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An Inequality of Subclasses of Univalent Functions Related to Complex Order by Convolution Method

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Abstract

The purpose of this paper to introduce the class $V(\lambda, \mu, A, B, b)$ *of analytic functions related to complex order b***.** *We investigate various properties of the univalent functions* $f(z)$ *belonging to this class and prove an inequality of the subclasses of univalent function by convolution method.*

Keywords: *Analytic function, Univalent function, Complex order, Convolution technique and the class* $V(\lambda, \mu, A, B, b)$.

1. Introduction

Analytic function is the special branch of mathematics, which played a very important role in the development of certain subclasses of univalent function. The theory of univalent functions is one of the most beautiful subjects in Geometric Function Theory. The purpose of this paper to describe only those aspects of the theory of univalent function in which we have pursued the study further. The work done on various researchers are discussed brief in this paper.

In the present paper an attempt has been made to verify various results of subclasses of univalent functions by employing different techniques, on the other hand we have prove an inequality of subclasses of univalent function by convolution method. The present paper is devoted a unified study of various subclasses of univalent functions.

For this purpose, let f (z) and g (z) are two analytic functions in class A such that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.2}$$

Then the convolution of f (z) and g (z) is denoted by f * g and it is defined by the power series

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \tag{1.3}$$

Now, we introduced the class V (λ, μ, A, B, b) of analytic functions of complex order b, by using convolution technique, as defined below:

A function f (z) of A belongs to the class V (λ, μ, A, B, b) if and only if there exists a function w (z) belonging to class H such that

$$1 + \frac{1}{b}\left\{\frac{D^{\lambda+1} f(z) - 1}{z}\right\} = (1-\mu) + \mu\left\{\frac{1+Aw(z)}{1+Bw(z)}\right\} \tag{1.4}$$

Where $-1 \le B < A \le 1$, $0 < \mu \le 1$, $\lambda > -1$ and b is a non-zero complex number.

D^{λ} f (z) is the differential operator defined as

$$D^{\lambda} f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) = \frac{z\left(z^{\lambda-1} f(z)\right)^{\lambda}}{\lambda!} \tag{1.5}$$

It is easy to find that the condition (1.4) is equivalent to

$$\left| \frac{\left\{ \dfrac{D^{\lambda+1} f(z)}{z - 1} \right\}}{\mu (A - B) b - B \left\{ \dfrac{D^{\lambda+1} f(z)}{z} - 1 \right\}} \right| < 1 \tag{1.6}$$

By giving specific values to λ, μ, A, B and b in (1.6), we obtain the following important subclasses of analytic function studied by various researchers in earlier works:

(i) For μ = 1, b = 1, and λ = 0, we obtain the class of functions f (z) satisfying the condition

$$\left|\frac{f'(z)-1}{Bf'(z)-A}\right|<1,$$ z ϵU studied by Goel and Mehrok [7].

(ii) For μ = 1, A = δ, B = -δ, b = 1, and λ = 0, we obtain the class of functions f (z) satisfying the condition

$$\left|\frac{f'(z)-1}{f'(z)+1}\right|<\delta ,$$ z ϵU studied by Caplinger and Cauchy [8].

(iii) For μ = 1, A = (1-2ρ) δ, B = - δ, b = 1, and λ = 0, we obtain the class of functions f (z) satisfying the Condition

$$\left|\frac{f'(z)-1}{f'(z)+1-2\rho}\right|<\delta,$$ z ϵU studied by Juneja and Mogra [5].

Where 0 ≤ ρ <1 and 0 < $\delta \leq 1$.

Thus the study of the class V (λ, μ, A, B, b) provides a unified approach for the various subclasses univalent functions studied by Goel and Mehrok [7], Caplinger and Cauchy [8], Juneja and Mogra [5]

2. Preliminaries

Before giving the main result we first quote a Lemma due to Jack [1] and we will prove a Lemma [2.2].

Lemma (2.1) ***[1]*** *If the function w is analytic for* $|z| \leq r < 1$,

$$w(0) and \; |w(z_0)| = \max_{|z|-r} |w(z)|$$

Then $z_0 w'(z_0) = \xi w(z_0)$, *where* $\xi \geq 1$.

Now we will prove the following Lemma which is an important tool to prove our main result.

Lemma (2.2) *A function f (z) belongs to the class V (λ, μ, A, B, b), where -1 ≤ B < A ≤ 1, If and only if*

$$|H(z)-m| < M \tag{1.7}$$

where

$$m = 1 - \frac{B\mu(A-B)}{(1-B^2)} \tag{1.8}$$

$$H(z) = 1 + \frac{1}{b}\left\{\frac{D^{\lambda+1} f(z)}{z} - 1\right\}, \tag{1.9}$$

and $$M = \frac{\mu(A-B)}{1-B^2} \qquad (2.0)$$

Proof. Suppose that f belongs to the class V (λ, μ, A, B, b), and then from (1.4) we get

$$H(z) = \frac{1+\{B+\mu(A-B)\}w(z)}{1+w(z)}$$, where H (z) is defined by (1.9)

$$H(z) - m = \frac{(1-m)+\{B+\mu(A-B)-Bm\}\,w(z)}{1+Bw(z)} \qquad (2.1)$$

$$M\left[\frac{B+w(z)}{1+Bw(z)}\right] = Mh(z)$$

It is clear that the function h (z) satisfies $|h(z)|<1$. Hence (1.7) follows from (2.1). Conversely, suppose that the condition (1.7) holds.
Then we have

$$\left|\frac{H(z)}{M} - \frac{m}{M}\right| < 1$$

Let $$g(z) = \frac{H(z)}{M} - \frac{m}{M}$$, then by (2.3.5)

$$w(z) = \frac{g(z)-g(0)}{1-g(0)g(z)} \qquad (2.2)$$

$$w(z) = \frac{\{H(z)-1\}}{\mu(A-B)-B\{H(z)-1\}}$$

Clearly w (0) = 0 and $|w(z)|<1$. Rearranging (2.2), we arrive at (1.4). Hence f (z) belongs to the class V (λ, μ, A, B, b)**.** This proves the above lemma.

3. Main Results

Theorem 3.1 *If f (z) is univalent function defined by* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *belongs to the class V (λ, μ, A, B, b), then*

$$|a_n| \le \frac{\mu(A-B)|b|}{\alpha(\lambda, n)}, \quad n = 2, 3\ldots \qquad (2.3)$$

where $\alpha(\lambda, n) = \left[\begin{matrix}\lambda + n \\ \lambda + 1\end{matrix}\right]$, *(2.4)*

Then the inequality (3.1.1) is sharp.

Proof. Since f (z) belongs to the class V (λ, μ, A, B, b), we have from (1.4)

$$1+\frac{1}{b}\left\{\frac{D^{\lambda+1}f(z)-1}{z}\right\}=(1-\mu)+\mu\left\{\frac{1+Aw(z)}{1+Bw(z)}\right\} \tag{2.5}$$

and

$$\frac{D^{\lambda+1}f(z)}{z}=\frac{1+\{\mu(A-B)b+B\}w(z)}{1+Bw(z)} \tag{2.6}$$

where w belonging to the class H. From (2.6), we have

$$\frac{D^{\lambda+1}f(z)}{z}-1=\left[\mu(A-B)b-B\{\frac{D^{\lambda+1}f(z)}{z}-1\}\right]w(z)$$

or

$$\sum_{j=2}^{\infty}\alpha(\lambda, j)a_j z^{j-1}=\left[\mu(A-B)-B\sum_{j=2}^{\infty}\alpha(\lambda, j)a_j z^{j-1}\right]w(z) \tag{2.7}$$

where $\qquad w(z)=\sum_{j=1}^{\infty}t_j z^j$

Now equating corresponding coefficients on both sides of (2.7), we find out that the coefficient a_n on the left hand side of (2.7) depends only on $a_2, a_{3,}...., a_{n-1}$ on the right hand side of (2.7). Hence for n ≥ 2, it follows from (2.7) that

$$\sum_{j=2}^{n}\alpha(\lambda, j)a_j z^{j-1}+\sum_{j=n+1}^{\infty}d_j z^{j-1}=\left[\mu(A-B)-B\sum_{j=2}^{n-1}\alpha(\lambda, j)a_j z^{j-1}\right]w(z)$$

where d_j are some complex numbers. Since $|w(z)|<1$, by using Parseval's identity, we get

$$\sum_{j=2}^{n}\{\alpha(\lambda, j)\}^2 |a_j|^2 r^{2(j-1)} + \sum_{j=n+1}^{\infty}|d_j|^2 r^{2(j-1)}$$

$$\leq \mu^2(A-B)^2|b|^2 + B^2\sum_{j=2}^{n-1}\{\alpha(\lambda, j)\}^2|a_j|^2 r^{2(j-1)}$$

$$\leq \mu^2(A-B)^2|b|^2 + B^2\sum_{j=2}^{n-1}\{\alpha(\lambda, j)\}^2|a_j|^2$$

Letting r →1 on the left hand side of the above inequality, we obtain

$$\sum_{j=2}^{n}\{\alpha(\lambda, j)\}^2 \left|a_j\right|^2 \leq \mu^2 (A-B)^2 \left|b\right|^2 + B^2 \sum_{j=2}^{n-1}\{\alpha(\lambda, j)\}^2 \left|a_j\right|^2$$

Thus

$$\{\alpha(\lambda, n)\}^2 \left|a_n\right|^2 \leq \mu^2 (A-B)^2 \left|b\right|^2 - (1-B^2)\sum_{j=2}^{n-1}\{\alpha(\lambda, j)\}^2 \left|a_j\right|^2$$

$$\{\alpha(\lambda, n)\}^2 \left|a_n\right|^2 \leq \mu^2 (A-B)^2 \left|b\right|^2$$

Hence $\qquad \left|a_n\right| \leq \dfrac{\mu(A-B)\left|b\right|}{\alpha(\lambda, n)} \qquad$ n=2, 3,

In order to establish the sharpness, we consider the function f (z) given by

$$1+\frac{1}{b}\left\{\frac{D^{\lambda+1} f(z)}{z}-1\right\}=(1-\mu)+\mu\left\{\frac{1+Az^{n-1}}{1+Bz^{n-1}}\right\}, n=2,3,....$$

We observe that

$$.\left|\frac{\left\{\dfrac{D^{\lambda+1} f(z)}{z-1}\right\}}{\mu(A-B)b - B\left\{\dfrac{D^{\lambda+1} f(z)}{z}-1\right\}}\right| < 1$$

This shows that the function f (z) belongs to the class V (λ, μ, A, B, b). It is easy to compute that the function f (z) has the expansion of the given form

$f(z) = z + \dfrac{\mu(A-B)\left|b\right|}{\alpha(\lambda, n)} z^n + ...$ Showing that the estimate (2.3) is sharp. This proves our main result.

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