Gen. Math. Notes, Vol. 16, No. 1, May, 2013, pp. 26-32
ISSN 2219-7184; Copyright © ICSRS Publication, 2013
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Fixed Point Theorems for Generalized Contraction Mappings in Complete Fuzzy 2-Metric Spaces

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(Received: 9-1-13 / Accepted: 2-3-13)

Abstract

In this paper we establish some fixed point theorems for Generalized Contraction mappings in complete fuzzy 2-metric space which are the extension of some known result of many authors.

Keywords: *Fuzzy 2-Metric Spaces, Sequence, Fixed Point.*

1 Introduction

In 1965, the concept of fuzzy set was introduced initially by Zadeh [11] since then many authors have expansively developed the theory of fuzzy sets and applications. Especially Deng [1], Ereeg [2], kaleva and seikkala [6], kramosil and Michalek [7] have introduced the concept of fuzzy metric spaces in different ways. Recently, many authors such as Fang[3], Grabiec [5], George and Veeramani [4], Mishra Sharma and Singh [8] established some fixed point theorems in fuzzy metric spaces. In 2007 Singh and Jain [10] has given the concept of fuzzy 2-

metric spaces. In the present paper we establish some results on common fixed points for generalized contraction mappings on fuzzy 2-metric spaces.

2 Preliminaries

We quote some definitions and statements of a few theorems which will be needed in the sequel.

Definition 2.1: *A fuzzy set A in X is function with domain X and values in [0, 1].*

Definition2.2: *A binary operation * : [0.1] × [0,1] × [0,1]→ [0,1] is called a t-norm of {[0,1],*} is an abelian topological monoid with unit 1 such that* $a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2$ *whenever* $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$ *for all* $a_1, a_2, b_1, b_2, c_1, c_2 \in [0,1]$.

Definition2.3: *The 3-triple (X, M,*) is said to be fuzzy 2- metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in* $X^3 \times [0, \infty)$ *satisfying the following conditions:*
For all x ,y ,z ∈ X and s ,t > 0

[FM-1] $M(x, y, z, 0) = 0$,

[FM-2] $M(x, y, z, t) = 1$ *for all t > 0 and when at least two of the three points are equal,*

[FM-3] $M(x, y, z, t) = M(y, x, z, t) = M(z, x, y, t)$ *symmetry about three variabls,*

[FM-4] $M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3) \leq M(x, y, z, t_1 + t_2 + t_3)$, $\forall$ $x, y, z, u \in X$ *and* $t_1, t_2, t_3 > 0$

[FM-5] $M(x, y, z, .) : [0,\infty) \to [0,1]$ *is left continuous ,*

[FM-6] $\lim_{t\to\infty} M(x, y, z, t) = 1$

The function value M (x ,y ,z ,t) may be interpreted as the probability that the area of triangle is less than t.

Definition2.4: *Let (X , M ,*) be a fuzzy 2-metric space.*

1. *A sequence* $\{x_n\}$ *in fuzzy 2-metric space X is said to be convergent to a point x ∈ X if*
   $\lim_{n\to\infty} M(x, y, z, t) = 1$ *for all a ∈ X and t > 0*
2. *A sequence* $\{x_n\}$ *in fuzzy 2-metric space X is called Cauchy sequence if*
   $\lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1$ *for all a ∈ X and t > 0, p > 0*
3. *A fuzzy 2-metric space in which every Cauchy sequence is convergent is said to complete*

Lemma 2.5: *[8] let $\{y_n\}$ be a sequence in a fuzzy 2- metric space (X , M ,*) with t * t ≥ t for all t ∈ [0,1] and the condition (FM-6). If there exists a number q ∈ (0,1) such that $M(y_{n+2}, y_{n+1}, w, qt) \geq M(y_{n+1}, y_n, w, t)$ for all t > 0 and n = 1,2…….. then $\{y_n\}$ is a Cauchy sequence in X.*

Lemma 2.6*: [8] If for all x , y ∈ X , t > 0 and for some a number q ∈ (0,1)*

$M(x, y, w, qt) \geq M(x, y, w, t)$ then x = y.

3 Main Result

Theorem 3.1: *Let T be a self mapping from a complete fuzzy metric space X into itself such that*

$$M^2(Tx,Ty,w,t) \geq M^2\left(x,y,w,\frac{t_1}{ka}\right) + M\left(x,Tx,w,\frac{t_2}{kb}\right).M\left(x,Tx,w,\frac{t_2}{kb}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (1)$$

For all x , y ,w ∈ X , where $t = t_1 + t_2$, a + b = 1 and 0 < k < 1. Then T has a unique fixed point.

Proof: Let $x_0 \in X$ be any arbitrary fixed element. We construct a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for n = 0, 1, 2…

Putting $x = x_{n-1}$, $y = x_n$ and $t_1 = at$, $t_2 = bt$ in (1) we have

$$\begin{aligned} M^2(x_n,x_{n+1},w,t) &\geq M^2\left(x_n,x_{n-1},w,\frac{t}{k}\right) + M\left(x_{n-1},x_n,w,\frac{t}{k}\right).M\left(x_n,x_{n+1},w,\frac{t}{k}\right) \\ &\geq M^2\left(x_n,x_{n-1},w,\frac{t}{k}\right) \\ &\quad + M\left(x_{n-1},x_n,w,\frac{t}{k}\right).M(x_n,x_{n+1},w,t) \; for\ all\ t > 0 \end{aligned}$$

Dividing by $M^2\left(x_{n-1},x_n,w,\frac{t}{k}\right)$ on both side and putting $r = \frac{M(x_n x_{n+1},w,t)}{M\left(x_{n-1} x_n, w \frac{t}{k}\right)}$ we get

$r^2 = 1 + r$, which implies $r^2 - r - 1 \geq 0$

Suppose r < 1, then

$$r^2 - r - 1 < 0 \; (since\ r > 0) \; which\ is\ contradiction\ to\ r^2 - r - 1 \geq 0$$
$$thus\ r \geq 1$$

Therefore

$$\frac{M(x_n x_{n+1} w,t)}{M\left(x_{n-1} x_n w \frac{t}{k}\right)} \geq 1 \ hence\ M(x_n, x_{n+1}, w, t) \geq M\left(x_{n-1}, x_n, w, \frac{t}{k}\right)\ for\ all\ n\ and\ for\ all\ t > 0$$

By lemma (2.5) $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete fuzzy metric space, $\{x_n\} \rightarrow x$ in X.

Now we prove x is a fixed point of T.

Now consider,

$$\begin{aligned} M^2(x,Tx,w,t) &= \lim_{n\to\infty} M^2(Tx_n,Tx,w,t) \\ &\geq \lim_{n\to\infty}\left[M^2\left(x_n,x,w,\frac{t}{k}\right) + M\left(x,Tx,w,\frac{t}{k}\right) M\left(x_n,x_{n+1},w,\frac{t}{k}\right)\right] \\ &= 1 + M\left(x,Tx,w,\frac{t}{k}\right) \end{aligned}$$

Thus

$M^2(x,Tx,w,t) \geq 1$.$Hence\ M(x,Tx,w,t) = 1\ for\ all\ t > 0. therefore\ Tx = x.$

Uniqueness: Suppose there exist $y \in X$ such that $Ty = y$

Now consider,

$$\begin{aligned} M^2(x,y,w,t) &= M^2(Tx,Ty,w,t) \\ &\geq M^2\left(x,y,w,\frac{t}{k}\right) + M\left(x,x,w,\frac{t}{k}\right) M\left(y,y,w,\frac{t}{k}\right) \\ &= M^2\left(x,y,w,\frac{t}{k}\right) + 1 \\ &\geq M^2\left(x,y,w,\frac{t}{k}\right) \end{aligned}$$

$$hence\ M(x,y,w,t) \geq M\left(x,y,w,\frac{t}{k}\right)$$

Hence by lemma (2.6) x = y this completes the proof.

Remark: Putting $t_2 = 0$ and t_1 = at in theorem (3.1) we get the following theorem as corollary.

Corollary 3.2: *Let T be a mapping from a complete fuzzy 2-metric space X in to itself such that* $M(Tx,Ty,w,t) \geq M\left(x,y,w,\frac{t}{k}\right)$ *for all x , y, w* $\in$ *X , t* ≥ 0 *and 0 < k < 1. Then T has a unique fixed point.*

Remark: Putting $t_1 = 0$ and $t_2 = bt$ in theorem (3.1) we get the following theorem as corollary.

Corollary 3.3: *Let T be a mapping from a complete fuzzy 2-metric space X in to itself such that* $M^2(Tx,Ty,w,t) \geq M\left(x,Tx,w,\frac{t}{k}\right)M\left(y,Ty,w,\frac{t}{k}\right)$ *for all x , y, w* $\in X$ *, t ≥ 0 and 0 < k < 1. Then T has a unique fixed point.*

Theorem 3.4: *Let T be a mapping from a complete fuzzy metric space X in to itself such that*

$$M^2(Tx,Ty,w,t) \geq M^2\left(x,y,w,\frac{t_1}{ka}\right) + M\left(x,Tx,w,\frac{t_2}{kb}\right).M\left(y,Ty,w,\frac{t_2}{kb}\right) + M\left(y,Tx,w,\frac{t_3}{kc}\right).M\left(x,Ty,w,\frac{t_3}{kc}\right)$$

For all x, y, w $\in X$*, where* $t = t_1 + t_2 + t_3$ *, a + b + 2c = 1 and 0 < k < 1, then T has a unique fixed point.*

Proof: Let $x_0 \in X$ be any arbitrary fixed element. We construct a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for n = 0, 1, 2…

Putting $x = x_{n-1}$, $y = x_n$ and $t_1 = at$, $t_2 = bt$ and $t_3 = 2ct$ in (2) we have

$$M^2(x_n,x_{n+1},w,t) \geq M^2\left(x_n,x_{n-1},w,\frac{t}{k}\right) + M\left(x_{n-1},x_n,w,\frac{t}{k}\right)M\left(x_n,x_{n+1},w,\frac{t}{k}\right) + M\left(x_n,x_n,w,\frac{t}{k}\right)\ M\left(x_{n-1},x_{n+1},w,\frac{2t}{k}\right)$$

$$\geq M^2\left(x_n,x_{n-1},w,\frac{t}{k}\right) + M\left(x_{n-1},x_n,w,\frac{t}{k}\right)M\left(x_n,x_{n+1},w,\frac{t}{k}\right) + M\left(x_{n-1},x_n,w,\frac{t}{k}\right)\ M(x_n,x_{n+1},w,\frac{t}{k})$$

$$\geq M^2\left(x_n,x_{n-1},w,\frac{t}{k}\right) + 2M\left(x_{n-1},x_n,w,\frac{t}{k}\right)M\left(x_n,x_{n+1},w,\frac{t}{k}\right)$$

For all t > 0

Dividing by $M^2\left(x_{n-1},x_n,w,\frac{t}{k}\right)$ *on both sides and putting* $r = \frac{M(x_n,x_{n+1},w,t)}{M\left(x_{n-1},x_n,w,\frac{t}{k}\right)}$

We get, $r^2 > 1 + 2r$ *which implies* $r^2 - 2r - 1 \geq 0$

Suppose
$r < 1, then\ r^2 - 2r - 1 < 0\ (since\ r > 0)\ , which\ is\ contradiction\ to\ r^2 - 2r - 1 \geq 0$

Thus $r \geq 1. Hence\ M(x_n, x_{n+1}, w, t) \geq$

$M\left(x_{n-1}, x_n, w, \frac{t}{k}\right)\ for\ all\ n\ and\ for\ all\ t > 0$

By Lemma 2.5 $\{x_n\}$ is a Cauchy sequence in X. Since X is a Complete fuzzy metric space, $\{x_n\} \rightarrow x$ in X.

Now we prove that x is a fixed point for T. Now consider

$$M^2(x, Tx, w, t) = \lim_{n\to\infty} M^2(x_{n+1}, Tx, w, t)$$

$$= \lim_{n\to\infty} M^2(Tx_n, Tx, w, t)$$

$$\geq \lim_{n\to\infty} M^2\left(x_n, x, w, \frac{t}{k}\right) + M\left(x, Tx, w, \frac{t}{k}\right) M\left(x_n, x_{n+1}, w, \frac{t}{k}\right)$$

$$+ M\left(x_n, Tx, w, \frac{t}{k}\right)\ M(x_n, x_{n+1}, w, \frac{2t}{k})$$

$$= 1 + 2M(x, Tx, w, \frac{t}{k})$$

$$= 1 + 2M(x, Tx, w, t)$$

$Hence\ M^2(x, Tx, w, t) \geq 1\ for\ all\ t > 0. which\ implies\ M(x, Tx, w, t) = 1$

Thux Tx = x

Uniqueness: Suppose there exist $y \in X$ such that Ty = y

Now consider,

$$M^2(x, y, w, t) = M^2(Tx, Ty, w, t)$$
$$\geq M^2\left(x, y, w, \frac{t}{k}\right) + M\left(x, x, w, \frac{t}{k}\right) M(y, y, w, \frac{t}{k}) +$$
$$M\left(x, y, w, \frac{2t}{k}\right) M(y, x, w, \frac{t}{k})$$

Hence $M(x, y, w, t) \geq M(x, y, w, \frac{t}{k})$
Hence by lemma (2.6) x = y .This completes the proof.

Remark: Putting $t_2 = 0$, $t_3 = 0$ and t_1 = at in theorem (3.4) we get the following theorem as corollary.

Corollary 3.5: *Let T be a mapping from a complete fuzzy metric spaces X in to itself such that*
$M(Tx,Ty,w,t) \geq M(x,y,w,\frac{t}{k})$ for all x , y , w ∈ X , t ≥ 0 and 0 < k < 1 . thus T has a unique fixed point.

Remark: Putting $t_1 = 0$, $t_3 = 0$ and $t_2 = bt$ theorem (3.4) we get the following theorem as corollary.

Collollary 3.6: *Let T be a mapping from a complete fuzzy metric spaces X in to itself such that*
$M^2(Tx,Ty,w,t) \geq M(x,Tx,w,\frac{t}{k})M(y,Ty,w,\frac{t}{k})$ for all x , y , w ∈ X , t ≥ 0 and 0 < k < 1 . thus T has a unique fixed point.

Remark: Putting $t_1 = 0$, $t_2 = 0$ and $t_3 = 2ct$ in theorem (3.4) we get the following theorem as corollary.

Collollary 3.7: *Let T be a mapping from a complete fuzzy metric spaces X in to itself such that*
$M^2(Tx,Ty,w,t) \geq M\left(x,Ty,w,\frac{t}{k}\right)M(y,Tx,w,\frac{2t}{k})$ for all x , y , w ∈ X , t ≥ 0 and 0 < k < 1 . Thus T has a unique fixed point.

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