Gen. Math. Notes, Vol. 21, No. 1, March 2014, pp.43-51
ISSN 2219-7184; Copyright ©ICSRS Publication, 2014
www.i-csrs.org
Available free online at http://www.geman.in

Spectrum of One Sturm-Liouville Type Problem on Two Disjoint Intervals

K. Aydemir[1] and O. Sh. Mukhtarov[2]

[1,2]Department of Mathematics, Faculty of Arts and Science
Gaziosmanpaşa University
[2]Institute of Mathematics and Mechanics, Azerbaijan
National Academy of Sciences, Baku, Azerbaijan
[1]E-mail: kadriye.aydemir@gop.edu.tr
[2]E-mail: omukhtarov@yahoo.com

(Received: 17-1-14 / Accepted: 27-2-14)

Abstract

In this study by modifying some techniques of classical Sturm-Liouville theory and suggesting own approaches we investigate some spectral properties of one Sturm-Liouville type problem on two disjoint intervals.

Keywords: *Sturm-Liouville problems, eigenvalue, eigenfunction, asymptotics of eigenvalues and eigenfunction.*

1 Introduction

Many interesting applications of mathematical physics require investigation of eigenvalues and eigenfunction of boundary value problems. For instance, physical and chemical properties of a surface are determined by its surface structure. Two basic questions addressed are: (1) Determine the atomic structure at the surfaces of such materials; and (2) Determine basic physical characteristics such as how the surface will react with various chemicals. Pandey [2] obtains this type of information by computing eigenfunctions of Schrodingers equation. Using these computations, he has shown that the accepted buckling reconstruction mechanism for the configuration of atoms at surfaces is valid only for heteropolar surfaces. Kerner [4] addresses the question of the stability of plasmas which are confined magnetically. Such plasmas play a key role in the research on controlled nuclear fusion. This is an application where large scale eigenvalue and eigenvector computations provide new insight into basic physical behavior. The most dangerous instabilities in a plasma are macroscopic in nature and can be described by the basic resistive magnetohydrodynamic model. A well-chosen discretization of this model transforms this model into generalized eigenvalue problems: $Ax = \lambda Bx$ where A-is a general matrix and B is a real symmetric and positive definite matrix. The eigenvalues and eigenvectors of these systems provide knowledge about the behavior of the plasma. Haller and Koppel [3] describe applications where eigenvalue and eigenvector computations are used to obtain basic physical properties of molecular systems and the matrices involved are complex symmetric. Electric power systems problems yield some of the most difficult nonsymmetric eigenvalue and eigenvector problems.

In this study we shall investigate one Sturm-Liouville type problem which consist of the equation

$$-y''(x,\lambda) + q(x)y(x,\lambda) = \lambda y(x,\lambda) \tag{1}$$

to hold on two disjoint intervals $[-\pi, 0)$ and $(0, \pi]$, where discontinuity in y and y' at the interface point $x = 0$ are prescribed by transmission conditions

$$a_1 y'(0+,\lambda) + a_2 y(0+,\lambda) + a_3 y'(0-,\lambda) + a_4 y(0-,\lambda) = 0, \tag{2}$$

$$b_1 y'(0+,\lambda) + b_2 y(0+,\lambda) + b_3 y'(0-,\lambda) + b_4 y(0-,\lambda) = 0, \tag{3}$$

together with the boundary conditions

$$\cos\alpha y(-\pi,\lambda) + \sin\alpha y'(-\pi,\lambda) = 0, \tag{4}$$

$$\cos\beta y(\pi,\lambda) + \sin\beta y'(\pi,\lambda) = 0. \tag{5}$$

where the potential $q(x)$ is real-valued continuous function in each of the intervals $[-\pi, 0)$ and $(0, \pi]$, and has a finite limits $q(c \mp 0)$, λ is a complex parameter, a_i and b_i $(i = 1, 2, 3, 4)$ are real numbers. These boundary conditions are of great importance for theoretical and applied studies and have a definite mechanical or physical meaning (for instance, of free ends). Also the problems with transmission conditions arise in mechanics, such as thermal conduction problems for a thin laminated plate, which studied in [6].

2 The Fundamental Solutions

Denote the determinant of the k-th and j-th columns of the matrix

$$H = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

by ρ_{kj} $(1 \leq k < j \leq 4)$. Note that throughout this study we shall assume that $\rho_{12} > 0$ and $\rho_{34} > 0$. With a view to constructing the characteristic function we define two solutions $\phi_1(x,\lambda)$, $\chi_1(x,\lambda)$ on left interval $[-\pi,0)$ and two solutions $\phi_2(x,\lambda)$, $\chi_2(x,\lambda)$ on right interval $(0,\pi]$ as follows. Denote by $\phi_1(x,\lambda)$ and $\chi_2(x,\lambda)$ the solutions of the equation (1) satisfying the initial conditions

$$y(-\pi,\lambda) = \sin\alpha,\ y'(-\pi,\lambda) = -\cos\alpha \tag{6}$$

and

$$y(\pi,\lambda) = -\sin\beta,\ y'(\pi,\lambda) = \cos\beta \tag{7}$$

respectively. It is known that these solutions are entire functions of $\lambda \in C$ for each fixed $x \in [-\pi,0)$ and $x \in (0,\pi]$ respectively (see, for example, [7]). Now, denote by $\phi_2(x,\lambda)$ and $\chi_1(x,\lambda)$ the solutions of the equation (1) on $(0,\pi]$ and $[-\pi,0)$ satisfying the initial conditions

$$y(0+,\lambda) = \frac{1}{\rho_{12}}(\rho_{23}\phi_1(0-,\lambda) + \rho_{24}\frac{\partial\phi_1(0-,\lambda)}{\partial x}) \tag{8}$$

$$y'(0+,\lambda) = \frac{-1}{\rho_{12}}(\rho_{13}\phi_1(0-,\lambda) + \rho_{14}\frac{\partial\phi_1(0-,\lambda)}{\partial x}). \tag{9}$$

and

$$y(0-,\lambda) = \frac{-1}{\rho_{34}}(\rho_{14}\chi_2(0+,\lambda) + \rho_{24}\frac{\partial\chi_2(0+,\lambda)}{\partial x}), \tag{10}$$

$$y'(0-,\lambda) = \frac{1}{\rho_{34}}(\rho_{13}\chi_2(0+,\lambda) + \rho_{23}\frac{\partial\chi_2(0+,\lambda)}{\partial x}). \tag{11}$$

respectively. By applying the method used in [1] we can prove that these solutions are entire functions of spectral parameter $\lambda \in C$ for each fixed x.

3 Asymptotic Approximation Formulas for Fundamental Solutions

By applying the method of variation of parameters we can prove that the next integral and integro-differential equations are hold for $k = 0$ and $k = 1$.

$$\begin{aligned}\frac{d^k}{dx^k}\phi_1(x,\lambda) &= \sin\alpha\frac{d^k}{dx^k}\cos\left[s\left(x+\pi\right)\right] - \frac{\cos\alpha}{s}\frac{d^k}{dx^k}\sin\left[s\left(x+\pi\right)\right] \\ &+\frac{1}{s}\int\limits_{-\pi}^{x}\frac{d^k}{dx^k}\sin\left[s\left(x-z\right)\right]q(z)\phi_1(z,\lambda)dz \end{aligned}\tag{12}$$

$$\begin{aligned}\frac{d^k}{dx^k}\chi_1(x,\lambda) &= -\frac{1}{\rho_{34}}(\rho_{14}\chi_2(0+,\lambda)+\rho_{24}\frac{\partial\chi_2(0+,\lambda)}{\partial x})\frac{d^k}{dx^k}\cos[sx] \\ &+\frac{1}{s\rho_{34}}(\rho_{13}\chi_2(0+,\lambda)+\rho_{23}\frac{\partial\chi_2(0+,\lambda)}{\partial x})\frac{d^k}{dx^k}\sin[sx] \\ &+\frac{1}{s}\int_x^0\frac{d^k}{dx^k}\sin[s(x-z)]\,q(z)\chi_1(z,\lambda)dz \qquad (13)\end{aligned}$$

for $x\in[-\pi,0)$ and

$$\begin{aligned}\frac{d^k}{dx^k}\phi_2(x,\lambda) &= \frac{1}{\rho_{12}}(\rho_{23}\phi_1(0-,\lambda)+\rho_{24}\frac{\partial\phi_1(0-,\lambda)}{\partial x})\frac{d^k}{dx^k}\cos[sx] \\ &-\frac{1}{s\rho_{12}}(\rho_{13}\phi_1(0-,\lambda)+\rho_{14}\frac{\partial\phi_1(0-,\lambda)}{\partial x})\frac{d^k}{dx^k}\sin[sx] \\ &+\frac{1}{s}\int_0^x\frac{d^k}{dx^k}\sin[s(x-z)]\,q(z)\phi_2(z,\lambda)dz \qquad (14)\\ \frac{d^k}{dx^k}\chi_2(x,\lambda) &= -\sin\beta\frac{d^k}{dx^k}\cos[s(\pi-x)]-\frac{\cos\beta}{s}\frac{d^k}{dx^k}\sin[s(\pi-x)] \\ &+\frac{1}{s}\int_x^\pi\frac{d^k}{dx^k}\sin[s(x-z)]\,q(z)\chi_2(z,\lambda)dz \qquad (15)\end{aligned}$$

for $x\in(0,\pi]$.

Lemma 3.1 *Let $\lambda=s^2$, $Ims=t$. Then if $\sin\alpha\neq 0$*

$$\begin{aligned}\frac{d^k}{dx^k}\phi_1(x,\lambda) &= \sin\alpha\frac{d^k}{dx^k}\cos[s(x+\pi)]+O\left(|s|^{k-1}e^{|t|(x+\pi)}\right) \qquad (16)\\ \frac{d^k}{dx^k}\phi_2(x,\lambda) &= \frac{\rho_{24}}{\rho_{12}}\sin\alpha s\sin[s\pi]\frac{d^k}{dx^k}\cos[sx]+O\left(|s|^k e^{|t|(x+\pi)}\right) \qquad (17)\end{aligned}$$

as $|\lambda|\to\infty$, while if $\sin\alpha=0$

$$\begin{aligned}\frac{d^k}{dx^k}\phi_1(x,\lambda) &= -\frac{\cos\alpha}{s}\frac{d^k}{dx^k}\sin[s(x+\pi)]+O\left(|s|^{k-2}e^{|t|(x+\pi)}\right) \qquad (18)\\ \frac{d^k}{dx^k}\phi_2(x,\lambda) &= -\frac{\rho_{24}}{\rho_{12}}\cos\alpha\cos[s\pi]\frac{d^k}{dx^k}\cos[sx]+O\left(|s|^{k-1}e^{|t|(x+\pi)}\right) \qquad (19)\end{aligned}$$

as $|\lambda|\to\infty$ $(k=0,1)$. Each of these asymptotic equalities hold uniformly for x.

Proof. Multiplying both side of (12) by $e^{-|t|(x+\pi)}$ and denoting $\widetilde{\phi}(x):=\max_{x\in[-\pi,0)}|e^{-|t|(x+\pi)}\phi_1(x,\lambda)|$ we can derive that $\widetilde{\phi}(\lambda)=O(1)$ as $|\lambda|\to\infty$. Consequently $\phi_1(x,\lambda)=O(e^{|t|(x+\pi)})$ as $|\lambda|\to\infty$. Substituting this asymptotic expression of $\phi_1(x,\lambda)$ in the integral term of the right hand of (12) we get (16) for $k=0$. Other formulas are derived similarly.

Similarly we can prove the next Lemma.

Lemma 3.2 *Let* $\lambda = s^2$, $Ims = t$. *Then* *if* $\sin\beta \neq 0$

$$\frac{d^k}{dx^k}\chi_2(x,\lambda) = \sin\beta \frac{d^k}{dx^k}\cos\left[s\left(\pi - x\right)\right] + O\left(|s|^{k-1} e^{|t|(\pi-x)}\right) \tag{20}$$

$$\frac{d^k}{dx^k}\chi_1(x,\lambda) = -\frac{\rho_{24}}{\rho_{34}}\sin\beta s \sin\left[s\pi\right] \frac{d^k}{dx^k}\cos\left[sx\right] + O\left(|s|^{k} e^{|t|(\pi-x)}\right) \tag{21}$$

as $|\lambda| \to \infty$, *while* *if* $\sin\beta = 0$

$$\frac{d^k}{dx^k}\chi_2(x,\lambda) = -\frac{\cos\beta}{s}\frac{d^k}{dx^k}\sin\left[s(\pi - x)\right] + O\left(|s|^{k-2} e^{|t|(\pi-x)}\right) \tag{22}$$

$$\frac{d^k}{dx^k}\chi_1(x,\lambda) = -\frac{\rho_{24}}{\rho_{34}}\cos\beta \cos\left[s\pi\right] \frac{d^k}{dx^k}\cos\left[sx\right] + O\left(|s|^{k-1} e^{|t|(\pi-x)}\right) \tag{23}$$

as $|\lambda| \to \infty$ $(k = 0, 1)$. *Each of these asymptotic equalities holds uniformly for* x.

4 The Eigenvalues

It is well-known from ordinary differential equation theory that the Wronskians $w_1(\lambda) := W[\phi_1(x,\lambda), \chi_1(x,\lambda)]$ and $w_2(\lambda) := W[\phi_2(x,\lambda), \chi_2(x,\lambda)]$ are independent of variable x. By using (8)-(11) we have

$$\begin{aligned} w_1(\lambda) &= \phi_1(0-,\lambda)\frac{\partial \chi_1(0-,\lambda)}{\partial x} - \chi_1(0-,\lambda)\frac{\partial \phi_1(0-,\lambda)}{\partial x} \\ &= \frac{\rho_{12}}{\rho_{34}}(\phi_2(0+,\lambda)\frac{\partial \chi_2(0+,\lambda)}{\partial x} - \chi_2(0+,\lambda)\frac{\partial \phi_2(0+,\lambda)}{\partial x}) \\ &= \frac{\rho_{12}}{\rho_{34}} w_2(\lambda) \end{aligned}$$

for each $\lambda \in C$. It is convenient to introduce the notation

$$w(\lambda) := \rho_{34} w_1(\lambda) = \rho_{12} w_2(\lambda). \tag{24}$$

Slightly modifying the standard method we prove that all eigenvalues of the problem $(1) - (5)$ are real.

Theorem 4.1 *The eigenvalues of the boundary-value-transmission problem* $(1) - (5)$ *are real.*

Proof. Let λ_0 be any eigenvalue and $y_0(x)$ be eigenfunction corresponding to this eigenvalue. By applying the well-known Lagrange's identity [7] we obtain that

$$\begin{aligned}
& \rho_{34}\int_{-\pi}^{0}(\lambda_0 y_0(x))\overline{y_0(x)}dx+\rho_{12}\int_{0}^{\pi}(\lambda_0 y_0(x))\overline{y_0(x)}dx \\
= \ & \rho_{34}\int_{-\pi}^{0}(-y_0''(x)+q(x)y_0(x))\overline{y_0(x)}dx+\rho_{12}\int_{0}^{\pi}(-y_0''(x)+q(x)y_0(x))\overline{y_0(x)}dx \\
= \ & \{\rho_{34}\int_{-\pi}^{0} y_0(x)\overline{\lambda_0 y_0(x)}dx+\rho_{12}\int_{0}^{\pi} y_0(x)\overline{\lambda_0 y_0(x)}dx\} \\
+ \ & \rho_{34}W[y_0,\overline{y_0};0-]-\rho_{34}\ W[y_0,\overline{y_0};-\pi]+\rho_{12}W[y_0,\overline{y_0};\pi] \\
- \ & \rho_{12}W[y_0,\overline{y_0};0+] \qquad (25)
\end{aligned}$$

Since the eigenfunction $y_0(x)$ is satisfied the boundary and transmission conditions $(2)-(5)$ it is easy to verify that

$$W(y_0,\overline{y_0};-\pi)=W(y_0,\overline{y_0};\pi)=0 \qquad (26)$$

$$W(y_0,\overline{y_0};0-)=\tfrac{\rho_{12}}{\rho_{34}}\ W(y_0,\overline{y_0};0+). \qquad (27)$$

By substituting these equations in (25) we have

$$(\lambda_0-\overline{\lambda_0})[\rho_{34}\int_{-\pi}^{0}(y_0(x))^2dx \quad + \quad \rho_{12}\int_{0}^{\pi}(y_0(x))^2dx]=0$$

Since $\rho_{12}>0$ and $\rho_{34}>0$ we get $\lambda_0=\overline{\lambda_0}$. Consequently all eigenvalues of the problem $(1)-(5)$ are real. The proof is complete.

Corollary 4.2 *Let $u(x)$ and $v(x)$ be eigenfunctions corresponding to distinct eigenvalues. Then they are orthogonal in the sense of the following equality*

$$\rho_{34}\int_{-\pi}^{0}u(x)v(x)dx+\rho_{12}\int_{0}^{\pi}u(x)v(x)dx=0. \qquad (28)$$

Theorem 4.3 *The geometric multiplicity of each eigenvalue of the problem* $(1)-(5)$ *(i.e. the maximal number of linearly independent eigenfunctions corresponding to this eigenvalue) is one.*

Proof. Let u_1 and u_2 are two eigenfunctions for the same eigenvalue λ_0 . From the boundary condition (4) it follows that $u_1(-\pi)u_2'(-\pi)-u_1'(-\pi)u_2(-\pi)=0$. Consequently $u_1(-\pi)=ku_2(-\pi)$ and $u_1'(-\pi)=ku_2'(-\pi)$ for some $k\in R$ $(k\neq 0)$. By the uniqueness theorem for solutions of ordinary differential equation we have that $u_1(x) = ku_2(x)$ for all $x \in [-\pi, 0)$. Similarly we deduce that $u_1 = \ell u_2$ on $(0, \pi]$ for some real $\ell \neq 0$. Substituting u_1 and u_2 in the transmission conditions (2)-(3) we see that $k = \ell$, i.e. u_1 and u_2 are linearly dependent on whole $[-\pi, 0) \cup (0, \pi]$. Thus, the geometric multiplicity of λ_0 is one. The proof is complete.

5 Asymptotic Behaviour of Eigenvalues and Eigenfunctions

Since the Wronskians of $\phi_2(x, \lambda)$ and $\chi_2(x, \lambda)$ are independent of x, in particular, by putting $x = \pi$ we have

$$\begin{aligned} w(\lambda) &= \phi_2(\pi, \lambda)\chi_2'(\pi, \lambda) - \phi_2'(\pi, \lambda)\chi_2(\pi, \lambda) \\ &= \cos\beta\phi_2(\pi, \lambda) + \sin\beta\phi_2'(\pi, \lambda). \end{aligned} \tag{29}$$

Let $\lambda = s^2$, $Ims = t$. By substituting (16)-(19) in (29) we obtain easily the following asymptotic representations

(i) If $\sin\beta \neq 0$ and $\sin\alpha \neq 0$, then

$$w(\lambda) = -\frac{\rho_{24}}{\rho_{12}} \sin\alpha \sin\beta\ s^2 \sin^2[s\pi] + O\left(|s|\, e^{2\pi|t|}\right) \tag{30}$$

(ii) If $\sin\beta \neq 0$ and $\sin\alpha = 0$, then

$$w(\lambda) = \frac{\rho_{24}}{\rho_{12}} \cos\alpha \sin\beta\ s\cos[s\pi]\sin[s\pi] + O\left(e^{2\pi|t|}\right) \tag{31}$$

(iii) If $\sin\beta = 0$ and $\sin\alpha \neq 0$, then

$$w(\lambda) = \frac{\rho_{24}}{\rho_{12}} \sin\alpha \cos\beta\ s\sin[s\pi]\cos[s\pi] + O\left(e^{2\pi|t|}\right) \tag{32}$$

(iv) If $\sin\beta = 0$ and $\sin\alpha = 0$, then

$$w(\lambda) = -\frac{\rho_{24}}{\rho_{12}} \cos\beta \cos\alpha \cos^2[s\pi] + O\left(\frac{1}{|s|} e^{2\pi|t|}\right) \tag{33}$$

Now we are ready to derive the needed asymptotic formulas for eigenvalues and eigenfunctions.

Theorem 5.1 *The boundary-value-transmission problem* (1)-(5) *has an precisely numerable many real eigenvalues, whose behavior may be expressed by* $\{\lambda_n\}$ *with following asymptotic as* $n \to \infty$

(i) *If* $\sin\beta \neq 0$ *and* $\sin\alpha \neq 0$*,then*

$$s_n = (\frac{n-1}{2}) + O\left(\frac{1}{n}\right) \tag{34}$$

(ii) *If* $\sin\beta \neq 0$ *and* $\sin\alpha = 0$, *then*

$$s_n = \frac{n}{2} + O\left(\frac{1}{n}\right), \tag{35}$$

(iii) *If* $\sin\beta = 0$ *and* $\sin\alpha \neq 0$, *then*

$$s_n = \frac{n}{2} + O\left(\frac{1}{n}\right), \tag{36}$$

(iv) *If* $\sin\beta = 0$ *and* $\sin\alpha = 0$, *then*

$$s_n = \frac{n+1}{2} + O\left(\frac{1}{n}\right), \tag{37}$$

where $\lambda_n = s_n^2$.

Proof. By applying the well-known Rouche theorem (see, [5]) we can show that $\omega(\lambda)$ has the same number of zeros inside the appropriate large contours as the leading term

$$\omega_0(\lambda) = -\frac{\rho_{24}}{\rho_{12}} \sin\alpha \sin\beta \; s^2 \sin^2[s\pi]$$

provided that each zero is counted according to its multiplicity. Consequently, if $\lambda_n = s_n^2$ are zeros of $\omega(\lambda)$, which numbered as $\lambda_1 \leq \lambda_2 \leq \lambda_3 ...$ we have

$$s_n = (\frac{n-1}{2}) + \delta_n$$

where $\mid \delta_n \mid < \frac{\pi}{4}$ for sufficiently large n. By substituting in (30) we have $\delta_n = O(n^{-1})$. The proof for the first case is complete. The other cases can be proved similarly.

Using these asymptotic expressions of eigenvalues we can easily obtain the corresponding asymptotic expressions for corresponding eigenfunctions of the problem (1)-(5). Indeed, taking in view, that the function $\phi_n(x)$ defined on whole $[-\pi, 0) \cup (0, \pi]$ by

$$\phi_n(x) = \begin{cases} \phi_1(x, \lambda_n) & for\ x \in [-\pi, 0) \\ \phi_2(x, \lambda_n) & for\ x \in (0, \pi] \end{cases} \tag{38}$$

is an eigenfunction according to the eigenvalue $\lambda_n = s_n^2$ and by putting (34)-(37) in the (16)-(19) we obtain the next Theorem.

Theorem 5.2 ***(i)*** *If* $\sin\alpha \neq 0$, *then*

$$\phi_n(x) = \sin\alpha \cos[s_n(x+\pi)] + O\left(\frac{1}{n}\right) \quad for\ x \in [-\pi, 0) \tag{39}$$

$$\phi_n(x) = \frac{\rho_{24}}{\rho_{12}} \sin \alpha s_n \sin [s_n \pi] \cos [s_n x] + O(1) \quad for\ x \in (0, \pi] \tag{40}$$

(ii) *If* $\sin \alpha = 0$*, then*

$$\phi_n(x) = \frac{-\cos \alpha}{s_n} \sin [s_n(x + \pi)] + O\left(\frac{1}{n^2}\right) for\ x \in [-\pi, 0)$$

$$\phi_n(x) = -\frac{\rho_{24}}{\rho_{12}} \sin \alpha \cos [s_n \pi] \cos [s_n x] + O\left(\frac{1}{n}\right) \quad for\ x \in (0, \pi] \tag{41}$$

as $n \to \infty$*. Each of these asymptotic equalities holds uniformly for* x*.*

Acknowledgements: We would like to express our gratitude to the anonymous reviewers and editors for their valuable comments and suggestions which improve the quality of present paper.

References

[1] K. Aydemir and O. Sh. Mukhtarov, Green's function method for self-adjoint realization of boundary-value problems with interior singularities, *Abstract and Applied Analysis*, Article ID 503267(2013), 7 pages.

[2] K.C. Pandey, Reconstruction of semiconductor surfaces: Si(111)-2 x 1, Si(111)- 7 x 7 and GaAs(110), *J. Vacuum Science and Technology*, A1(2) (1983), 1099-1100.

[3] E. Haller and H. Koppel, *Investigation of Nuclear Dynamics in Molecules by Means of the Lanczos Algorithm*, Elsevier Science Publishers B.V. (NorthHolland), (1986).

[4] W. Kerner, *Computing the Complex Eigenvalue Spectrum for Resistive Magnetohydrodynamics*, Elsevier Science Publishers B.V. (North-Holland), (1986).

[5] H.L. Royden, *Real Analysis*, Macmillan and Prentice Hall, (1968).

[6] A.N. Tikhonov and A.A. Samarskii, *Equations of Mathematical Physics*, Oxford and New York, Pergamon, (1963).

[7] E.C. Titchmarsh, *Eigenfunctions Expansion Associated with Second Order Differential Equations I (Second edn.)*, Oxford Univ. Press, London, (1962).