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Coefficient Inequality for Functions Whose Derivative has a Positive Real Part

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Abstract

Recently, Acu and Owa [1] further studied the work of Kanas and Ronning [2] by investigating the classes of close - to - convex and $\alpha-$ convex functions normalised with $f(w) = f'(w) - 1 = 0$ and w is a fixed point in E. Ghanim and Darus introduced another subclass using the fixed point. Necessary and sufficient conditions were provided for this class. The aim of this paper is to continue the investigation by extending this class of functions to the class $S_n(\alpha)$ defined by the Salagean [4], our result extends some existing ones and new ones are derived.

Keywords: *univalent functions, starlike function, convex function, close-to-convex function, $\alpha-$ convex function .*

1 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the unit disk $E = [z = |z| < 1]$. Let $S \subset A$ be the class of functions univalent in E

Here we recall the following definitions of the well known classes of starlike, convex, close-to-convex and $\alpha-$ convex functions.

$S^* = \left\{ f \in A : Re \left[\frac{zf'(z)}{f(z)} \right] > 0, \quad z \in E \right\}$

$S^c = \left\{ f \in A : Re \left[1 + \frac{zf''(z)}{f'(z)} \right] > 0, \quad z \in E \right\}$

$CC = \left\{ f \in A : \exists \quad g \in S^*, Re \left[\frac{zf'(z)}{g(z)} \right] > 0, \quad z \in E \right\}$

Let w be a fixed point in E and $A(w) = \{ f \in A(w) : f(w) = f'(w) - 1 = 0 \}$

The following classes were introduced in [2] and further studied in [1]

$S(w) = \{ f \in A(w) : f \in S \}$

$ST(w) = S^*(w) = \left\{ f \in S(w) : Re \left[\frac{(z-w)f'(z)}{f(z)} \right] > 0, \quad z \in E \right\}$

$CV(w) = S^c(w) = \left\{ f \in S(w) : Re \left[1 + \frac{(z-w)f''(z)}{f'(z)} \right] > 0, \quad z \in E \right\}$

See details in [1].

2 Preliminary Notes

Let $p(w)$ denote the class of all functions

$$p(z) = 1 + \sum_{k=1}^{\infty} B_k (z-w)^k \qquad (2)$$

that are regular in E and satisfying $p(w) = 1$ and $Re p(z) > 0$ for $z \in E$, where

$$|B_k| \leq \frac{2}{(1+d)(1-d)^k} \qquad (3)$$

and $d = |w|$ and $k \geq 1$. See [1, 2, 3].

Definition 1.1 [4]: A function $f(z) \in A(w)$ is said to be in the class $S_n^w(\alpha)$ if and only if

$$Re \frac{D^{n+1} f(z)}{D^n f(z)} > \alpha \qquad (4)$$

where $0 \leq \alpha < 1, n = 0, 1, 2, 3, \ldots$ and $z \in E$ and D^n is the Salagean differential operator and it is defined as follows:

$D^0 f(z) = f(z), D' f(z) = z f'(z), \ldots, D^n f(z) = z \left(D^{n-1} f(z) \right)'$

3 Main Results

These are the main results of the paper.

Theorem 3.1. *Let $f \in S_n^w(\alpha)$ and $f(z) = (z-w) + \sum_{k=2}^{\infty} a_k (z-w)^k$. Then*

$$|a_2| \leq \frac{1-\alpha}{2^{n-1}(1-d^2)} \tag{5}$$

$$|a_3| \leq \frac{(1-\alpha)(1+d) + 2(1-\alpha)^2}{3^n (1-d^2)^2} \tag{6}$$

$$|a_4| \leq \frac{2(1-\alpha)(1+d)^2 + 2(1-\alpha)^2(5+3d-2\alpha)}{3.4^n (1-d^2)^3} \tag{7}$$

$$|a_5| \leq \frac{3(1-\alpha)(1+d)^3 + 11(1-\alpha)^2(1+d)^2 + 4(1-\alpha)^3(4+3d-\alpha)}{2.3.5^n (1-d^2)^4} \tag{8}$$

where $d = |w|$

Proof. Let us define

$$\frac{\frac{D^{n+1}f(z)}{D^n f(z)} - \alpha}{1-\alpha} = p(z) \tag{9}$$

From equation(9) we have

$$\frac{D^{n+1}f(z) - \alpha D^n f(z)}{(1-\alpha) D^n f(z)} = p(z) \tag{10}$$

which readily yields

$$D^{n+1}f(z) = \alpha D^n f(z) + (1-\alpha) D^n f(z) p(z) \tag{11}$$

On comparing the coefficients in equation (11) the results follows.

Remarks Putting $\alpha = 0$ and $n = 0$ in the results of Theorem 3.1 above, we obtain the coefficient bounds of Kanas and Ronning [2] immediately. With different choices of α and n different coefficients bounds can be obtained.

Theorem 3.2. *Let $f \in S_n^w(\alpha)$. Then,*

$$|a_3 - \mu a_2^2| \leq \frac{(1-\alpha)(1+d) + 2(1-\alpha)^2}{3^n (1-d^2)^2} - \frac{\mu (1-\alpha)^2}{2^{2(n-1)} (1-d^2)^2} \tag{12}$$

$$|a_2 a_4 - a_3^2| \leq \frac{(1-\alpha)^2(1+d)^2 + (1-\alpha)^3(5+3d-2\alpha)}{3.2^{3n-2}(1-d^2)^4} - \frac{(1-\alpha)^2(1+d)^2 + 4(1-\alpha)^4}{3^{2n}(1-d^2)^4} \tag{13}$$

where $\mu \geq 1$

Proof. The proof is immediate from Theorem 3.1

Theorem 3.3. *Let w be a fixed point in E and $f \in S_n^w(\alpha)$ and*

$$f(z) = (z - w) + \sum_{k=2}^{\infty} b_k (z - w)^k \tag{14}$$

with respect to function $g(z) \in S^*(w)$, where

$$g(z) = (z - w) + \sum_{k=2}^{\infty} a_k (z - w)^k \tag{15}$$

Then,

$$|b_k| \leq \frac{1}{k^{n+1}} \left[k^n |a_k| + \sum_{\rho=1}^{k-1} (k-\rho)^n |a_{k-\rho}| \frac{2(1-\alpha)}{(1+d)(1-d)^{k-\rho}} \right] \tag{16}$$

where $d = |w|, k \geq 2$ and $a_\rho = 1$

Proof. Let $f \in S_n^w(\alpha)$ with respect to the function $g \in S^*(w)$
Then there exists a function $p \in P(w)$ such that

$$\frac{\frac{D^{n+1}f(z)}{D^n g(z)} - \alpha}{1 - \alpha} = p(z) \tag{17}$$

where $p(z) = 1 + \sum_{k=1}^{\infty} B_k (z - w)^k$
Using the hypothesis through identification of $(z - w)^k$ coefficients, we have

$$k^{n+1} b_n = k^n a_k + \sum_{\rho=1}^{k-1} (1-\alpha)(k-\rho)^n a_{k-\rho} B_{k-\rho} \tag{18}$$

where a_1 and $k \geq 2$. From equation(18) we have the result.

Remarks If we use the estimates in Theorem 3.1, we obtain some estimates for the coefficients $b_k, k = 2, 3, 4, \ldots$. Also, at $\alpha = 0, n = 0$, we obtain the results of Acu and Owa [2].

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