin^{2\beta}\theta d\theta.$$

$$I_{x,y} = [-|x|-|y|, -||x|-|y||] \cup [||x|+|y||, |x|+|y|],$$
$$\rho_\theta(x,y,z) = 1 - \sigma^\theta_{x,y,z} + \sigma^\theta_{z,x,y} + \sigma^\theta_{z,y,x}$$

$$\sigma^\theta_{x,y,z} = \begin{cases} \dfrac{\cosh(x)+\cosh(y)-\cosh(z)\cos(\theta)}{\sinh(x)\sinh(y)} & , \text{ if } \ xy \neq 0; \\ 0 & , \text{ if } \ xy = 0. \end{cases}$$

for all $x, y, z \in \mathbb{R}$, $\theta \in [0, \pi]$.

$$g_\theta(x,y,z) = 1 - \cosh^2 x - \cosh^2 y - \cosh^2 z + 2\cosh x \cosh y \cosh z \cos\theta \ .$$

$$t_+ = \begin{cases} t & , \text{ if } \ t > 0; \\ 0 & , \text{ if } \ t \leq 0. \end{cases}$$

and

$$M_{\alpha,\beta} = \begin{cases} \dfrac{2^{-2\rho}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} & , \text{ if } \ \alpha > \beta; \\ 0 & , \text{ if } \ \alpha = \beta. \end{cases}$$

We have

$$\mathcal{F}_{\alpha,\beta}(\tau_h f)(\lambda) = \psi^{\alpha,\beta}_\lambda(h).\mathcal{F}_{\alpha,\beta}(f)(\lambda) \quad ; \quad h, \lambda \in \mathbb{R} \ . \tag{12}$$

Let $g \in L^2(\sigma_{\alpha,\beta})$. Then the distribution $T_{g\sigma_{\alpha,\beta}}$ defined by

$$\langle T_{g\sigma_{\alpha,\beta}}, \varphi\rangle = \int_{\mathbb{R}} g(\lambda)\varphi(\lambda)d\sigma_{\alpha,\beta}(\lambda), \quad \varphi \in \mathcal{D}(\mathbb{R}), \tag{13}$$

belongs to $\mathcal{S}'(\mathbb{R})$.

Let $f \in L^2(A_{\alpha,\beta})$. Then the distribution $T_{fA_{\alpha,\beta}}$ defined by

$$\langle T_{fA_{\alpha,\beta}}, \varphi\rangle = \int_{\mathbb{R}} f(x)\varphi(x)A_{\alpha,\beta}(x)dx, \quad \varphi \in \mathcal{S}^1(\mathbb{R}), \tag{14}$$

belongs to $(\mathcal{S}^1(\mathbb{R}))'$.

Via the correspondance $f \mapsto T_{fA_{\alpha,\beta}}$, we identify $L^2(A_{\alpha,\beta})$ as a subspace of $(\mathcal{S}^1(\mathbb{R}))'$.

The jacobi-dunkl transform of a distribution $T \in (\mathcal{S}^1(\mathbb{R}))'$ is defined by:

$$\langle \mathcal{F}_{\alpha,\beta}(T), \varphi\rangle = \langle T, \mathcal{F}^{-1}_{\alpha,\beta}(\check{\varphi})\rangle, \ \varphi \in \mathcal{S}(\mathbb{R}), \tag{15}$$

where $\check{\varphi}$ is given by $\check{\varphi}(x) = \varphi(-x)$.

It is clear that $\mathcal{F}_{\alpha,\beta}(T) \in \mathcal{S}'(\mathbb{R})$.

The jacobi-dunkl transform of a distribution defined by $f \in L^2(A_{\alpha,\beta})$ is given by the distribution $T_{\mathcal{F}_{\alpha,\beta}(f)\sigma_{\alpha,\beta}}$; i.e.

$$\mathcal{F}_{\alpha,\beta}(T_{fA_{\alpha,\beta}}) = T_{\mathcal{F}_{\alpha,\beta}(f)\sigma_{\alpha,\beta}} . \tag{16}$$

We identify the tempered distribution given by $\mathcal{F}_{\alpha,\beta}(f)$ and the function $\mathcal{F}_{\alpha,\beta}(f)$. Let $T \in (\mathcal{S}^1(\mathbb{R}))'$ and consider the distribution $\Lambda_{\alpha,\beta}T$ defined by

$$\langle \Lambda_{\alpha,\beta}(T), \varphi \rangle = -\langle T, \Lambda_{\alpha,\beta}(\varphi) \rangle , \text{ for all } \varphi \in \mathcal{S}^1(\mathbb{R}) . \tag{17}$$

(Note that $\mathcal{S}^1(\mathbb{R})$ is unvariant under $\Lambda_{\alpha,\beta}$).
By using (9) it is easy to see that

$$\mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta}(T)) = i\lambda \mathcal{F}_{\alpha,\beta}(T) . \tag{18}$$

For $f \in L^2(A_{\alpha,\beta})$, we define the finite differences of first and higher order as follows:

$$\begin{aligned} \Delta_h^1 f &= \Delta_h f = \tau_h f + \tau_{-h} f - 2f = (\tau_h + \tau_{-h} - 2E)f; \\ \Delta_h^k f &= \Delta_h(\Delta_h^{k-1})f = (\tau_h + \tau_{-h} - 2E)^k f , \quad k = 2, 3, ...; \end{aligned}$$

where E is the unit operator in $L^2(A_{\alpha,\beta})$.

Lemma 2.1. *The following inequalities are valids for Jacobi functions* $\varphi_\mu^{\alpha,\beta}(h)$

1. $|\varphi_\mu^{(\alpha,\beta)}(h)| \leq 1$;
2. $|1 - \varphi_\mu^{(\alpha,\beta)}(h)| \leq h^2\lambda^2$; *where* $\lambda^2 = \mu^2 + \rho^2$.

Proof. (See [11], Lemmas 3.1-3.2) □

For $\alpha \geq \frac{-1}{2}$, we introduce the Bessel normalized function of the first kind defined by

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n!\Gamma(n+\alpha+1)} \quad , \ z \in \mathbb{C}.$$

We see that $\lim_{z\to 0} \frac{j_\alpha(z) - 1}{z^2} \neq 0$, by consequence, there exists $c_1 > 0$ and $\eta > 0$ satisfying

$$|z| \leq \eta \Rightarrow \quad |j_\alpha(z) - 1| \geq c_1 |z|^2 . \tag{19}$$

Lemma 2.2. *Let* $\alpha \geq \beta \geq \frac{-1}{2}$, $\alpha \neq \frac{-1}{2}$. *Then for* $|v| \leq \rho$, *there exists a positive constant* c_2 *such that*

$$|1 - \varphi_{\mu+iv}^{(\alpha,\beta)}(t)| \geq c_2 |1 - j_\alpha(\mu t)| .$$

Proof. (See [6], Lemma 9) □

3 Main Results

We denote by $W^{2,k}_{\alpha,\beta}$, $k \in \mathbb{N}$, the Sobolev space constructed by the operator $\Lambda_{\alpha,\beta}$; i.e.

$$W^{2,k}_{\alpha,\beta} = \left\{ f \in L^2(A_{\alpha,\beta});\ \Lambda^j_{\alpha,\beta} f \in L^2(A_{\alpha,\beta}),\ j = 0, 1, 2, ..., k \right\}; \tag{20}$$

where, $\Lambda^0_{\alpha,\beta} f = f$, $\Lambda^1_{\alpha,\beta} f = \Lambda_{\alpha,\beta} f$, $\Lambda^r_{\alpha,\beta} f = \Lambda_{\alpha,\beta}(\Lambda^{r-1}_{\alpha,\beta} f)$, $r = 2, 3, ...$

Definition 3.1. *Let $\delta \in (0, 1)$ and $k \in \mathbb{N}$. A function $f \in W^{2,k}_{\alpha,\beta}$ is said to be in the k-Jacobi-Dunkl-Lipschitz class, denoted by $Lip(\delta, 2, k, r)$, if*

$$\left\| \Delta^{k+1}_h \Lambda^r_{\alpha,\beta} f \right\|_{L^2(A_{\alpha,\beta})} = O(h^\delta), \quad as\ h \longrightarrow 0,$$

where $r = 0, 1, ..., k$.

Lemma 3.2. *Let $f \in W^{2,k}_{\alpha,\beta}$, $k \in \mathbb{N}$. Then*

$$\left\| \Delta^{k+1}_h \Lambda^r_{\alpha,\beta} f \right\|^2_{L^2(A_{\alpha,\beta})} = 2^{2k+2} \int_{\mathbb{R}} \lambda^{2r} |1 - \varphi_\mu(h)|^{2k+2} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda),$$

where $r = 0, 1, ..., k$.

Proof. We have

$$\mathcal{F}_{\alpha,\beta}(\tau_h f + \tau_{-h} f - 2f)(\lambda) = (\psi^{(\alpha,\beta)}_\lambda(h) + \psi^{(\alpha,\beta)}_\lambda(-h) - 2).\mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

Since $\psi^{(\alpha,\beta)}_\lambda(h) = \varphi^{(\alpha,\beta)}_\mu(h) + i\dfrac{\lambda}{4(\alpha+1)} \sinh(2h) \varphi^{(\alpha+1,\beta+1)}_\mu(h)$,

$\psi^{(\alpha,\beta)}_\lambda(-h) = \varphi^{(\alpha,\beta)}_\mu(-h) - i\dfrac{\lambda}{4(\alpha+1)} \sinh(2h) \varphi^{(\alpha+1,\beta+1)}_\mu(-h)$,

and $\varphi^{(\alpha,\beta)}_\mu$ is even [See (2)]; then:

$$\mathcal{F}_{\alpha,\beta}(\tau_h f + \tau_{-h} f - 2f)(\lambda) = 2(\varphi^{(\alpha,\beta)}_\mu(h) - 1).\mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

and

$$\mathcal{F}_{\alpha,\beta}(\Delta^{k+1}_h f)(\lambda) = 2^{k+1}(\varphi^{(\alpha,\beta)}_\mu(h) - 1)^{k+1}.\mathcal{F}_{\alpha,\beta}(f)(\lambda). \tag{21}$$

From formula (18), we obtain

$$\mathcal{F}_{\alpha,\beta}(\Lambda^r_{\alpha,\beta} f)(\lambda) = (i\lambda)^r \mathcal{F}_{\alpha,\beta}(f)(\lambda). \tag{22}$$

Using the formulas (21) and (22) we get

$$\mathcal{F}_{\alpha,\beta}(\Delta^{k+1}_h \Lambda^r_{\alpha,\beta} f)(\lambda) = 2^{k+1}(i\lambda)^r.(\varphi^{(\alpha,\beta)}_\mu(h) - 1)^{k+1}.\mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

By the Plancherel formula (7), we have the result. □

Theorem 3.3. *Let $f \in W^{2,k}_{\alpha,\beta}$, $k \in \mathbb{N}$. Then the following are equivalents:*

1. $f \in Lip(\delta, 2, k, r)$;
2. $\int_s^\infty \lambda^{2r} \left|\mathcal{F}_{\alpha,\beta}(f)(\lambda)\right|^2 d\sigma_{\alpha,\beta}(\lambda) = O(s^{-2\delta})$, *as* $s \to +\infty$.

Proof. (1) ⇒ (2): Assume that $f \in Lip(\delta, 2, k, r)$; then

$$\left\|\Delta_h^{k+1}\Lambda^r_{\alpha,\beta} f\right\|_{L^2(A_{\alpha,\beta})} = O(h^\delta) \quad as\ h \longrightarrow 0.$$

by lemma 3.2, we have

$$\begin{aligned}\int_{\mathbb{R}} \lambda^{2r}|1-\varphi_\mu(h)|^{2k+2}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) &= \frac{1}{4^{k+1}}||\Delta_h^{k+1}\Lambda^r_{\alpha,\beta} f||^2\\ &= O(h^{2\delta})\end{aligned}$$

If $|\lambda| \in [\frac{\eta}{2h}, \frac{\eta}{h}]$ then $|\mu h| \le \eta$ (recall that $\lambda^2 = \mu^2 + \rho^2$).
We get by (19):

$$|j_\alpha(\mu h) - 1| \ge c_1 \mu^2 h^2.$$

From $|\lambda| \ge \dfrac{\eta}{2h}$ we have,

$$\mu^2 h^2 \ge \frac{\eta^2}{4} - \rho^2 h^2;$$

then we can find an absolute constant $c_3 = c_3(\eta, \alpha, \beta)$ such that $\mu^2 h^2 \ge c_3$ (take $h < 1$) ; thus,

$$|j_\alpha(\mu h) - 1| \ge c_1 c_3.$$

this inequality and lemma 2.2 implys that:

$$|1 - \varphi_\mu^{(\alpha,\beta)}(h)| \ge c_1 c_2 c_3 = C$$

Hence,

$$1 \le \frac{1}{C^{2k+2}}|1 - \varphi_\mu^{(\alpha,\beta)}(h)|^{2k+2}.$$

So,

$$\begin{aligned}\int_{\frac{\eta}{2h}\le|\lambda|\le\frac{\eta}{h}} \lambda^{2r}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) &\le \frac{1}{C^{2k+2}}\int_{\frac{\eta}{2h}\le|\lambda|\le\frac{\eta}{h}} \lambda^{2r}|1-\varphi_\mu^{(\alpha,\beta)}(h)|^{2k+2}\\ &\qquad\times|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda)\\ &\le \frac{1}{C^{2k+2}}\int_{\mathbb{R}} \lambda^{2r}|1-\varphi_\mu^{(\alpha,\beta)}(h)|^{2k+2}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda)\\ &= O(h^{2\delta}).\end{aligned}$$

Then we have,

$$\int_{s\leq|\lambda|\leq 2s}\lambda^{2r}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda)=O(s^{-2\delta})\ ,\quad as\ s\to+\infty.$$

Or equivalently

$$\int_{s\leq|\lambda|\leq 2s}\lambda^{2r}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda)\leq K_1 s^{-2\delta}\ ,\quad as\ s\to+\infty,$$

where K_1 is some absolute constant . It follows that,

$$\begin{aligned}\int_{|\lambda|\geq s}\lambda^{2r}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) &= \sum_{i=0}^{\infty}\int_{2^i s\leq|\lambda|\leq 2^{i+1}s}\lambda^{2r}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda)\\ &\leq K_1\sum_{i=0}^{\infty}\left(2^i s\right)^{-2\delta}\\ &\leq Ks^{-2\delta}.\end{aligned}$$

which proves that:

$$\int_{|\lambda|\geq s}\lambda^{2r}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda)=O(s^{-2\delta})\quad ,\ as\ s\to+\infty.$$

(2) $\Rightarrow$ (1) : Suppose now that

$$\int_{|\lambda|\geq s}\lambda^{2r}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda)=O(s^{-2\delta})\quad ,\ as\ s\to+\infty.$$

we have to show that:

$$\int_{\mathbb{R}}\lambda^{2r}|1-\varphi_\mu^{(\alpha,\beta)}(h)|^{2k+2}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda)=O(h^{2\delta})\quad ,\ as\ h\to 0.$$

Write:

$$\int_{\mathbb{R}}\lambda^{2r}|1-\varphi_\mu^{(\alpha,\beta)}(h)|^{2k+2}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda)=I_1+I_2,$$

where:

$$\begin{aligned}I_1 &= \int_{|\lambda|\leq\frac{1}{h}}\lambda^{2r}|1-\varphi_\mu^{(\alpha,\beta)}(h)|^{2k+2}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda)\,;\\ I_2 &= \int_{|\lambda|>\frac{1}{h}}\lambda^{2r}|1-\varphi_\mu^{(\alpha,\beta)}(h)|^{2k+2}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda).\end{aligned}$$

Estimate I_1 and I_2. From (1) of lemma 2.1 we can write,

$$\begin{aligned} I_2 &\leq 4^{k+1}\int_{|\lambda|>\frac{1}{h}} \lambda^{2r}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda)\,, \quad (s=\frac{1}{h}) \\ &= O(h^{2\delta}). \end{aligned}$$

Using the inequalities (1) and (2) of lemma 2.1 we get

$$\begin{aligned} I_1 &= \int_{|\lambda|\leq\frac{1}{h}} \lambda^{2r}|1-\varphi_\mu^{(\alpha,\beta)}(h)|^{2k+2}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) \\ &\leq 2^{2k+1}\int_{|\lambda|\leq\frac{1}{h}} \lambda^{2r}|1-\varphi_\mu^{(\alpha,\beta)}(h)|.|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) \\ &\leq 2^{2k+1}h^2\int_{|\lambda|\leq\frac{1}{h}} \lambda^{2r}.\lambda^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda). \end{aligned}$$

Consider the function

$$\psi(s)=\int_s^\infty \lambda^{2r}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda).$$

An integration by parts gives:

$$\begin{aligned} 2^{2k+1}h^2\int_0^{\frac{1}{h}} \lambda^{2r}.\lambda^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) &= 2^{2k+1}h^2\int_0^{\frac{1}{h}} \left(-s^2\psi'(s)\right) ds \\ &= 2^{2k+1}h^2\left(-\frac{1}{h^2}\psi(\frac{1}{h})+2\int_0^{\frac{1}{h}} s\psi(s)ds\right) \\ &\leq 2^{2k+2}h^2\int_0^{\frac{1}{h}} s\psi(s)ds. \end{aligned}$$

Since $\psi(s)=O(s^{-2\delta})$, we get

$$\begin{aligned} \int_0^{\frac{1}{h}} s\psi(s)ds &= O\left(\int_0^{\frac{1}{h}} s^{1-2\delta}ds\right) \\ &= O(h^{2\delta-2}). \end{aligned}$$

Hence,

$$\begin{aligned} 2^{2k+1}h^2\int_0^{\frac{1}{h}} \lambda^{2r}.\lambda^2|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) &\leq 2^{2k+2}h^2 O(h^{2\delta-2}). \\ &= O(h^{2\delta}) \end{aligned}$$

Finally,

$$\begin{aligned}\int_{\mathbb{R}} \lambda^{2r}|1-\varphi_{\mu}^{(\alpha,\beta)}(h)|^{2k+2}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) &= I_1 + I_2 \\ &= O(h^{2\delta}) + O(h^{2\delta}) \\ &= O(h^{2\delta})\end{aligned}$$

Which completes the proof of the theorem.

□

Corollary 3.4. *Let* $f \in W^{2,k}_{\alpha,\beta}$ *such that* $f \in Lip(\delta, 2, k, r)$ *. Then:*

$$\int_{|\lambda|\geq s} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) = O\left(s^{-2\delta-2r}\right) \quad , \ as\ s \to +\infty\,.$$

If we take $k = 0$ in theorem 3.3, we deduce an analog of Titchmarsh's theorem (theorem 1.1) for the Jacobi-Dunkl transform:

Corollary 3.5. *Let* $\delta \in (0,1)$ *and* $f \in L^2(A_{\alpha,\beta})$ *. Then the following are equivalents:*

1. $\|\tau_h f + \tau_{-h} f - 2f\|_{L^2(A_{\alpha,\beta})} = O(h^{\delta}) \quad , \ as\ h \to 0\,.$

2. $\displaystyle\int_{|\lambda|\geq s} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^2 d\sigma_{\alpha,\beta}(\lambda) = O(s^{-2\delta}) \quad , \ as\ s \to +\infty\,.$

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