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On Smarandache TN Curves in Terms of Biharmonic Curves in the Special Three-Dimensional $\phi -$ Ricci Symmetric Para-Sasakian Manifold P

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Abstract

In this paper, we study Smarandache **TN** *curves in terms of biharmonic curves in the special three-dimensional $\phi -$ Ricci symmetric para-Sasakian manifold* P*. We define a special case of such curves and call it Smarandache* **TN** *curves in the special three-dimensional $\phi -$ Ricci symmetric para-Sasakian manifold* P*. We construct parametric equations of Smarandache* **TN** *curves in terms of biharmonic curve in the special three-dimensional $\phi -$ Ricci symmetric para-Sasakian manifold* P*.*

Keywords: *Biharmonic curve, curvature, para-Sasakian manifold, Smarandache* **TN** *curves, torsion.*

1 Introduction

The main interest in harmonic maps, Eells and Sampson also envisaged some generalizations and defined biharmonic maps $\varphi:(M,g)\to(N,h)$ between Riemannian manifolds as critical points of the bienergy functional

$$E_2(\varphi)=\frac{1}{2}\int_M|\tau(\varphi)|^2 v_g,$$

where $\tau(\varphi)$ = trace $\nabla d\varphi$ is the tension field of J that vanishes on harmonic maps. The Euler- Lagrange equation corresponding to E_2 is given by the vanishing of the bitension field

$$\tau_2(\varphi)=-\mathrm{J}^{\varphi}(\tau(\varphi))=-\Delta\tau(\varphi)-\mathrm{trace}R^N(d\varphi,\tau(\varphi))d\varphi, \tag{1.1}$$

where J^{φ} is the Jacobi operator of φ . The equation $\tau_2(f)=0$ is called the biharmonic equation. Since J^{φ} is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

Although E_2 has been on the mathematical scene since the early '60 (when some of its analytical aspects have been discussed) and regularity of its critical points is nowadays a welldeveloped field, a systematic study of the geometry of biharmonic maps has started only recently.

In this paper, we study Smarandache **TN** curves in terms of biharmonic curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold P. We define a special case of such curves and call it Smarandache **TN** curves in the special three-dimensional $\phi-$ Ricci symmetric para-Sasakian manifold P. We construct parametric equations of Smarandache **TN** curves in terms of biharmonic curve in the special three-dimensional $\phi-$ Ricci symmetric para-Sasakian manifold P.

2 Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold P

An n-dimensional differentiable manifold M is said to admit an almost para-contact Riemannian structure (ϕ,ξ,η,g), where ϕ is a (1,1) tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that

$$\phi\xi=0,\eta(\xi)=1,g(X,\xi)=\eta(X), \tag{2.1}$$

$$\phi^2(X)=X-\eta(X)\xi, \tag{2.2}$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{2.3}$$

for any vector fields X, Y on M [3].

Definition 2.1. *A para-Sasakian manifold* M *is said to be locally* ϕ*-symmetric if*

$$\phi^2\left((\nabla_W R)(X,Y)Z\right)=0,$$

for all vector fields X,Y,Z,W *orthogonal to* ξ *[3].*

Definition 2.2 *A para-Sasakian manifold* M *is said to be* ϕ*-symmetric if*

$$\phi^2\left((\nabla_W R)(X,Y)Z\right)=0,$$

for all vector fields X,Y,Z,W *on* M.

Definition 2.3 *A para-Sasakian manifold* M *is said to be* ϕ*-Ricci symmetric if the Ricci operator satisfies*

$$\phi^2\left((\nabla_X Q)(Y)\right)=0,$$

for all vector fields X *and* Y *on* M *and* $S(X,Y)=g(QX,Y)$.

If X,Y *are orthogonal to* ξ*, then the manifold is said to be locally* ϕ*-Ricci symmetric.*
We consider the three-dimensional manifold

$$\mathrm{P}=\left\{\left(x^1,x^2,x^3\right)\in\mathbb{R}^3:\left(x^1,x^2,x^3\right)\neq(0,0,0)\right\},$$

where $\left(x^1,x^2,x^3\right)$ *are the standard coordinates in* $\mathbb{R}^3$. *We choose the vector fields*

$$\mathbf{e}_1=e^{x^1}\frac{\partial}{\partial x^2},\ \mathbf{e}_2=e^{x^1}\left(\frac{\partial}{\partial x^2}-\frac{\partial}{\partial x^3}\right),\ \mathbf{e}_3=-\frac{\partial}{\partial x^1} \tag{2.4}$$

are linearly independent at each point of P. *Let* g *be the Riemannian metric defined by*

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=g(\mathbf{e}_3,\mathbf{e}_3)=1, \tag{2.5}$$
$$g(\mathbf{e}_1,\mathbf{e}_2)=g(\mathbf{e}_2,\mathbf{e}_3)=g(\mathbf{e}_1,\mathbf{e}_3)=0.$$

Let η *be the 1-form defined by*

$$\eta(Z)=g(Z,\mathbf{e}_3)\text{for any } Z\in\chi(\mathrm{P}).$$

Let be the (1,1) tensor field defined by

$$\phi(\mathbf{e}_1) = \mathbf{e}_2, \phi(\mathbf{e}_2) = \mathbf{e}_1, \phi(\mathbf{e}_3) = 0. \tag{2.6}$$

Then using the linearity of and g *we have*

$$\eta(\mathbf{e}_3) = 1, \tag{2.7}$$

$$\phi^2(Z) = Z - \eta(Z)\mathbf{e}_3, \tag{2.8}$$

$$g(\phi Z, \phi W) = g(Z,W) - \eta(Z)\eta(W), \tag{2.9}$$

for any $Z,W \in \chi(\mathrm{P})$. *Thus for* $\mathbf{e}_3 = \xi$, (ϕ, ξ, η, g) *defines an almost para-contact metric structure on* P.

Let ∇ *be the Levi-Civita connection with respect to* g. *Then, we have*

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1, [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_2.$$

The Riemannian connection ∇ *of the metric* g *is given by*

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y)$$

$$- g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]),$$

which is known as Koszul's formula.

Taking $\mathbf{e}_3 = \xi$ *and using the Koszul's formula, we obtain*

$$\begin{aligned} &\nabla_{\mathbf{e}_1}\mathbf{e}_1 = -\mathbf{e}_3, \nabla_{\mathbf{e}_1}\mathbf{e}_2 = 0, \quad \nabla_{\mathbf{e}_1}\mathbf{e}_3 = \mathbf{e}_1, \\ &\nabla_{\mathbf{e}_2}\mathbf{e}_1 = 0, \quad \nabla_{\mathbf{e}_2}\mathbf{e}_2 = -\mathbf{e}_3, \nabla_{\mathbf{e}_2}\mathbf{e}_3 = \mathbf{e}_2, \\ &\nabla_{\mathbf{e}_3}\mathbf{e}_1 = 0, \quad \nabla_{\mathbf{e}_3}\mathbf{e}_2 = 0, \quad \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0. \end{aligned} \tag{2.10}$$

Moreover we put

$$R_{ijk} = R(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_k, \; R_{ijkl} = R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l),$$

where the indices i, j, k *and* l *take the values* 1,2 *and* 3.

$$R_{122} = -\mathbf{e}_1, R_{133} = -\mathbf{e}_1, R_{233} = -\mathbf{e}_2,$$

and

$$R_{1212} = R_{1313} = R_{2323} = 1. \tag{2.11}$$

3 Biharmonic Curves in the Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold P

Biharmonic equation for the curve γ reduces to

$$\nabla^3_{\mathbf{T}}\mathbf{T} - R(\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T})\mathbf{T} = 0, \tag{3.1}$$

that is, γ is called a biharmonic curve if it is a solution of the equation (3.1).
Let us consider biharmonicity of curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold P. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet--Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.2}$$

where κ is the curvature of γ and τ its torsion and

$$\begin{aligned} &g(\mathbf{T}, \mathbf{T}) = 1, g(\mathbf{N}, \mathbf{N}) = 1, g(\mathbf{B}, \mathbf{B}) = 1, \\ &g(\mathbf{T}, \mathbf{N}) = g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T}\times\mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{3.3}$$

Theorem 3.1 $\gamma : I \to \mathsf{P}$ *is a biharmonic curve if and only if*

$$\begin{aligned} &\kappa = \text{constant} \neq 0, \\ &\kappa^2 + \tau^2 = 1, \\ &\tau = \text{constant}. \end{aligned} \tag{3.4}$$

Proof. Using (3.1) and Frenet formulas (3.2), we have (3.4).

Theorem 3.2 *All of biharmonic curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold P are helices.*

4 Smarandache TN Curve in the Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold P

Definition 4.1 *Let $\gamma : I \to \mathsf{P}$ be a unit speed regular curve in the special three-dimensional $\phi-$ Ricci symmetric para-Sasakian manifold P, whose position* *vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve.*

Now, let us define a special form of Definition 4.1.

Definition 4.2 *Let $\gamma : I \to \mathrm{P}$ be a unit speed regular curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold P and $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be its moving Frenet-Serret frame. Smarandache* **TN** *curves are defined by*

$$\Omega = \frac{1}{\sqrt{2\kappa^2+\tau^2}}\left(\mathbf{T}+\mathbf{N}\right). \tag{4.1}$$

Theorem 4.3 *Let $\gamma : I \to \mathrm{P}$ be a unit speed spacelike biharmonic curve and Ω its Smarandache* **TN** *curve on P. Then, the parametric equations of Ω are*

$$x_\Omega^1(s) = \frac{1}{\sqrt{2\kappa^2+\tau^2}}\left(-\cos\varphi - \frac{\sin^2\varphi}{2\kappa}s^2 + \frac{\overline{C}_1}{\kappa}s + \frac{\overline{C}_2}{\kappa}\right),$$

$$\begin{aligned} x_\Omega^2(s) = {} & \frac{1}{\sqrt{2\kappa^2+\tau^2}}\left(\sin\varphi e^{x^1}\left(\sin[\mathsf{k}s+C]+\cos[\mathsf{k}s+C]\right)\right. \\ & +\frac{1}{\kappa}\left(e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}\left(\mathsf{k}\sin\varphi\sin[\mathsf{k}s+C]+\cos\varphi\sin\varphi\cos[\mathsf{k}s+C]\right)\right) \\ & +\frac{1}{\kappa}\left(e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}\left(-\mathsf{k}\sin\varphi\cos[\mathsf{k}s+C]+\cos\varphi\sin\varphi\sin[\mathsf{k}s+C]\right)\right), \end{aligned} \tag{4.2}$$

$$\begin{aligned} x_\Omega^3(s) = {} & \sin\varphi e^{x^1}\sin[\mathsf{k}s+C] \\ & -\frac{1}{\kappa}e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1 s+\overline{C}_2}\left(-\mathsf{k}\sin\varphi\cos[\mathsf{k}s+C]+\cos\varphi\sin\varphi\sin[\mathsf{k}s+C]\right), \end{aligned}$$

where C, $\overline{C}_1$, $\overline{C}_2$, C_1, C_2, C_3 are constants of integration and $\mathsf{k} = \dfrac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}$.

Proof. Since γ is biharmonic, γ is a helix. So, without loss of generality, we take the axis of γ is parallel to the vector $\mathbf{e}_3$. Then,

$$g(\mathbf{T},\mathbf{e}_3) = T_3 = \cos\varphi, \tag{4.3}$$

where φ is constant angle.

The tangent vector can be written in the following form

$$\mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3. \qquad (4.4)$$

From (4.3), we have the following equation

$$\mathbf{T} = \sin\varphi\cos\mu\mathbf{e}_1 + \sin\varphi\sin\mu\mathbf{e}_2 + \cos\varphi\mathbf{e}_3. \qquad (4.5)$$

Since $|\nabla_{\mathbf{T}}\mathbf{T}| = \kappa,$ we obtain

$$\mu = \frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}s + C, \qquad (4.6)$$

where $C \in \mathbb{R}$.

Thus (4.5) and (4.6), imply

$$\mathbf{T} = \sin\varphi\cos[\mathsf{k}s + C]\mathbf{e}_1 + \sin\varphi\sin[\mathsf{k}s + C]\mathbf{e}_2 + \cos\varphi\mathbf{e}_3,$$

where $\mathsf{k} = \frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}$.

Using (2.4) in above equation, we obtain

$$\mathbf{T} = (-\cos\varphi, \sin\varphi e^{x^1}(\sin[\mathsf{k}s + C] + \cos[\mathsf{k}s + C]), \sin\varphi e^{x^1}\sin[\mathsf{k}s + C]). \qquad (4.7)$$

Using (4.4), we have

$$\nabla_{\mathbf{T}}\mathbf{T} = (T_1' + T_1T_3)\mathbf{e}_1 + (T_2' + T_2T_3)\mathbf{e}_2 + (T_3' - (T_1^2 - T_2^2))\mathbf{e}_3. \qquad (4.8)$$

From (3.1) and (4.8), we get

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= \sin\varphi(-\mathsf{k}\sin[\mathsf{k}s + C] + \cos\varphi\cos[\mathsf{k}s + C])\mathbf{e}_1 \\ &+ \sin\varphi(\mathsf{k}\cos[s + C] + \cos\varphi\sin[\mathsf{k}s + C])\mathbf{e}_2 \\ &- \sin^2\varphi\mathbf{e}_3,\end{aligned} \qquad (4.9)$$

where $\mathsf{k} = \frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}$.

By the use of Frenet formulas (3.2), we get

$$\begin{aligned}\mathbf{N} &= \frac{1}{\kappa}\nabla_{\mathbf{T}}\mathbf{T} \\ &= \frac{1}{\kappa}[(\mathsf{k}\sin\varphi\sin[\mathsf{k}s + C] + \cos\varphi\sin\varphi\cos[\mathsf{k}s + C])\mathbf{e}_1 \\ &+ (-\mathsf{k}\sin\varphi\cos[\mathsf{k}s + C] + \cos\varphi\sin\varphi\sin[\mathsf{k}s + C])\mathbf{e}_2\end{aligned} \qquad (4.10)$$

$$-\sin^2\varphi\mathbf{e}_3].$$

Substituting (2.4) in (4.10), we have

$$\mathbf{N}=\frac{1}{\kappa}(-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1s+\overline{C}_2,$$
$$e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1s+\overline{C}_2}\left(\mathrm{k}\sin\varphi\sin[\mathrm{k}s+C]+\cos\varphi\sin\varphi\cos[\mathrm{k}s+C]\right) \tag{4.11}$$
$$+e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1s+\overline{C}_2}\left(-\mathrm{k}\sin\varphi\cos[\mathrm{k}s+C]+\cos\varphi\sin\varphi\sin[\mathrm{k}s+C]\right),$$
$$-e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1s+\overline{C}_2}\left(-\mathrm{k}\sin\varphi\cos[\mathrm{k}s+C]+\cos\varphi\sin\varphi\sin[\mathrm{k}s+C]\right)),$$

where $\overline{C}_1,\overline{C}_2$ are constants of integration.

Lemma 4.4 *Let $\gamma:I\rightarrow\mathrm{P}$ be a unit speed spacelike biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold P. Then*

$$\kappa=\cos\Lambda,$$
$$\tau=\sin\Lambda,$$

where Λ is arbitrary angle.

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