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A Two-Sided Multiplication Operator Norm

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Abstract

Let $\mathcal{A}$ be a C^*-algebra and define an elementary operator $T_{a,b} : \mathcal{A} \to \mathcal{A}$ by $T_{a,b}(x) = \Sigma_{i=1}^{n} a_i x b_i$, $\forall\ x \in \mathcal{A}$ where a_i and b_i are fixed in $\mathcal{A}$ or multiplier algebra $M(\mathcal{A})$ of $\mathcal{A}$. Here, we determine the norm of a two-sided multiplication operator.

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1 Introduction

Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Then $T : B(H) \to B(H)$ is an elementary operator if T has a representation $T_{a,b}(x) = \Sigma_{i=1}^{n} a_i x b_i$, $\forall\ x \in B(H)$, where a_i and b_i are fixed in $B(H)$. Some examples of elementary operators are the left multiplication $L_a(x) = ax$; the right multiplication $R_b(x) = xb$; the generalized derivation $\delta_{a,b} = L_a - R_b$; the inner derivation, the two-sided multiplication operator

$M_{a,b} = L_a R_b$ and the Jordan elementary operator $\mathcal{U}_{a,b} = M_{a,b} + M_{b,a}$. Determining the lower estimate of the norm of elementary operators has attracted a lot of interest from many mathematicians (see [1-5, 7-18]). Clearly, every elementary operator is bounded. For the lower estimates of the norms, there have been several results obtained by different mathematicians. For example, Mathieu [6] proved that for a prime C*- algebra $\mathcal{A}$, $\|\mathcal{U}_{a,b}|\mathcal{A}\| \geq \frac{2}{3}\|a\|\|b\|$, Cabrera and Rodriguez [4] proved that for JB* algebras, $\|\mathcal{U}_{a,b}|\mathcal{A}\| \geq \frac{1}{20412}\|a\|\|b\|$, while Stacho and Zalar [12] obtained results for standard operator algebras on Hilbert spaces i.e. they showed that $\|\mathcal{U}_{a,b}|\mathcal{A}\| \geq 2(\sqrt{2}-1)\|a\|\|b\|$. Recently, Timoney [15, 16] demonstrated that $\|\mathcal{U}_{a,b}|\mathcal{A}\| \geq \|a\|\|b\|$. He [18] also gave a formula for the norm of an elementary operator on a C*-algebra using the notion of matrix valued numerical ranges and a kind of tracial geometric mean.

Theorem 1.1. *For $a = [a_1, ..., a_n] \in B(H)^n$ (a row matrix of operators $a_i \in B(H)$), $b = [b_1, ..., b_n] \in B(H)^n$ (a column matrix of operators $b_i \in B(H)$) and $T_{a,b}(x) = \Sigma_{i=1}^n a_i x b_i$, $\forall\ x \in B(H)$, an elementary operator, we have*

$$\|T\| = sup\{tgm(Q(a^*, \xi), Q(b, \eta)) : \xi,\ \eta \in H,\ \|\xi\| = 1,\ \|\eta\| = 1\}.$$

For proof, see [18, Theorem 1.4].

Interestingly, for Calkin algebras, it has been easy to calculate the norms of elementary operators as shown by Mathieu [7]. Considering a two-sided multiplication operator $M_{a,b}$, it has been shown in [2], the necessary and sufficient conditions for any pair of operators $a, b \in B(H)$ to satisfy the equation $\|I + M_{a,b}\| = 1 + \|a\|\|b\|$.

Definition 1.2. *Let $T \in B(H)$. The maximal numerical range of T is defined by $W_0(T) = \{\lambda : \langle Tx_n, x_n\rangle \to \lambda,$ where $\|x_n\| = 1$ and $\|Tx_n\| \to \|T\|\}$ and the normalized maximal numerical range is given by*

$$W_N(T) = \begin{cases} W_0(\frac{T}{\|T\|}), & \text{if } T \neq 0, \\ 0, & \text{if } T = 0. \end{cases}$$

The set $W_0(T)$ is nonempty, closed, convex and contained in the closure of the numerical range, see [14].

Theorem 1.3. *For $a, b \in B(H)$ the following are equivalent:*
(1) $\|I + M_{a,b}\| = 1 + \|a\|\|b\|$,
(2) $W_N(a^*) \cap W_N(b) \neq \emptyset$.

See [2] for proof.

Conjecture 1.4. *Let $\mathcal{A}$ be a standard operator subalgebra of $B(H)$. The estimate of M, such that $\|M_{a,b}x\| = \|a\|\|b\|$ holds for every $a,\ b \in \mathcal{A}$.*

This conjecture was verified in the following cases :
(i) for $a,\ b \in B(H)$ such that $\inf_{\lambda\in\mathbb{C}} \|a+\lambda b\| = \|a\|$ or $\inf_{\lambda\in\mathbb{C}} \|b+\lambda a\| = \|b\|$,
(ii) in the Jordan algebra of symmetric operators. See [1, 13].

Nyamwala and Agure [8] used the spectral resolution theorem to calculate the norm of an elementary operator induced by normal operators in a finite dimensional Hilbert space. They gave the following result.

Theorem 1.5. *Let $T_{a,b} : B(H) \to B(H)$ be an elementary operator defined by $T_{a,b}(x) = \sum_{i=1}^{k} a_i x b_i$ where a_i and b_i are normal operators and H a finite $m-$dimensional Hilbert space then*

$$\|T\| = (\sum_{j=1}^{k}(\sum_{j=1}^{m} \mid \alpha_{i,j} \mid^2 \mid \beta_{i,j} \mid^2))^{\frac{1}{2}}$$

where $\alpha_{i,j}$ and $\beta_{i,j}$ are distinct eigenvalues of a_i and b_i respectively.

A specific example in [8, Example 2.3] shows that $\|T\| = 2$. In the next section, we determine the norm of a two-sided multiplication operator.

2 Two-sided Multiplication Operator Norm

In this section we concentrate on a complex Hilbert space over the field $\mathbb{K}$. We show that for a two-sided multiplication operator M, $\|M_{a,b}x\| = \|a\|\|b\|$.

Definition 2.1. *Let $\phi \in H^*$ and $\xi \in H$. We define $\phi \otimes \xi \in B(H)$ by*

$$(\phi \otimes \xi)\eta = \phi(\eta)\xi,\ \forall\, \eta \in H.$$

Theorem 2.2. *Let H be a complex Hilbert space, $B(H)$ the algebra of all bounded linear operators on H. Let $M_{a,b} : B(H) \to B(H)$ be defined by $M_{a,b}(x) = axb,\ \forall\, x \in B(H)$ where a, b are fixed in $B(H)$. Then $\|M_{a,b}x\| = \|a\|\|b\|$.*

Proof. By definition, $\|M_{a,b}|B(H)\| = \sup\{\|M_{a,b}(x)\| :\ x \in B(H),\ \|x\| = 1\}$.
This implies that $\|M_{a,b}|B(H)\| \geq \|M_{a,b}(x)\|,\ \forall\, x \in B(H),\ \|x\| = 1$.
So $\forall\, \epsilon > 0,\ \|M_{a,b}|B(H)\| - \epsilon < \|M_{a,b}(x)\|,\ \forall\, x \in B(H),\ \|x\| = 1$.
But, $\|M_{a,b}|B(H)\| - \epsilon < \|axb\| \leq \|a\|\|x\|\|b\| = \|a\|\|b\|$.
Since ϵ is arbitrary, this implies that

$$\|M_{a,b}|B(H)\| \leq \|a\|\|b\|. \quad (1)$$

On the other hand, let $\xi,\ \eta \in H,\ \ \|\xi\| = \|\eta\| = 1,\ \phi \in H^*$.
Now,

$$\|M_{a,b}|B(H)\| \geq \|M_{a,b}(x)\|,\ \forall\, x \in B(H),\ \|x\| = 1.$$

But,

$$\begin{aligned}\|M_{a,b}(x)\| &= \sup\{\|(M_{a,b}(x))\eta\| : \forall\,\eta \in H,\ \|\eta\| = 1\} \\ &= \sup\{\|(axb)\eta\| : \eta \in H,\ \|\eta\| = 1\}.\end{aligned}$$

Setting $a = (\phi \otimes \xi_1),\ \forall\,\xi_1 \in H,\ \|\xi_1\| = 1$ and
$b = (\varphi \otimes \xi_2),\ \forall\,\xi_2 \in H,\ \|\xi_2\| = 1$, we have,

$$\begin{aligned}\|M_{a,b}|B(H)\| &\geq \|M_{a,b}(x)\| \geq \|(M_{a,b}(x))\eta\| \\ &= \|(axb)\eta\| \\ &= \|((\phi \otimes \xi_1)x(\varphi \otimes \xi_2))\eta\| \\ &= \|(\phi \otimes \xi_1)x(\varphi(\eta)\xi_2)\| \\ &= \|(\phi \otimes \xi_1)\varphi(\eta)x(\xi_2)\| \\ &= |\varphi(\eta)|\|(\phi \otimes \xi_1)x(\xi_2)\| \\ &= |\varphi(\eta)|\|\phi(x(\xi_2))\xi_1\| \\ &= |\varphi(\eta)||\phi(x(\xi_2))|\|\xi_1\| \\ &= \|a\|\|b\|.\end{aligned}$$

Therefore,

$$\|M_{a,b}|B(H)\| \geq \|a\|\|b\|. \tag{2}$$

Hence by inequalities (1)and (2),

$$\|M_{a,b}|B(H)\| = \|a\|\|b\|.$$

This completes the proof. □

3 The Jordan Elementary Operator

Theorem 3.1. *Let H be a 2-dimensional complex Hilbert space, $B(H)$ the algebra of bounded linear operators on H. Let $T_{a,b} : B(H) \to B(H)$ be defined by $T_{a,b}(x) = axb + bxa,\ \forall\,x \in B(H)$ where a, b are fixed in $B(H)$ and $\{e_1,\ e_2\}$ an orthonormal basis for H. Then for a constant $C > 0$ such that $\|T_{a,b}\| \geq C\|a\|\|b\|$, $C = 1$.*

Proof. The proof of this theorem follows immediately from the results obtained in [3]. □

Remark 3.2. *From [13], we see that $C = 1$ is also true for symmetric operators (in this case, a and b are self adjoint).*

Theorem 3.3. *Let $a, b \in Symm(H)$. Then $\|\mathcal{U}_{a,b}|\mathcal{A}\| \geq \|a\|\|b\|$.*
See [13] for proof.

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