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# Some Strong Forms of Semiseparated Sets and Semidisconnected Space

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**Abstract**

*The concept of semi-open sets in topological spaces was first introduced by Levine. Also the concept of θ-semi-open sets in topological spaces was introduced by Noiri, which is stronger than semi-open sets. Now, we introduce a new type of separated sets called θ-semiseparated sets, which is stronger than semiseparated sets due to Dube and Panwar, and we give some properties of it, furthermore we introduce a new type of disconnectedness interms of θ-semiseparated sets called θ-semidisconnected space, which is stronger than semi-disconnectedness due to Dorsett. Moreover, we give some characterizations and properties of it. It is shown that, a space X is θ-semiconnected if and only if every θs-continuous function from X to the discrete space {0, 1} is constant.*

**Keywords**: *θ-semi-open sets, θ-semiseparated sets and θ-semidisconnected.*

## 1 Introduction

The symbols X and Y represent topological spaces with no separation axioms assumed unless explicitly stated. Let S be a subset of X, the interior and closure of S are denoted by Int(S) and Cl(S), respectively. A subset S of X is said to be semi-open [8] if and only if $S \subset Cl(Int((S)))$. A subset S of X is said to be θ-semi-open set [10] if for each $x \in S$, there exists a semi-open set G in X such that $x \in G \subset$

$Cl(G) \subset S$. The complement of each semi-open (resp. θ-semi-open) sets is called semi-closed (resp. θ-semi-closed). A point x is said to be in the θ-semi-closure of a set S [5], denoted by $sCl_\theta(S)$, if $S \cap Cl(G) \neq \phi$, for each $G \in SO(X)$ containing x. If $S = sCl_\theta(S)$, then S is called θ-semi-closed. For each $G \in SO(X)$, Cl(G) is θ-semi-open and hence every regular closed set is θ-semi-open. Therefore, $x \in sCl_\theta(S)$ if and only if $S \cap E \neq \phi$, for each θ-semi-open set E containing x. A space X is said to be semi-disconnected [2] if there exist two semi-open sets A and B such that $X = A \cup B$ and $A \cap B = \phi$, otherwise it is called semi-connected. Two non-empty subsets A and B of a topological space X are said to be semiseparated [3] if and only if $A \cap sCl(B) = sCl(A) \cap B = \phi$. In a topological space X, a set which can be expressed as the union of two semiseparated sets is called a semi-disconnected space [3]. A function $f : X \to Y$ is said to be θs-continuous [7] if for each $x \in X$ and each open set B of Y containing $f(x)$, there exists a semi-open set U of X containing x such that $f(Cl(U)) \subset B$.

## 2 θ-Semiseparated Sets

In this section we introduce a new type of separated sets called θ-semiseparated sets, and some characterizations and properties of it will be given.
We start this section with the following definition.

**Definition 2.1** *Two non-empty subsets A and B of a topological space X are said to be θ-semiseparated if* $A \cap sCl_\theta(B) = sCl_\theta(A) \cap B = \phi$.

**Lemma 2.2** *Every θ-semiseparated sets is semiseparated.*

**Lemma 2.3** *Every two θ-semiseparated sets in topological spaces are disjoint.*

**Proof** Assume that A and B are two θ-semiseparated sets. Then, $A \cap sCl_\theta(B) = sCl_\theta(A) \cap B = \phi$ and hence $(A \cap sCl_\theta(B)) \cup (B \cap sCl_\theta(A)) = \phi$. By Theorem 1.2.2 of [1], $sCl_\theta(C) = C \cup \theta sd(C)$. Therefore, $(A \cap (B \cup \theta sd(B))) \cup (B \cap (A \cup \theta sd(A))) = \phi$. Then, $((A \cap B) \cup (A \cap \theta sd(B)) \cup ((B \cap A) \cup (B \cap \theta sd(A)) = \phi$. Thus, $A \cap B = \phi$.
The converse of the above two lemmas are not true ingeneral as it is shown in the following examples.

**Example 2.4** Let X = {a, b, c, d} and τ = {ϕ, X, {a, b}, {c}, {a, b, c}}. Let A = {a} and B = {c, d} be two subsets of (X, τ). Then, SO(X, τ) = {ϕ, X, {c}, {a, b}, {c, d}, {a, b, c}, {a, b, d}} and θSO(X, τ) = {ϕ, X, {c, d}, {a, b, d}}. Therefore, {a} and {c, d} are two semiseparated sets, but they are not θ-semiseparated sets since $\{a\} \cap sCl_\theta(\{c, d\}) = \{a\} \cap X \neq \phi$.

**Example 2.5** If we use the same topology (X, τ) in Example 2.4 and we take Y = {a, b} and W = {c, d} are two subsets of (X, τ). Then, Y and W are disjoint, but they are not θ-semiseparated sets.

**Proposition 2.6** *If A and B are two $\theta$-semiseparated subsets of a topological space X, $C \subset A$ and $D \subset B$, then C and D are also $\theta$-semiseparated.*

**Proof** It is obvious.

**Theorem 2.7** *Two $\theta$-semi-closed subsets A and B of a topological space X are $\theta$-semiseparated if and only if they are disjoint.*

**Proof** The first direction follows from Lemma 2.3 and the second direction it is obvious.

**Theorem 2.8** *Two $\theta$-semi-open subsets A and B of a topological space X are $\theta$-semiseparated if and only if they are disjoint.*

**Proof** The first direction follows from Lemma 2.3.
Conversely, assume that A and B are disjoint. Since A and B are two θ-semi-open sets, then (X \ A) and (X \ B) are θ-semi-closed. Therefore, $sCl_\theta$ (X \ A) = (X \ A) and $sCl_\theta$ (X \ B) = (X \ B). Since A and B are disjoint, then $A \subset (X \setminus B)$ and $B \subset (X \setminus A)$. Therefore, $sCl_\theta (A) \subset sCl_\theta (X \setminus B)$ and $sCl_\theta (B) \subset sCl_\theta (X \setminus A)$. This implies that, $sCl_\theta (A) \subset (X \setminus B)$ and $sCl_\theta (B) \subset (X \setminus A)$. So, $(sCl_\theta (A) \cap B) \subset ((X \setminus B) \cap B) = \phi$ and $(A \cap sCl_\theta (B)) \subset (A \cap (X \setminus A)) = \phi$. Therefore, $sCl_\theta (A) \cap B = A \cap sCl_\theta (B) = \phi$. Hence A and B are θ-semiseparated sets.

## 3 θ-Semidisconnectedness and θ-Semiconnectedness

In this section we introduce two new types of disconnected and connected spaces interms of θ-semiseparated sets called θ-semidisconnected and θ-semiconnected spaces, some characterizations and properties of them will be given.
We start this section with the following definition.

**Definition 3.1** *Let X be a topological space, a subset A of X is said to be $\theta$-semidisconnected if it is the union of non empty $\theta$-semiseparated sets, that is there exist two non empty sets B and C such that $B \cap sCl_\theta (C) = \phi$, $sCl_\theta (B) \cap C = \phi$ and $A = B \cup C$. Also, we say that A is $\theta$-semiconnected if it is not $\theta$-semidisconnected.*
*It is obvious that every $\theta$-semidisconnected space is semidisconnected. But the converse is not true ingeneral, as it is shown in the following example.*

**Example 3.2** Let X = {a, b, c, d}and τ = {ϕ, X, {a, b}, {c}, {a, b, c}}. Then, SO(X, τ) = {ϕ, X, {c}, {a, b}, {c, d}, {a, b, c}, a, b, d}} and θSO(X, τ) = {ϕ, X, {c, d}, {a, b, d}}. Therefore, X is semidisconnected, but it is not θ-semidisconnected.

We give some characterizations of θ-semidisconnected space.

**Theorem 3.3** *A topological space X is θ-semidisconnected if and only if there exists a non empty proper subset of X which is both θ-semi-open and θ-semi-closed in X.*

**Proof** Let X be a θ-semidisconnected, so there exist two non empty subsets A and B of X such that $A \cap sCl_\theta(B) = \phi$, $sCl_\theta(A) \cap B = \phi$ and $X = A \cup B$. Since $B \subset sCl_\theta(B)$. Then, $(A \cap B) \subset (A \cap sCl_\theta(B)) = \phi$. Therefore, $A \cap B = \phi$ and $A \cup B = X$, so $A = (X \setminus B)$ (B is a non empty and A is a proper subset of X) because if $A = X$, then $(X \setminus B) = X$, which implies that $B = \phi$, this is contradiction. Now, $A \cup B = X$ and $B \subset sCl_\theta(B)$, then $X = (A \cup B) \subset (A \cup sCl_\theta(B))$. But, always $(A \cup sCl_\theta(B)) \subset X$. So, $A \cup sCl_\theta(B) = X$. Since $A \cap sCl_\theta(B) = \phi$. Therefore, $A = (X \setminus sCl_\theta(B))$. Likewise, we can show $B = (X \setminus sCl_\theta(A))$. Since $sCl_\theta(A)$ and $sCl_\theta(B)$ are θ-semi-closed sets. Also, $A = (X \setminus sCl_\theta(B))$. Thus, A is θ-semi-open. Also, $B = (X \setminus sCl_\theta(A))$. Thus, B is also θ-semi-open, and since $A = (X \setminus B)$, then A is θ-semi-closed. So, A is the required non empty subset of X which is both θ-semi-open and θ-semi-closed (infact B is also a non empty proper subset of X, which is both θ-semi-open and θ-semi-closed).
Conversely, let A be a non empty proper subset of X, which is both θ-semi-open and θ-semi-closed and $B = (X \setminus A)$. Now, $A \cup B = (A \cup (X \setminus A)) = X$. Also, $A \cap B = A \cap (X \setminus A) = \phi$. Since A is θ-semi-closed. Therefore, $sCl_\theta(A) = A$. Also, A is θ-semi-open. Then, $(X \setminus A)$ is θ-semi-closed.This implies that B is θ-semi-closed. Therefore, $sCl_\theta(B) = B$. Hence, $A \cap B = A \cap sCl_\theta(B) = \phi$ and $sCl_\theta(A) \cap B = \phi$. So, X is θ-semidisconnected.
Recall that, a space X is said to be θs-disconnected [1] if there exist two θ-semi-open sets A and B such that $X = A \cup B$ and $A \cap B = \phi$. In this case, we call $A \cup B$ is called a θs-disconnection of X, otherwise X is called θs-connection.
The above definition is equivalent to the Definition 3.1 as it is shown in the following result.

**Theorem 3.4** *A topological space X is θ-semidisconnected if and only if one of the following statements hold:*

*(i) X is the union of two non empty disjoint θ-semi-open sets.*
*(ii) X is the union of two non empty disjoint θ-semi-closed sets.*

**Proof** (i) Let X be a θ-semidisconnected, so by Theorem 3.3, there exists a non-empty proper subset A of X which is both θ-semi-open and θ-semi-closed. So, $(X \setminus A)$ is also both θ-semi-open and θ-semi-closed. Thus, A and $(X \setminus A)$ are two θ-semi-open sets such that $A \cap (X \setminus A) = \phi$ and $A \cup (X \setminus A) = X$. So, X is the union of two non empty disjoint θ-semi-open sets A and $X \setminus A$ of X.
Conversely, let $X = A \cup B$ and $A \cap B = \phi$, where A and B are two non empty θ-semi-open subsets of X. We want to show that X is θ-semidisconnected. Since $A \cap B = \phi$ and $X = A \cup B$. Therefore, $A = (X \setminus B)$, so A is θ-semi-closed.

Thus, A is a non empty proper subset of X (if A is not proper, then A = X and hence B = $\phi$, this is contradiction). Hence, A is a non empty proper subset of X, which is both $\theta$-semi-open and $\theta$-semi-closed, so by Theorem 3.3, X is $\theta$-semidisconnected.
(ii) We can show the equivalence between $\theta$-semidisconnectedness of X and the condition gives in (ii) by the same way.

**Theorem 3.5** *Let X be a topological space. If A and B are two non empty $\theta$-semiseparated sets, then $A \cup B$ is $\theta$-semidisconnected.*

**Proof** Since A and B are $\theta$-semiseparated sets, then $A \cap sCl_\theta(B) = \phi$ and $sCl_\theta(A) \cap B = \phi$. Let $G = (X \setminus sCl_\theta(B))$ and $H = (X \setminus sCl_\theta(A))$. Then, G and H are $\theta$-semi-open and $(A \cup B) \cap G = A$ and $(A \cup B) \cap H = B$ are non empty disjoint set whose union is $A \cup B$. Thus, G and H form a $\theta$-semidisconnection of $A \cup B$ and so $A \cup B$ is $\theta$-semidisconnected.

**Theorem 3.6** *Let $G \cup H$ be a $\theta$-semidisconnection of A. Then, $A \cap G$ and $A \cap H$ are $\theta$-semiseparated sets.*

**Proof** Now, $A \cap G$ and $A \cap H$ are disjoint; hence we need only show that each set contains no $\theta$s-limit point of the other. Let p be a $\theta$s-limit point of $A \cap G$ and suppose $p \in (A \cap H)$. Then, H is a $\theta$-semi-open set containing p and so H contains a point of $A \cap G$ distinct from p, that is, $(A \cap G) \cap H \neq \phi$. But $(A \cap G) \cap (A \cap H) = \phi = (A \cap G) \cap H$. Then, $p \notin (A \cap H)$. Likewise, if p is a $\theta$s-limit point of $A \cap H$, then $p \notin (A \cap G)$. Thus, $A \cap G$ and $A \cap H$ are $\theta$-semiseparated sets.

**Theorem 3.7** *Let $G \cup H$ be a $\theta$-semidisconnection of A and let B be a $\theta$-semiconnected subset of A. Then, either $B \cap H = \phi$ or $B \cap G = \phi$, and so either $B \subset G$ or $B \subset H$.*

**Proof** Now, $B \subset A$, and so $A \subset (G \cup H)$. Then, $B \subset (G \cup H)$ and $(G \cap H) \subset (X \setminus A)$. Therefore, $(G \cap H) \subset (X \setminus B)$. Thus, if both $B \cap G$ and $B \cap H$ are non empty, then $G \cup H$ forms a $\theta$-semidisconnection of B. But B is $\theta$-semiconnected, hence the conclusion follows.

**Theorem 3.8** *Let X be a topological space. If A and B are $\theta$-semiconnected sets which are not $\theta$-semiseparated, then $A \cup B$ is $\theta$-semiconnected.*

**Proof** Let $A \cup B$ be $\theta$-semidisconnected and $G \cup H$ be a $\theta$-semidisconnection of $A \cup B$. Since A is a $\theta$-semiconnected subset of $A \cup B$. Therefore, by Theorem 3.7, either $A \subset G$ or $A \subset H$. Likewise, either $B \subset G$ or $B \subset H$. Now, if $A \subset G$ and $B \subset H$ (or $B \subset G$ and $A \subset H$), then by Theorem 3.6, $(A \cup B) \cap G = A$ and $(A \cup B) \cap H = B$ are $\theta$-semiseparated sets. This contradicts the hypothesis; hence $(A \cup B) \subset G$ or $(A \cup B) \subset H$, and so $G \cup H$ is not a $\theta$-semidisconnection of $A \cup B$. In other words, $A \cup B$ is $\theta$-semiconnected.

**Theorem 3.9** *Let X be a topological space. If* $\boldsymbol{A} = \{A_i\}$ *is a class of θ-semiconnected subsets of X such that no two members of* $\boldsymbol{A}$ *are θ-semiseparated. Then,* $B = \cup_i A_i$ *is θ-semiconnected.*

**Proof** Assume that B is not θ-semiconnected and $G \cup H$ is a θ-semidisconnection of B. Now, each $A_i \in \boldsymbol{A}$ is θ-semiconnected and so by Theorem 3.7, is contained in either G or H and disjoint from the other. Futhermore, any two members $A_{i1}$, $A_{i2} \in \boldsymbol{A}$ are not θ-semiseparated and so by Theorem 3.8, $A_{i1} \cup A_{i2}$ is θ-semiconnected; then $A_{i1} \cup A_{i2}$ is contained in G or H and disjoint from the other. Therefore, all the members of **A**, and hence $B = \cup_i A_i$, must be contained in either G or H and disjoint from the other. This is contradictions the fact that $G \cup H$ is a θ-semidisconnection of B; hence B is θ-semiconnected.

**Theorem 3.10** *Let* $\boldsymbol{A} = \{A_i\}$ *be a class of θ-semiconnected subsets of X with a non empty intersection. Then,* $B = \cup_i A_i$ *is θ-semiconnected.*

**Proof** *Since* $\cap_i A_i \neq \phi$*, any two members of* $\boldsymbol{A}$ *are not disjoint and so are not θ-semiseparated; hence by Theorem 3.9,* $B = \cup_i A_i$ *is θ-semiconnected.*

**Theorem 3.11** *Let X be a topological space. If A is θ-semiconnected subset of X and* $A \subset B \subset sCl_\theta(A)$*, then B is θ-semiconnected and hence, inparticular,* $sCl_\theta(A)$ *is θ-semiconnected.*

**Proof** Suppose that B is θ-semidisconnected and suppose $G \cup H$ is a θ-semidisconnected of B. Now, A is a θ-semiconnected subset of B and so, by Theorem 3.7, either $A \cap H = \phi$ or $A \cap G = \phi$; say, $A \cap H = \phi$. Then, $(X \setminus H)$ is a θ-semi-closed superset of A and therefore, $A \subset B \subset sCl_\theta(A) \subset (X \setminus H)$. Consequently, $B \cap H = \phi$. This is contradicts the fact that $G \cup H$ is a θ-semidisconnection of B; hence B is θ-semiconnected.

**Theorem 3.12** *A topological space X is θ-semidisconnected if and only if there exists a θs-continuous function f from X onto the discrete space {0, 1}.*

**Proof** Suppose that X is θ-semidisconnected. Then, there exist two non empty disjoint θ-semi-open subsets $G_1$ and $G_2$ of X such that $X = G_1 \cup G_2$. Define a function $f : X \to \{0, 1\}$ as follows

$$f(x) = \begin{cases} 0 & \text{if } x \in G_1 \\ 1 & \text{if } x \in G_2 \end{cases}$$

Now, the only open sets in {0, 1} are $\phi$, {0}, {1} and {0, 1}. So, $f^{-1}(\phi) = \phi$, $f^{-1}(\{0\}) = G_1$, $f^{-1}(\{1\}) = G_2$ and $f^{-1}(\{0, 1\}) = X$, which are θ-semi-open sets in X. Thus, $f$ is θs-continuous surjection from X to the discrete space {0, 1}.
Conversely, let the hypothesis holds and if possible suppose that X is θ-semiconnected. Therefore, by [6, Corollary 17], $f(X)$ is connected. Thus, {0, 1} is connected, which is contradiction since {0, 1} is discrete space and every discrete space which contain more than one point is disconnected. So, X must be θ-semidisconnected.
Finally, we prove the following theorem.

**Theorem 3.13** *A topological space X is θ-semiconnected if and only if every θs-continuous function from X to the discrete space {0, 1} is constant.*

**Proof** Let X be θ-semiconnected and $f : X \to \{0, 1\}$ any θs-continuous function. Let $y \in f(X) \subset \{0, 1\}$, then $\{y\} \subset \{0, 1\}$ and since {0, 1} is discrete, so {y} is both open and closed in {0, 1}. Since $f$ is θs-continuous. Therefore, by [7, Theorem 2.3], $f^{-1}(\{y\})$ is both θ-semi-open and θ-semi-closed in X. Now, since $y \in f(X)$. Therefore, there exists $x \in X$ such that $y = f(x)$. Thus, $f(x) \in \{y\}$ and $x \in f^{-1}(\{y\})$.Thus, we obtain $f^{-1}(\{y\}) \neq \phi$. If $f^{-1}(\{y\}) \neq X$, then $f^{-1}(\{y\})$ is a non empty subset of X which is both θ-semi-open and θ-semi-closed, which implies that X is θ-semidisconnected, this is a contradiction, so $f^{-1}(\{y\}) = X$. Thus, $f(X) = \{y\}$, it means that $f(x) = y$, for each $x \in X$, so $f$ is constant.
Conversely, let the hypothesis be holds; if possible suppose that X is a θ-semidisconnected. Therefore, by Theorem 3.3, X has a non-empty proper subset of X which is both θ-semi-open and θ-semi-closed. So, (X \ A) is also a non empty proper subset of X which is both θ-semi-open and θ-semi-closed. Now, consider the characteristic function $\psi_A$ of A defined as

$$\psi_A(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in (X \setminus A) \end{cases}$$

$\psi_A^{-1}(\phi) = \phi$, $\psi_A^{-1}(\{0\}) = (X \setminus A)$, $\psi_A^{-1}(\{1\}) = A$ and $\psi_A^{-1}(\{0, 1\}) = X$, which are all θ-semi-open sets in X. So, $\psi_A$ is θs-continuous function from X to the discrete space {0, 1}. By hypothesis, $\psi_A$ must be constant, this is contradiction since $\psi_A$ is not constant function. So, X is θ-semiconnected, which completes the proof.

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