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φ- Pairs and a Unique Common Fixed Point Theorem for Six Maps in Cone Metric Spaces

K. Prudhvi

Department of Mathematics
University College of Science, Saifabad
Osmania University, Hyderabad
Andhra Pradesh, India
E-mail: prudhvikasani@rocketmail.com

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Abstract

In this paper, we prove φ-pairs and a unique common fixed point theorem for six maps in cone metric spaces. Our result generalizes and extends some recent results.

Keywords: *Coincidence points, Common fixed point, Cone metric space, φ-pairs.*

1 Introduction and Preliminaries

In 2007, Huang and Zhang [4] have introduced the concept of cone metric spaces and established some fixed point theorems for contractive mappings in these spaces. Subsequently Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also,[4] ,[6] and the references mentioned therein). In [3] Di Bari and Vetro have introduced the concept of φ-map and proved some fixed point theorems generalizing some known results. In this paper we extend the fixed point theorem for three maps of R.P. Pant et al., [5] into six maps in cone metric spaces**.**

Throughout this paper R_+ denotes the set of all non negative real numbers, E is a real Banach space, N = {1,2,3………}, the set of all natural numbers and θ denotes the zero element of E.

The following definitions are due to Huang and Zhang [4].

Definition 1.1: *Let B be a real Banach Space and P a subset of B .The set P is called a cone if and only if:*

(a). P is closed , non –empty and $P \neq \{\theta\}$;
(b). $a, b \in R$, $a, b \geq 0$, $x, y \in P$ *implies* $ax + by \in P$;
(c). $x \in P$ *and* $-x \in P$ *implies* $x = \theta$.

Definition 1.2: *Let P be a cone in a Banach Space B, define partial ordering '*$\leq$*' with respect to P by* $x \leq y$ *if and only if* $y-x \in P$ *.We shall write x<y to indicate* $x \leq y$ *but* $x \neq y$ *while X<<y will stand for* $y-x \in$ *Int P, where Int P denotes the interior of the set P. This Cone P is called an order cone.*

Definition 1.3: *Let B be a Banach Space and* $P \subset B$ *be an order cone. The order cone P is called normal if there exists K>0 such that for all* $x, y \in B$,

$$\theta \leq x \leq y \text{ implies } \|x\| \leq K \|y\| .$$

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 1.4: *Let X be a nonempty set of B .Suppose that the map* $d: X \times X \to B$ *satisfies:*

(d1). $\theta \leq d(x,y)$ for all $x,y \in X$ and

$d(x,y) = \theta$ if and only if x = y;

(d2). $d(x,y) = d(y,x)$ for all $x,y \in X$;

(d3). $d(x,y) \leq d(x,z) + d(y,z)$ for all $x,y,z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.
The concept of a cone metric space is more general than that of a metric space.

Definition 1.5: *Let (X, d) be a cone metric space .We say that* $\{x_n\}$ *is*

(i) a Cauchy sequence if for every c in B with $c >> \theta$*, there is N such that for all n, m>N,* $d(x_n, x_m) << c$;
(ii) a convergent sequence if for any $c >> \theta$*, there is an N such that for all n>N,* $d(x_n, x) << c$*, for some fixed x in X .*
We denote this $x_n \to x$ *(as* $n \to \infty$*).*

Definition 1.6. [8]: *Let f, g: X→X. Then the pair (f, g) is said to be (IT)-Commuting at z∈ X if f(g(z)) = g(f(z)) with f(z) = g(z).*

Definition 1.7: *Let P be an order cone. A non-decreasing function* $\varphi : P \to P$ *is called a* $\varphi -$ *map if*

(i). $\varphi(\theta) = \theta$ and $\theta < \varphi(\omega) < \omega$ for $\omega \in P \setminus \{\theta\}$,

(ii) . $\omega \in IntP$ implies $\omega - \varphi(\omega) \in IntP$,

(iii). $\lim_{n \to \infty} \varphi^{n}(\omega) = \theta$ for every $\omega \in P \setminus \{\theta\}$.

2 Common Fixed Point Theorem

In this section we prove φ-pairs and a unique common fixed point theorem for six maps in cone metric spaces, which generalizes and extends the results of R.P. Pant et al., [5]

We define common asymptotic regularity of two functions in the following way.

Definition 2.1: *Let f, g h and r, s, t be self-maps on a cone metric space (X, d).The pairs (f, g) and (r, s) are said to be common asymptotically regular with respect to h and t respectively at* $x_0 \in X$ *if there exists a sequence* $\{x_n\}$ *in X Such that*

$hx_{2n+1} = fx_{2n} = rx_{2n+2} = tx_{2n+3}$,
$hx_{2n+2} = gx_{2n+1} = sx_{2n+3} = tx_{2n+4}$, n = 0, 1,2,3……
and $\lim_{n\to\infty} d(hx_n, hx_{n+1}) = \theta = \lim_{n\to\infty} d(tx_n, tx_{n+1})$.

The following theorem is an extends and improve the Theorem 3.2 [5]

Theorem 2.2: *Let (X, d) be a cone metric space, P be an order cone and f, g, h and r, s, t be (self-maps) a φ-pair, that is, there exists a φ-map such*

(A1): $d(fx, gy) \le \varphi(d(hx, hy))$ for all x, y ∈ X,

(A2): $d(rx, sy) \le \varphi(d(tx, ty))$ for all x, y ∈ X.

If f(X) ∪ g(X) ∪ r(X) ∪ s(X) ⊆ h(X)(=t(X)) and h(X)(= t(X)) is a complete subspace of X, then the maps f, g ,h and r, s, t have a coincidence point in X. Moreover (f, h), (g, h), (r, t) and (s, t) are (IT)-commuting, then f, g, r, s and h, t have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X.

Since $f(X) \cup g(X) \cup r(X) \cup S(X) \subset h(X)(= t(X))$, then we can define a sequence$\{x_n\}$ in X such that

$$hx_{2n+1} = fx_{2n} = rx_{2n+2} = tx_{2n+3},$$
$$hx_{2n+2} = gx_{2n+1} = sx_{2n+3} = tx_{2n+4}. \quad n = 0, 1,2,\ldots.. \tag{1}$$

Applying the contractive condition (A1)

$$d(hx_{2n+1}, hx_{2n+2}) = d(fx_{2n}, gx_{2n+1})$$
$$\leq \varphi(d(hx_{2n}, hx_{2n+1})) \tag{2}$$

Similarly,
$$d(hx_{2n+2}, hx_{2n+3}) = d(fx_{2n+1}, gx_{2n+2})$$
$$\leq \varphi(d(hx_{2n+1}, hx_{2n+2})) \tag{3}$$

That is,
$$d(hx_{2n+2}, hx_{2n+3}) \leq \varphi(\varphi(d(hx_{2n+1}, hx_{2n+2}))) \tag{4}$$

From (2) and (3), by the induction, we obtain that

$$d(hx_{2n+1}, hx_{2n+2}) \leq \varphi(d(hx_{2n}, hx_{2n+1}))$$
$$\leq \varphi(\varphi(d(hx_{2n-1}, hx_{2n}))\leq \ldots\ldots \leq \varphi^{2n}(d(hx_0, hx_1)). \tag{5}$$

And

$$d(hx_{2n+2}, hx_{2n+3}) \leq \varphi^{2n+1}(d(hx_0, hx_1)). \tag{6}$$

Fix $\theta \ll \varepsilon$ and we choose a positive real number δ such that

$\varepsilon - \varphi(\varepsilon) + I(\theta, \delta) \subset \text{Int}P$, where $I(\theta, \delta) = \{y \in E : \|y\| < \delta\}$.

Also choose a natural number N such that

$\varphi^m((d((fx_0, gx_1))) \in I(\theta, \delta)$ for all $m \geq N$, then
$\varphi^m((d((fx_0, gx_1))) \ll \varepsilon - \varphi(\varepsilon)$ for all $m \geq N$.

Consequently, $d(hx_m, hx_{m+1}) \ll \varepsilon - \varphi(\varepsilon)$ for all $m \geq N$.
Fix $m \geq N$ and we prove

$$d((hx_m, hx_{n+1})) \ll \varepsilon \quad \text{for all } n \geq m . \tag{7}$$

We note that (7) holds when $n = m$. We assume that (7) holds for $n \geq m$.
Now we prove for n+1, then, we have, by the triangle inequality

$$\begin{aligned} d(hx_m , hx_{n+2})) &\leq d(hx_m, hx_{m+1}) + d(hx_{m+1}, hx_{n+2}) \\ &\ll \varepsilon - \varphi(\varepsilon) + \varphi (d (fx_m , gx_{n+1})) \\ &\ll \varepsilon - \varphi(\varepsilon) + \varphi(d(hx_m , hx_{n+1})) \\ &\ll \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) = \varepsilon \qquad \text{(by induction)} \end{aligned}$$

$d(hx_m , hx_{n+2}) \ll \varepsilon$ .Therefore (7) holds when, $n = n + 1$.

By induction we deduce (7) holds for all $n \geq m$.

Hence, $\{fx_n\}$ is a Cauchy sequence. Similarly we can prove $\{rx_n\}$ is a Cauchy sequence.

We shall show that

$$hu = fu = gu$$
and $tv = rv = sv$.

Firstly, let us estimate that

$d(hu, fu) = d(z, fu)$.

We have that by the triangle inequality

$$\begin{aligned} d(hu, fu) &\leq d(hu, hx_{2n+1})+d(hx_{2n+1} , fu) \\ &= d(z, hx_{2n+1})+d(fu, gx_{2n+1}) \end{aligned}$$

By the contraction condition
$d(fu, gx_{2n+1})$ may be negligible as $n\to\infty$.

Therefore,

$d(hu, fu)\leq d(z, hx_{2n+1})+d(fu, gx_{2n+1})\leq d(z, z)=\theta$.

Which leads to $d(hu, fu)\leq\theta$ and $hu = fu$. (8)

Similarly, we can find $hu = gu$. (9)

Since, $z = fu = gu = hu$, (10)

z is a coincidence point of f, g, h. (11)

Now we estimate that $d(tv, rv) = d(z,fz)$.

We have that by the triangle inequality

$d(tv, rv) \leq d(tv, tx_{2n+3}) + d(tx_{2n+3}, tv)$

$= d(z, tx_{2n+3}) + d(tv, sx_{2n+3})$

By the contraction condition

$d(tv, sx_{2n+3})$ may be negligible as $n \to \infty$.

Therefore, $d(tv, rv) \leq d(z, tx_{2n+3}) \leq d(z, z) = \theta$.

Which leads to $d(tv, rv) = \theta$ and $tv = rv$. (12)

Similarly we can find $tv = sv$. (13)

Since, $z = tv = rv = sv$, (14)

z is a coincidence point of r, s, t. (15)

In view of (11) and (15), we conclude that f, g, h and r, s, t have a coincidence point in X.

In view of (10) and (14), it follows that

$fu = gu = hu = tv = rv = sv = z$.

Since, (f,h), (g,h), (r,t),and (s,t), are (IT)-Commuting

$d(ffu, fu) = d(ffu, gu) \leq \varphi(d(hfu, hu))$

$< d(hfu, hu) = d(ffu, fu)$.

$\Rightarrow$ $ffu = fu = hfu = z$.

Therefore, fu(=z) is a common fixed point of f and h. (16)

Similarly, $ggu = gu = hgu = z$.

Therefore, gu(=z) is a common fixed point of g and h. (17)

Since, $fu = gu = (z)$.

Therefore, from (16) and (17), it follows that f, g, h have a common fixed point (18)

$d(rrv, rv) = d(rrv, sv) \leq \varphi(d(trv, tv))$

$< d(trv, tv) = d(rrv, rv)$

$\Rightarrow$ rrv = rv = trv (= z).

Therefore, rv(=z) is a common fixed point of r and t. (19)

Similarly, ssv = sv = tsv(=z).

Therefore, sv(=z) is a common fixed point of s and t. (20)

Since, rv = sv = (z).

Therefore, from (19) and (20), it follows that r, s,t have a common fixed point. (21)

From (18) and (21), it follows that f, g, h, and r, s, t have a common fixed point.

Uniqueness, let w be another common fixed point of f, g, h, and r, s, t.

Consider, d(z, w) = d(fz, gw)≤φ(d(hz, hw))<d(hz, hw)= d(z,w).
$\Rightarrow$ z = w.
Therefore, f, g, h, and r, s, t have a unique common fixed point.

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