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Periodic Solution of Integro-Differential Equations for the Second Order of the Operators

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Abstract

In this paper, we investigate the existence, uniqueness and stability of a periodic solution of integro-differential equations of second order with the operators by using the method of Samoilenko [7]. These investigations lead us to improving and extending the above method. Thus the integro-differential equations with the operators are more general and detailed than those introduced by Butris [2].

Keywords: *Numerical-analytic method, nonlinear system, existence, uniqueness and stability of periodic solution, integro-differential equations of second order with the operators.*

I Introduction

The integro-differential equations has been arisen in many mathematical and engineering field, so that solving this kind of problems are more efficient and useful in many research branches. Analytical solution of this kind of equation is not accessible in general form of equation and we can only get an exact solution only in special cases. But in industrial problems we have not spatial cases so that we try to solve this kind of equations numerically in general format. Many numerical schemes are employed to give an approximate solution with sufficient accuracy [1, 3, 4, 5, 6, 8, 10].

Butris [2] used numerical–analytic method for studying the periodic solution of integro-differential equations which has the form:

$$\frac{dx}{dt} = f\left(t, x, Ax, \int_0^{h(t)} g\big(s, x(s), Bx(s)\big)\, ds\right)$$

where $x \in D \subseteq R^n$, D is a closed and bounded domain.

In this paper, we investigate the existence, uniqueness and stability of periodic solution of integro-differential equations of second order with the operators by using the method of Samoilenko [7].

Consider the following problem:

$$\frac{d^2x}{dt^2} = f\left(t, x, \dot{x}, Ax, A\dot{x}, \int_0^{h(t)} g(s, x, \dot{x}, Bx, B\dot{x})\, ds\right) \quad \dots (1)$$

which are defined on the domain

$$\left.\begin{aligned} &(t, x, \dot{x}, y, \dot{y}, z) \in R^1 \times D \times D_1 \times D_2 \times D_3 \times D_4 = (-\infty, \infty) \times D \times D_1 \times D_2 \times D_3 \times D_4 \\ &(t, x, \dot{x}, w, \dot{w}) \in R^1 \times D \times D^* \times D^{**} = (-\infty, \infty) \times D \times D^* \times D^{**} \end{aligned}\right\} \quad \dots (2)$$

and continuous *in* $t, x, \dot{x}, y, \dot{y}, z, w, \dot{w}$ and periodic in t of a period T.

where $x \in D \subset R^n$, D is closed and bounded domain subset of Euclidean space R^n and $D_1, D_2, D_3, D_4\, D^*, D^{**}$ are bounded domains subset of Euclidean space R^m .

Suppose that the vector functions $f\,(t, x, \dot{x}, y, \dot{y}, z), g(t, x, \dot{x}, w, \dot{w})$ and the operators A and B satisfy the following inequalities:

$$\|f\,(t, x, \dot{x}, y, \dot{y}, z)\| \le M \ , \ \|g(t, x, \dot{x}, w, \dot{w})\| \le N \quad \cdots (3)$$

$$\left\|f\left(t,x_1,\dot{x}_1,y_1,\dot{y}_{1,}z_1\right)-f(t,x_2,\dot{x}_2,y_2,\dot{y}_2,z_2)\right\|\le K[\ \|x_1-x_2\|+\ \|\dot{x}_1-\dot{x}_2\|+$$

$$\|y_1-y_2\|+\left\|\dot{y}_{1,}-\dot{y}_2\right\|+\|z_1-z_2\|\ ] \qquad \cdots(4)$$

$$\|g(t,x_1,\dot{x}_1,w_1,\dot{w}_1)-g(t,x_2,\dot{x}_2,w_2,\dot{w}_2)\| \le P\ [\ \|x_1-x_2\|+\|\dot{x}_1-\dot{x}_2\|+\ \|w_1-w_2\| + \|\dot{w}_1-\dot{w}_2\|\ ] \qquad \cdots(5)$$

$$\|\ h(t)\ \|\le h<\infty \qquad \ldots(6)$$

$$\|Ax_1-Ax_2\|\le Q_1\|x_1-x_2\| \qquad \ldots(7)$$

$$\|A\dot{x}_1-A\dot{x}_2\|\le Q_2\|\dot{x}_1-\dot{x}_2\| \qquad \ldots(8)$$

$$\|Bx_1-Bx_2\|\le Q_3\|x_1-x_2\| \qquad \ldots(9)$$

$$\|B\dot{x}_1-B\dot{x}_2\|\le Q_4\|\dot{x}_1-\dot{x}_2\| \qquad \ldots(10)$$

for all $t\in R^1$, $x,x_1,x_2\ \in D$, $\dot{x},\dot{x}_1,\dot{x}_2\in D_1$, $y,y_1,y_2\ D_2$, $y,\dot{y}_1,\dot{y}_2\ \in D_3$ $z,z_1,z_2\in D_4$, $w,w_1,w_2\in D^*$, $\dot{w},\dot{w}_1,\dot{w}_2\in D^{**}$.

where $M,N,h,K,P,Q_1,Q_2,Q_3,Q_4$ are a positive constants. But A, B are operators where $A: R^1\longrightarrow R^1$ and $B: R^1\longrightarrow R^1$.

Moreover we define the non-empty sets as follows:

$$\left.\begin{aligned}
D_f &= D-M\ \frac{T^2}{4}\\
D_{1f} &= D_1-M\ \frac{T}{2}\\
D_{2f} &= D_2-\ Q_1M\ \frac{T^2}{4}\\
D_{3f} &= D_3-Q_2M\frac{T}{2}\\
D_{4f} &= D_4-\frac{hPMT}{2}\ [\ (1+Q_4)\ +\ (\frac{T+TQ_3}{2})\ ]
\end{aligned}\right\} \qquad \ldots(11)$$

We consider the matrix $\Lambda=\begin{pmatrix}\frac{\mathcal{H}_1\,T^2}{4} & \frac{\mathcal{H}_2\,T^2}{4}\\ \frac{\mathcal{H}_1\,T}{2} & \frac{\mathcal{H}_2\,T}{2}\end{pmatrix}$

where $\mathcal{H}_1\ =K(\ 1++P\ h+\ Q_3P\ h\ ),$
$\mathcal{H}_2\ =\ K(1+Q_2+P\ h+\ Q_4P\ h\ )$

Furthermore, we assume that the largest Eigen value λ_{max} of the matrix Λ does not exceed unity. That is

$$\lambda_{max}(\Lambda) = \frac{H_1 T^2 + 2 H_2 T}{4} < 1 . \qquad \dots(12)$$

Lemma 1: *Let $f(t)$ be a continuous vector function in the interval $0 \le t \le T$. then*

$$\left\| \int_0^t \left(f(s) - \frac{1}{T}\int_0^T f(s)ds \right) ds \right\| \le \alpha(t) \max_{t\in[0,T]} \|f(t)\|,$$

where $\alpha(t) = 2t(1 - \frac{t}{T})$. (For the proof see [7]) .

II Approximate of Periodic Solution

The study of the approximate periodic solution of integro-differential equation (1) be introduced by the following theorem.

Theorem 1: *Let t vector functions $f\,(t, x, \dot{x}, y, \dot{y}, z)$ and $g(t, x, \dot{x}, w, \dot{w})$ defined and continuous on the domain (2), satisfy the inequality (3) to (10) and the condition (12), then there exist a sequence of functions;*

$$x_{m+1}(t, x_0) = x_0 + L^2 f\left(t, x_m(t, x_0), \dot{x}_m(t, x_0), y_m(t, x_0), \dot{y}_m(t, x_0), z_m(t, x_0)\right) \dots (13)$$

with $x_0(t, x_0) = x_0$, *m=0,1,2,…*

and

$$\dot{x}_{m+1}(t, x_0) = \dot{x}_0 + L f\left(t, x_m(t, x_0), \dot{x}_m(t, x_0), y_m(t, x_0), \dot{y}_m(t, x_0), z_m(t, x_0)\right) \dots (14)$$

with $x_0(0, x_0) = \dot{x}_0$, $m = 0,1,2, \dots$, where

$$L^2 f\left(t, x_m(t, x_0), \dot{x}_m(t, x_0), y_m(t, x_0), \dot{y}_m(t, x_0), z_m(t, x_0)\right) = \int_0^t [\, Lf(s, x_m(s, x_0), Ax_m(s, x_0),$$

$$, \int_0^{h(s)} g(\tau, x_m(\tau, x_0), Bx_m(\tau, x_0)) d\tau] ds - \frac{1}{T}\int_0^T [\, Lf(s, x(s, x_0), Ax(s, x_0),$$

$$, \int_0^{h(s)} g\left(\tau, x(\tau, x_0), Bx(\tau, x_0)\right) d\tau]\, ds$$

and

$$L f\left(t, x_m(t, x_0), \dot{x}_m(t, x_0), y_m(t, x_0), \dot{y}_m(t, x_0), z_m(t, x_0)\right) = \int_0^t f(s, x_m(s, x_0), Ax_m(s, x_0),$$

$$\int_0^{h(s)} g(\tau, x_m(\tau, x_0), Bx_m(\tau, x_0))d\tau)ds \; - \frac{1}{T}\int_0^T f(s, x(s, x_0), Ax(s, x_0),$$

$$\int_0^{h(s)} g\big(\tau, x(\tau, x_0), Bx(\tau, x_0)\big)d\tau)\; ds$$

periodic in t of period T ,convergent uniformly as $m \to \infty$ in the domain

$(t, x_0, \dot{x}_0) \in R^1 \times D_f \times D_{1f}$ ….(15)

to the limit function $x^0(t, x_0)$ which is defined on the domain (15) and satisfy the following integral equation

$x(t, x_0) = x_0 + L^2 f\;\big(t, x_m(t, x_0), \dot{x}_m(t, x_0), y_m(t, x_0), \dot{y}_m(t, x_0)\;, z_m(t, x_0)\big)$ ……………………..(16)

which is a periodic solution of the problem (1), provided that

$\|x^0(t, x_0) - x_0\| \le M\,\frac{T^2}{4}$ ... (17)

and

$\|x^0(t, x_0) - x_m(t, x_0)\| \le \Lambda^m (E - \Lambda)^{-1} V_1$ … (18)

for all $m \ge 1$ and $t \in R^1$

Proof: By the lemma1 and using the sequence of functions (13), when $m = 0$, we get

$$\|x_1(t, x_0) - x_0\| \le (1 - \frac{t}{T})\int_0^t \left\| Lf(s, x_0, \dot{x}_0, Ax_0, A\dot{x}_0, \int_0^{h(s)} g(\tau, x_0, , \dot{x}_0, Bx_0, B\dot{x}_0)d\tau \right\| ds$$

$$+ \frac{t}{T}\int_t^T \left\| Lf(s, x_0, \dot{x}_0, Ax_0, A\dot{x}_0, \int_0^{h(s)} g(\tau, x_0, , \dot{x}_0, Bx_0, B\dot{x}_0)d\tau \right\| ds \le M\,\frac{T^2}{4}$$

Therefore, $x_1(t, x_0) \in D$, *for all* $t \in [0, T]$

Then, by mathematical induction we can prove that

$\|x_m(t, x_0) - x_0\| \le M\,\frac{T^2}{4}$. … (19)

From (19) we obtain the estimate

$\|Ax_m(t,x_0) - A\,x_0\| \leq Q_1 M\,\frac{T^2}{4}$

which given $x_m(t, x_0) \in D$, $Ax_m(t, x_0) \in D_2$ *for all* $t \in [0,T]$ and $x_0 \in D_f$, $Ax_0(t, x_0) \in D_{2f}$.

And by the lemma 1 and using (14), when $m = 0$, *we* have

$$\|\dot{x}_1(t,x_0) - \dot{x}_0\| \leq (1-\frac{t}{T})\int_0^t \left\| f(s,x_0,\dot{x}_0,Ax_0,A\dot{x}_0,\int_0^{h(s)} g(\tau,x_0,,\dot{x}_0,Bx_0,B\dot{x}_0)d\tau\right\| ds$$

$$+\ \frac{t}{T}\int_t^T \left\| f(s,x_0,\dot{x}_0,Ax_0,A\dot{x}_0,\int_0^{h(s)} g(\tau,x_0,,\dot{x}_0,Bx_0,B\dot{x}_0)d\tau\ \right\| ds \leq\ M\,\frac{T}{2}.$$

Therefore, $\dot{x}_1(t, x_0) \in D_1$, *for all* $t \in [0,T]$

Then also by mathematical induction we can prove that

$$\|\dot{x}_m(t,x_0) - \dot{x}_0\| \leq M\,\frac{T}{2} \qquad \ldots (20)$$

From *(20)* we obtain that

$\|A\,\dot{x}_m(t,x_0) - A\,\dot{x}_0\| \leq Q_2 M\,\frac{T}{2}$.

That *is* $\dot{x}_m(t, x_0) \in D_1$, $A\dot{x}_m(t, x_0) \in D_3$ *for all* $t \in [0,T]$ and $\dot{x}_0 \in D_{1f}$, $A\dot{x}_0(t, x_0) \in D_{3f}$.

Also

$\|z_1(t,x_0) - z_0(t,x_0)\|$

$$\leq \int_0^{h(t)} \left\| P\left[\ \|x_1(s,x_0) - x_0\| + \|\dot{x}_1(s,x_0) - \dot{x}_0\| + \|Bx_1(s,x_0) - Bx_0\| + \left\|B^{\cdot} x_1(s,x_0) - B\dot{x}_0\right\|\ \right]\right\| ds$$

$\leq \frac{hPMT}{2}\,[\,(1+Q_4) + (\frac{T+TQ_3}{2})\,]$

That is $z_1(t, x_0) \in D_4$ for all $t \in [0,T]$ and $z_0 \in D_{4f}$.

Then also by mathematical induction we can prove the following

$\|z_m(t,x_0) - z_0(t,x_0)\| \leq \frac{hPMT}{2}\,[\,(1+Q_4) + (\frac{T+TQ_3}{2})\,]$

which given $z_m(t, x_0) \in D_4$ *for all* $t \in [0, T]$ and $z_0 \in D_{4f}$.

Next, we shall to prove that the sequence of functions (13) converges uniformly on the domain (2).

By the lemma 1 and using the sequence of functions (14), when $m = 1$, we get

$$\|\dot{x}_2(t,x_0) - \dot{x}_1(t,x_0)\| \leq (1-\frac{t}{T})\int_0^t K\,[\,\|x_1(s,x_0)-x_0\| + \|\dot{x}_1(s,x_0)-\dot{x}_0\| + Q_1\|x_1(s,x_0)-x_0\| + Q_2\|\dot{x}_1(s,x_0)-\dot{x}_0\|$$

$$+ Ph\,(\|x_1(s,x_0)-x_0\| + \|\dot{x}_1(s,x_0)-\dot{x}_0\|\,] + Q_3\|x_1(s,x_0)-x_0\| + Q_4\|\dot{x}_1(s,x_0)-\dot{x}_0\|\,]\,ds$$

$$+ \frac{t}{T}\int_t^T K[\|x_1(s,x_0)-x_0\| + \|\dot{x}_1(s,x_0)-\dot{x}_0\| + Q_1\|x_1(s,x_0)-x_0\| + Q_2\|\dot{x}_1(s,x_0)-\dot{x}_0\|$$

$$+Ph(\|x_1(s,x_0)-x_0\| + \|\dot{x}_1(s,x_0)-\dot{x}_0\| + Q_3\|x_1(s,x_0)-x_0\| + Q_4\|\dot{x}_1(s,x_0)-\dot{x}_0\|\,]\,ds$$

$$\leq \frac{H_1 T}{2}\,\|x_1(t,x_0) - x_0\| + \frac{H_2\,T}{2}\,\|\dot{x}_1(t,x_0) - \dot{x}_0\| \,. \qquad \ldots (21)$$

And by using the same method above, the following inequality holds

$$\|\dot{x}_{m+1}(t,x_0) - \dot{x}_m(t,x_0)\| \leq \frac{H_1 T}{2}$$

$$\|x_m(t,x_0) - x_{m-1}(t,x_0)\| + \frac{H_2 T}{2}\,\|\dot{x}_m(t,x_0) - \dot{x}_{m-1}(t,x_0)\| \qquad \ldots (22)$$

By using the sequence of functions (13) ,when $m = 1$, we get

$$\|x_2(t,x_0) - x_1(t,x_0)\| \leq \alpha^2(t)\,[\,H_1\,\|x_1(t,x_0) - x_0\| + H_2\,\|\dot{x}_1(t,x_0) - \dot{x}_0\|\,]$$

$$\leq \frac{H_1 T^2}{4}\,\|x_1(t,x_0) - x_0\| + \frac{H_2 T^2}{4}\,\|\dot{x}_1(t,x_0) - \dot{x}_0\| \qquad \ldots (23)$$

And also

$$\|\dot{x}_{m+1}(t,x_0) - \dot{x}_m(t,x_0)\| \leq \frac{H_1 T^2}{4}$$

$$\|x_m(t,x_0) - x_{m-1}(t,x_0)\| + \frac{H_2 T^2}{4}\,\|\dot{x}_m(t,x_0) - \dot{x}_{m-1}(t,x_0)\| \qquad \ldots (24)$$

for all $t \in [0, T]$ and all $m \geq 1$.

Rewrite, the inequalities (22) and (24) in a vector from

$$\begin{pmatrix} \|x_{m+1}(t,x_0) - x_m(t,x_0)\| \\ \|\dot{x}_{m+1}(t,x_0) - \dot{x}_m(t,x_0)\| \end{pmatrix} \leq \begin{pmatrix} \frac{\mathcal{H}_1 T^2}{4} & \frac{\mathcal{H}_2 T^2}{4} \\ \frac{\mathcal{H}_1 T}{2} & \frac{\mathcal{H}_2 T}{2} \end{pmatrix}$$

$$\begin{pmatrix} \|x_m(t,x_0) - x_{m-1}(t,x_0)\| \\ \|\dot{x}_m(t,x_0) - \dot{x}_{m-1}(t,x_0)\| \end{pmatrix}$$

That is

$$V_{m+1}(t,x_0) \leq \Lambda(t)\, V_m(t,x_0)$$

where $V_{m+1}(t,x_0) = \begin{pmatrix} \|x_{m+1}(t,x_0) - x_m(t,x_0)\| \\ \|\dot{x}_{m+1}(t,x_0) - \dot{x}_m(t,x_0)\| \end{pmatrix}$,

$$\Lambda(t) = \begin{pmatrix} \frac{\mathcal{H}_1 t^2}{4} & \frac{\mathcal{H}_2 t^2}{4} \\ \frac{\mathcal{H}_1 t}{2} & \frac{\mathcal{H}_2 t}{2} \end{pmatrix}$$

and

$$V_m(t,x_0) = \begin{pmatrix} \|x_m(t,x_0) - x_{m-1}(t,x_0)\| \\ \|\dot{x}_m(t,x_0) - \dot{x}_{m-1}(t,x_0)\| \end{pmatrix}.$$

If we assuming $(\Lambda = \max_{t \in [0,T]} \Lambda(t))$

We have the estimate

$$\sum_{i=1}^{m} V_i \leq \sum_{i=1}^{m} \Lambda^{i-1}\, V_1 \qquad \ldots (25)$$

where $V_1 = \begin{pmatrix} \frac{M\,T^2}{4} \\ \frac{M\,T}{2} \end{pmatrix}$

Since the matrix Λ has maximum eigen-values

$$\lambda_1 = 0 \ \ and \ \ \lambda_2 = \frac{\mathcal{H}_1 T^2 + 2\mathcal{H}_2 T}{4} < 1\,.$$

Then the series (25) is uniformly convergent, i. e.

$$\lim_{m\to\infty}\sum_{i=1}^{m}\Lambda^{i-1}V_1=\sum_{i=1}^{\infty}\Lambda^{i-1}V_1=(E-\Lambda)^{-1}V_1 \quad \cdots (26)$$

The limiting relation (26) signifies a uniform convergence of

$x_m(t,x_0)$ and $\dot{x}_m(t,x_0)$ in the domain (3) as $m\to\infty$.

Putting

$$\left.\begin{aligned}\lim_{m\to\infty} x_m(t,x_0)&=x(t,x_0)\\ \lim_{m\to\infty}\dot{x}_m(t,x_0)&=\dot{x}(t,x_0)\end{aligned}\right] \quad \cdots (27)$$

Next, we need to prove $x(t,x_0)\in D$ and $\dot{x}_\infty(t,x_0)\in D_1$, for all $t\in[0,T]$
Taking

$$\Big\|\int_0^t[Lf(s,x_m(s,x_0),\dot{x}_m(s,x_0),Ax_m(s,x_0),A\dot{x}_m(s,x_0),\int_0^{h(s)}g(\tau,x_m(\tau,x_0),\ \dot{x}_m(\tau,x_0),$$

$$Bx_m(\tau,x_0),B\dot{x}_m(\tau,x_0))\,d\tau\,]ds-\frac{1}{T}\int_t^T Lf(s,x_m(s,x_0),\dot{x}_m(s,x_0),Ax_m(s,x_0),A\dot{x}_m(s,x_0)$$

$$,\int_0^{h(s)}g(\tau,x_m(\tau,x_0)\,,\ \dot{x}_m(\tau,x_0),Bx_1(\tau,x_0),B\dot{x}_m(\tau,x_0))\,d\tau\,]ds-$$

$$\int_0^t[Lf(s,x(s,x_0),\dot{x}(s,x_0),Ax(s,x_0),A\dot{x}(s,x_0),\int_0^{h(s)}g(\tau,x(\tau,x_0),\dot{x}(\tau,x_0),$$

$$Bx(\tau,x_0),B\dot{x}(\tau,x_0))d\tau]ds-\frac{1}{T}\int_t^T[Lf(s,x(s,x_0),\dot{x}(s,x_0),Ax(s,x_0),A\dot{x}(s,x_0)$$

$$,\int_0^{h(s)}g(\tau,x(\tau,x_0),\dot{x}(\tau,x_0)\,,Bx(\tau,x_0),B\dot{x}(\tau,x_0))d\tau]ds\Big\|$$

$$\le\alpha(t)\Big\|\int_0^t[f(s,x_m(s,x_0),\dot{x}_m(s,x_0),Ax_m(s,x_0),A\dot{x}_m(s,x_0),\int_0^{h(s)}g(\tau,x_m(\tau,x_0),$$

$$\dot{x}_m(\tau,x_0),Bx_m(\tau,x_0),B\dot{x}_m(\tau,x_0))\,d\tau\,]ds-\frac{1}{T}\int_t^T f(s,x_m(s,x_0),\dot{x}_m(s,x_0),$$

$$,Ax_m(s,x_0),A\dot{x}_m(s,x_0),\int_0^{h(s)}g(\tau,x_m(\tau,x_0)\,,\dot{x}_m(\tau,x_0),Bx_1(\tau,x_0),B\dot{x}_m(\tau,x_0))\,d\tau\,]ds$$

$$-\int_0^t [f(s,x(s,x_0),\dot{x}(s,x_0),Ax(s,x_0),A\dot{x}(s,x_0),\int_0^{h(s)} g(\tau,x(\tau,x_0),\dot{x}(\tau,x_0),$$

$$Bx(\tau,x_0),B\dot{x}(\tau,x_0))d\tau]ds-\frac{1}{T}\int_t^T [f(s,x(s,x_0),\dot{x}(s,x_0),Ax(s,x_0),A\dot{x}(s,x_0)$$

$$,\int_0^{h(s)} g(\tau,x(\tau,x_0),\dot{x}(\tau,x_0)\,,Bx(\tau,x_0),B\dot{x}(\tau,x_0))d\tau]ds\Big\|$$

$$\leq \alpha^2(t)\ [\ H_1\ \|x_m(t,x_0)-x(t,x_0)\| + H_2\ \|\dot{x}_m(t,x_0)-\dot{x}(t,x_0)\|\ ]$$

$$\leq \frac{H_1T^2}{4}\ \|x_m(t,x_0)-x(t,x_0)\| + \frac{H_2T^2}{4}\ \|\dot{x}_m(t,x_0)-\dot{x}(t,x_0)\|\ .$$

From (27), we find that

$$\|x_m((t,x)-\ x(t,x)\ \| \leq \in_1 \quad and \quad \|\dot{x}_m((t,x)-\ \dot{x}(t,x)\ \| \leq \in_1$$

Therefore

$$\Big\|\int_0^t [Lf(s,x_m(s,x_0),\dot{x}_m(s,x_0),Ax_m(s,x_0),A\dot{x}_m(s,x_0),\int_0^{h(s)} g(\tau,x_m(\tau,x_0),\ \dot{x}_m(\tau,x_0),$$

$$Bx_m(\tau,x_0),B\dot{x}_m(\tau,x_0))\ d\tau\ ]ds-\frac{1}{T}\int_t^T Lf(s,x_m(s,x_0),\dot{x}_m(s,x_0),Ax_m(s,x_0),$$

$$A\dot{x}_m(s,x_0)\,,\int_0^{h(s)} g(\tau,x_m(\tau,x_0)\,,\ \dot{x}_m(\tau,x_0),Bx_1(\tau,x_0),B\dot{x}_m(\tau,x_0))\ d\tau\ ]ds-$$

$$\int_0^t [Lf(s,x(s,x_0),\dot{x}(s,x_0),Ax(s,x_0),A\dot{x}(s,x_0),\int_0^{h(s)} g(\tau,x(\tau,x_0),\dot{x}(\tau,x_0),$$

$$Bx(\tau,x_0),B\dot{x}(\tau,x_0))d\tau]ds-\frac{1}{T}\int_t^T [Lf(s,x(s,x_0),\dot{x}(s,x_0),Ax(s,x_0),$$

$$A\dot{x}(s,x_0)\,,\int_0^{h(s)} g(\tau,x(\tau,x_0),\dot{x}(\tau,x_0)\,,Bx(\tau,x_0),B\dot{x}(\tau,x_0))d\tau]ds\Big\|$$

$$\leq \frac{H_1T^2}{4}\ \in_1 + \frac{H_2T^2}{4}\ \in_1$$

$$\leq \in_1\ \Big(\ \frac{(H_1+H_2)\,T^2}{4}\Big)$$

$\leq \epsilon$, for all $m \geq 0$, where $\in_1 = \frac{4\in}{(H_1+H_2)\,T^2}$

So that

$$\lim_{m\to\infty}\int_0^t [\, Lf(s, x_m(s,x_0), \dot{x}_m(s,x_0), Ax_m(s,x_0) A\dot{x}_m(s,x_0), \int_0^{h(s)} g(\tau, x_m(\tau,x_0),$$

$$\dot{x}_m(\tau,x_0), Bx_m(\tau,x_0), B\dot{x}_m(\tau,x_0)\; d\tau \,]\; ds - \frac{1}{T}\int_t^T Lf(s, x_m(s,x_0), \dot{x}_m(s,x_0),$$

$$Ax_m(s,x_0) A\dot{x}_m(s,x_0)\,, \int_0^{h(s)} g(\tau, x_m(\tau,x_0)\,,\; \dot{x}_m(\tau,x_0), Bx_1(\tau,x_0), B\dot{x}_m(\tau,x_0))\; d\tau \,]ds$$

$$= \int_0^t [\, Lf(s, x(s,x_0), \dot{x}(s,x_0), Ax(s,x_0), A\dot{x}(s,x_0), \int_0^{h(s)} g(\tau, x(\tau,x_0), \dot{x}(s,x_0),$$

$$Bx(\tau,x_0), B\dot{x}(\tau,x_0))d\tau]ds - \frac{1}{T}\int_t^T [Lf(s, x(s,x_0), \dot{x}(s,x_0), Ax(s,x_0),$$

$$A\dot{x}(s,x_0)\,, \int_0^{h(s)} g(\tau, x(\tau,x_0), \dot{x}(\tau,x_0)\,, Bx(\tau,x_0), B\dot{x}(\tau,x_0))d\tau]ds$$

Thus $x(t, x_0) \in D$, $\dot{x}_\infty(t, x_0) \in D_1$ and $x(t, x_0)$ is a periodic solutions of (1).

III Uniqueness of Periodic Solution

The study of the uniqueness periodic solution of problem (1) is introduced by:

Theorem 2: *If the right side of the problem (1) satisfying all condition and inequalities of theorem 1, then there exist a unique continuous periodic solution of the problem (1).*

Proof: Suppose that $r(t, x_0)$ is another periodic solution of the problem (1), then

$$r(t,x_0) = x_0 + L^2 f\left(t, x_m(t,x_0), \dot{x}_m(t,x_0), y_m(t,x_0), \dot{y}_m(t,x_0)\,, z_m(t,x_0)\right)$$

and

$$\dot{r}(t,x_0) = x_0 + L\, f\left(t, x_m(t,x_0), \dot{x}_m(t,x_0), y_m(t,x_0), \dot{y}_m(t,x_0)\,, z_m(t,x_0)\right)$$

where

$$L^2 f\left(t, x_m(t,x_0), \dot{x}_m(t,x_0), y_m(t,x_0), \dot{y}_m(t,x_0)\,, z_m(t,x_0)\right)$$

$$= \int_0^t [\, Lf(s, r(s,x_0), \dot{r}(s,x_0), Ar(s,x_0), A\dot{r}(s,x_0),$$

$$\int_0^{h(s)} g(\tau, r(\tau,x_0),\ \dot r(s,x_0), Br(\tau,x_0), B\dot r(\tau,x_0))d\tau]ds$$

$$-\frac{1}{T}\int_t^T [\ Lf(s, r(s,x_0), \dot r(s,x_0)\ , Ar(s,x_0), A\dot r(s,x_0),$$

$$\int_0^{h(s)} g(\tau, r(\tau,x_0), \dot r(\tau,x_0)\ , Br(\tau,x_0), B\dot r(\tau,x_0))d\tau]ds$$

and

$$L\ f\ \big(t, x_m(t,x_0), \dot x_m(t,x_0), y_m(t,x_0), \dot y_m(t,x_0)\ , z_m(t,x_0)\big)$$

$$= \int_0^t f(s, r(s,x_0), \dot r(s,x_0), Ar(s,x_0), A\dot r(s,x_0),$$

$$\int_0^{h(s)} g(\tau, r(\tau,x_0), \dot r(s,x_0), Br(\tau,x_0), B\dot r(\tau,x_0))d\tau]ds$$

$$-\frac{1}{T}\int_t^T [f(s, r(s,x_0), , \dot r(s,x_0), Ar(s,x_0) A\dot r(s,x_0)\ ,$$

$$\int_0^{h(s)} g(\tau, r(\tau,x_0), \dot r(\tau,x_0)\ , Br(\tau,x_0), B\dot r(\tau,x_0))d\tau]ds$$

Taking

$$\|x(t,x_0) - \ r(t,x_0)\ \|$$

$$\leq \alpha^2(t)\ [\ \mathcal{H}_1\ \|x(t,x_0) - r(t,x_0)\| + \mathcal{H}_2\ \|\dot x(t,x_0) - \dot r(t,x_0)\|\ ]$$

$$\leq \frac{\mathcal{H}_1 T^2}{4}\ \|x(t,x_0) - r(t,x_0)\| + \frac{\mathcal{H}_2 T^2}{4}\ \|\dot x(t,x_0) - \dot r(t,x_0)\| \qquad \dots (28)$$

and

$$\|\dot x(t,x_0) - \ \dot r(t,x_0)\ \|\ + \mathcal{H}_2\ \|\dot x(t,x_0) - \dot r(t,x_0)\|\ ]$$

$$\leq \frac{\mathcal{H}_1 T}{2}\ \|x(t,x_0) - r(t,x_0)\| + \frac{\mathcal{H}_2 T}{2}\ \|\dot x(t,x_0) - \dot r(t,x_0)\| \qquad \dots (29)$$

From (28) and (29) we have

$$\begin{pmatrix} \|x(t,x_0) - r(t,x_0)\ \| \\ \|\dot x(t,x_0) - \dot r(t,x_0)\ \| \end{pmatrix} \leq \begin{pmatrix} \frac{\mathcal{H}_1 T^2}{4} & \frac{\mathcal{H}_2 T^2}{4} \\ \frac{\mathcal{H}_1 T}{2} & \frac{\mathcal{H}_2 T}{2} \end{pmatrix} \begin{pmatrix} \|x(t,x_0) - \ r(t,x_0)\ \| \\ \|\dot x(t,x_0) - \ \dot r(t,x_0)\ \| \end{pmatrix}$$

By the condition $\lambda_{max}(\Lambda) < 1$, then

$$\begin{pmatrix} \|x(t,x_0) - r(t,x_0)\| \\ \|\dot{x}(t,x_0) - \dot{r}(t,x_0)\| \end{pmatrix} < \begin{pmatrix} \|x(t,x_0) - r(t,x_0)\| \\ \|\dot{x}(t,x_0) - \dot{r}(t,x_0)\| \end{pmatrix}.$$

We get contradiction, then $\begin{pmatrix} \|x(t,x_0) - r(t,x_0)\| \\ \|\dot{x}(t,x_0) - \dot{r}(t,x_0)\| \end{pmatrix} = \binom{0}{0}$

Therefore, $x(t,x_0) = r(t,x_0)$, $\dot{x}(t,x_0) = \dot{r}(t,x_0)$ and hence $x(t,x_0)$ is a unique periodic solution of the problem (1).

IV Existence of Periodic Solution

The problem of existence of periodic solution of a period T of (1) is uniquely connected with the existence of zeros of the function $\Delta(t,x_0)$ which has the form

$$\Delta\,(0,x_0) = \frac{2}{T^2}\int_0^T\int_0^T [\, f(s,x^0(s,x_0),\dot{x}^0(s,x_0),Ax^0(s,x_0),A\dot{x}^0(s,x_0)$$

$$\int_0^{h(s)} g\big(\tau,x^0(\tau,x_0),\dot{x}^0(s,x_0),Bx^0(\tau,x_0),B\dot{x}^0(s,x_0)\big)d\tau\,]\,ds\,ds \qquad \dots (30)$$

where

$\Delta: D_f \longrightarrow R^1$

and $x^0(t,x_0)$ is the limiting function of (13). Then the equation (30) is approximation determined by the sequence of functions

$$\Delta_m(0,x_0) = \frac{2}{T^2}\int_0^T\int_0^T [\, f(s,x_m(s,x_0),\dot{x}_m(s,x_0),Ax_m(s,x_0),A\dot{x}_m(s,x_0),$$

$$\int_0^{h(s)} g(\tau,x_m(\tau,x_0),\dot{x}_m(s,x_0),Bx_m(\tau,x_0),B\dot{x}_m(s,x_0))d\tau]\,ds\,ds \qquad \dots (31)$$

where

$\Delta_m: D_f \longrightarrow R^1$, *m=0,1,2,…*

Theorem 3: *Under the hypothesis of theorems 1 and 2, the following inequality*

$$\|\Delta(0,x_0) - \Delta_m(0,x_0)\| \le \eta_m \qquad \cdots (32)$$

is hold, where $\eta_m = \langle \begin{pmatrix} 'H_1 \\ 'H_2 \end{pmatrix}, \Lambda^m (E-\Lambda)^{-1} V_1 \rangle$, *for all* $m \geq 0$.

Proof: Assume that

$\|\Delta(0,x_0) - \Delta_m(0,x_0)\|$

$\leq \frac{2}{T^2}\int_0^T \int_0^T K \, [\, \|x^0(s,x_0) - x_m(s,x_0)\| + \|\dot{x}^0(s,x_0) - \dot{x}_m(s,x_0)\| + Q_1\|x^0(s,x_0) - x_m(s,x_0)\|$

$+Q_2\|\dot{x}^0(s,x_0) - \dot{x}_m(s,x_0)\| + Ph(\|x^0(s,x_0) - x_m(s,x_0)\| + \|\dot{x}^0(s,x_0) - \dot{x}_m(s,x_0)\|$

$+Q_3\|x^0(s,x_0) - x_m(s,x_0)\| + Q_4\|\dot{x}^0(s,x_0) - \dot{x}_m(s,x_0)\| \,]\, ds$

$\leq \langle \begin{pmatrix} 'H_1 \\ 'H_2 \end{pmatrix}, \Lambda^m (E-\Lambda)^{-1} V_1 \rangle = \eta_m$

where $\langle . \rangle$ denotes the ordinary scalar product .

By using the theorem 3 , we can state and proof the following theorem

Theorem 4: *Let the functions* $f\,(t,x,\dot{x},y,\dot{y},z)$ *and* $g(t,x,\dot{x},w,\dot{w})$ *be defined on the domain* $G = \{0 \leq s \leq t \leq T, a \leq x \leq b \quad, e \leq y,z \leq f\} \subseteq R^1$, *suppose that the sequence of functions* $\Delta_m(0,x_0)$*which is is defined in (31) satisfy the inequalities:*

$$\left.\begin{aligned} \min_{a+P_1 \leq x_0 \leq b-P_1} \Delta_m(0,x_0) \leq -\eta_m \quad , \\ \max_{a+P_1 \leq x_0 \leq b-P_1} \Delta_m(0,x_0) \leq \eta_m \quad . \end{aligned}\right\} \quad \cdots (33)$$

for all $m \geq 0$, where $P_1 = M(h_2 - h_1)$ and $\eta_m = \|\Omega^{m+1}(1-\Omega)^{-1}M\|$.

Then the problem (1) has periodic solution $x = x(t,x_0)$ for which

$x_0 \in [a + P_1, b - P_1]$.

Proof: Let x_1, x_2 be any points in the interval $x_0 \in [a + P_1, b - P_1]$ such that

$$\left.\begin{aligned} \Delta_m(0,x_1) = \min_{a+P_1 \leq x_0 \leq b-P_1} \Delta_m(0,x_0) \quad , \\ \Delta_m(0,x_2) = \max_{a+P_1 \leq x_0 \leq b-P_1} \Delta_m(0,x_0) \quad , \end{aligned}\right\} \quad \cdots (34)$$

From the inequalities (32) and (33), we have

$$\left.\begin{aligned} \Delta(0,x_1) = \Delta_m(0,x_1) + [\Delta(0,x_1) - \Delta_m(0,x_1)] \leq 0 \quad , \\ \Delta(0,x_2) = \Delta_m(0,x_2) + [\Delta(0,x_2) - \Delta_m(0,x_2)] \geq 0 \quad . \end{aligned}\right\} \quad \cdots (35)$$

It follows from (35) and the continuity of the function $\Delta(0, x_0)$, that there exists an isolated singular point x^0, $x^0 \in [x_1, x_2]$, such that $\Delta(0, x^0) = 0$. This means that the system (1) has a periodic solution $x = x(t, x_0)$ for which $x_0 \in [a + P_1, b - P_1]$.

V Stability of Periodic Solution

In this section, we shall prove the theorem of stability periodic solution for the problem (1).

Theorem 5: *If the function* $\Delta(0, x_0)$ *be defined by* (20), *where* $x^0(t, x_0)$ *is a limit function of* $\{x_m(0, x_0)\}_{m=0}^{\infty}$, *then the following inequalities*

$$\|\Delta(0, x_0)\| \leq M \qquad \dots (36)$$

and

$$\|\Delta(0, x_0^1) - \Delta(0, x_0^2)\| \leq F_1\, F_2 \left('H_1 \,(1 - \frac{T^2}{4}\ 'H_2) + \frac{T}{2}\ 'H_1\, 'H_2 \right) \|x_0^1(t) - x_0^2(t)\|$$

$$+ F_1\, 'H_2 \left(\frac{T^2}{4}\ 'H_1\, F_2 + (1 - \frac{T}{2}\ 'H_1)\,[1 + \frac{T^3}{8} F_1\, F_2\, 'H_1\, 'H_2]\right) \|\dot{x}_0^1(t) - \dot{x}_0^2(t)\| \quad \dots (37)$$

are holds for all $x^0, x_0^1, x_0^2 \in D_f$, $\dot{x}^0, \dot{x}_0^1, \dot{x}_0^2 \in D_{1f}$

where $F_1 = [\,(1 - \frac{T^2}{4}\ 'H_1)\,(1 - \frac{T^2}{4}\ 'H_2)\,]^{-1}$ *and* $F_2 = (1 - \frac{T^3}{8}\ 'H_1\, 'H_2\, F_1)^{-1}$

Proof: From the equation (30), we get

$$\|\Delta(0, x_0)\| \leq \frac{2}{T} \int_0^T \int_0^T \|f(s, x^0(s, x_0), \dot{x}^0(s, x_0), Ax^0(s, x_0), A\dot{x}^0(s, x_0)$$

$$, \int_0^{h(s)} g\big(\tau, x^0(\tau, x_0), \dot{x}^0(s, x_0), Bx^0(\tau, x_0), B\dot{x}^0(s, x_0)\big) d\tau \Big\| ds\, ds$$

$$\leq \frac{2}{T} \int_0^T \int_0^T (M)\, ds\, ds$$

$$\leq M$$

And by using (31), we find that

$$\|\Delta(0,x_0^1)-\Delta(0,x_0^2)\| \le \acute{H}_1 \|x^0(t,x_0^1)-x^0(t,x_0^2)\| + \acute{H}_2 \|\dot{x}^0(t,x_0^1)-\dot{x}^0(t,x_0^2)\| \quad \ldots(38)$$

where $x^0(t,x_0^1)$, $x^0(t,x_0^2)$, $\dot{x}^0(s,x_0^1)$, $\dot{x}^0(s,x_0^2)$ are the periodic solutions of the following integral equations

$$x(t,x_0^k) = x_0^k(t) + \int_0^t [\, Lf(s,x^0(s,x_0^k),\dot{x}^0(s,x_0^k),Ax^0(s,x_0^k),A\dot{x}^0(s,x_0^k)$$

$$\int_0^{h(s)} g\left(\tau,x^0(\tau,x_0^k),\dot{x}^0(s,x_0^k),Bx^0(\tau,x_0^k),B\dot{x}^0(s,x_0^k)\right) d\tau - \frac{1}{T}\int_0^T [\, Lf(s,x^0(s,x_0^k),\dot{x}^0(s,x_0^k),$$

$$Ax^0(s,x_0^k),A\dot{x}^0(s,x_0^k),\int_0^{h(s)} g(\tau,x^0(\tau,x_0^k)\dot{x}^0(s,x_0^k),Bx^0(s,x_0^k),B\dot{x}^0(s,x_0^k)\,]\,ds \quad \ldots(39)$$

And

$$\dot{x}(t,x_0^k) = \dot{x}_0^k(t) + \int_0^t [\, f(s,x^0(s,x_0^k),\dot{x}^0(s,x_0^k),Ax^0(s,x_0^k),A\dot{x}^0(s,x_0^k)$$

$$\int_0^{h(s)} g\left(\tau,x^0(\tau,x_0^k),\dot{x}^0(s,x_0^k),Bx^0(\tau,x_0^k),B\dot{x}^0(s,x_0^k)\right) d\tau - \frac{1}{T}\int_0^T [\, f(s,x^0(s,x_0^k),\dot{x}^0(s,x_0^k),$$

$$Ax^0(s,x_0^k)A\dot{x}^0(s,x_0^k),\int_0^{h(s)} g(\tau,x^0(\tau,x_0^k),\dot{x}^0(s,x_0^k),Bx^0(s,x_0^k),B\dot{x}^0(s,x_0^k)\,]\,ds \quad \ldots(40)$$

where $k = 1, 2.$

Now, by using (39), we have

$$\|x^0(t,x_0^1) - x^0(t,x_0^2)\|$$

$$\le \|x_0^1(t)-x_0^2(t)\| + \frac{\acute{H}_1 T^2}{4}\|x^0(s,x_0^1)-x^0(s,x_0^2)\| + \frac{\acute{H}_2 T^2}{4}\|\dot{x}^0(s,x_0^1)-\dot{x}^0(s,x_0^2)\|$$

$$\|x^0(t,x_0^1)-x^0(t,x_0^2)\| \le \left(1-\frac{T^2}{4}\acute{H}_1\right)^{-1}\|x_0^1(t)-x_0^2(t)\|$$

$$+\frac{T^2}{4}\acute{H}_2\left(1-\frac{T^2}{4}\acute{H}_1\right)^{-1}\|\dot{x}^0(s,x_0^1)-\dot{x}^0(s,x_0^2)\| \quad \ldots(41)$$

And from (40), we have

$$\|\dot{x}^0(t,x_0^1)-\dot{x}^0(t,x_0^2)\| \le \|\dot{x}_0^1(t)-\dot{x}_0^2(t)\|+\frac{'\mathrm{H}_1 T}{2}\|x^0(s,x_0^1)-x^0(s,x_0^2)\| + \frac{'\mathrm{H}_2 T}{2}\|\dot{x}^0(s,x_0^1)-\dot{x}^0(s,x_0^2)\|$$

Therefore

$$\|\dot{x}^0(t,x_0^1)-\dot{x}^0(t,x_0^2)\| \le (1-\frac{T}{2}\ '\mathrm{H}_2)^{-1}\|\dot{x}_0^1(t)-\dot{x}_0^2(t)\| + \frac{T}{2}\ '\mathrm{H}_1\ (1-\frac{T}{2}\ '\mathrm{H}_2)^{-1}\|x^0(s,x_0^1)-x^0(s,x_0^2)\| \qquad \dots (42)$$

By substituting inequality (42) in (41), we get

$$\|x^0(t,x_0^1)-x^0(t,x_0^2)\| \le (1-\frac{T^2}{4}\ '\mathrm{H}_1)^{-1}\|x_0^1(t)-x_0^2(t)\|$$

$$+\frac{T^2}{4}\ '\mathrm{H}_2\ [(1-\frac{T^2}{4}\ '\mathrm{H}_1)(1-\frac{T^2}{4}\ '\mathrm{H}_2)]^{-1}\|\dot{x}_0^1(t)-\dot{x}_0^2(t)\|$$

$$+\frac{T^3}{8}\ '\mathrm{H}_1\ '\mathrm{H}_2[(1-\frac{T^2}{4}\ '\mathrm{H}_1)(1-\frac{T^2}{4}\ '\mathrm{H}_2)]^{-1}\|x^0(s,x_0^1)-x^0(s,x_0^2)\|$$

Putting $F_1=[(1-\frac{T^2}{4}\ '\mathrm{H}_1)(1-\frac{T^2}{4}\ '\mathrm{H}_2)]^{-1}$,

then $F_1\left(1-\frac{T^2}{4}\ '\mathrm{H}_2\right)=\left(1-\frac{T^2}{4}\ '\mathrm{H}_1\right)^{-1}$.

So that

$$\|x^0(t,x_0^1)-x^0(t,x_0^2)\| \le F_1(1-\frac{T^2}{4}\ '\mathrm{H}_2)\|x_0^1(t)-x_0^2(t)\| + \frac{T^2}{4}\ '\mathrm{H}_2\ F_1\|\dot{x}_0^1(t)-\dot{x}_0^2(t)\|$$

$$+\frac{T^3}{8}\ '\mathrm{H}_1\ '\mathrm{H}_2\ F_1\|x^0(s,x_0^1)-x^0(s,x_0^2)\|$$

$$\le F_1(1-\frac{T^2}{4}\ '\mathrm{H}_2)(1-\frac{T^3}{8}\ '\mathrm{H}_1\ '\mathrm{H}_2\ F_1)^{-1}\|x_0^1(t)-x_0^2(t)\|$$

$$+\frac{T^2}{4}\ '\mathrm{H}_2\ F_1(1-\frac{T^3}{8}\ '\mathrm{H}_1\ '\mathrm{H}_2\ F_1)^{-1}\|\dot{x}_0^1(t)-\dot{x}_0^2(t)\|$$

Putting $F_2 = (\, 1 - \frac{T^3}{8}\ \text{'H}_1\,\text{'H}_2\, F_1)^{-1}$

This implies that

$$\|x^0(t,x_0^1) - x^0(t,x_0^2)\| \leq\ F_1\ F_2\ \ (1 - \frac{T^2}{4}\ \text{'H}_2)\ \|x_0^1(t) - x_0^2(t)\| + \frac{T^2}{4}\,\text{'H}_2\, F_1\, F_2\, \|\dot{x}_0^1(t) - \dot{x}_0^2(t)\| \quad \dots (43)$$

Also substituting the inequalities (43) in (42), we get

$$\|\dot{x}^0(t,x_0^1) - \dot{x}^0(t,x_0^2)\| \leq (\,1 - \frac{T}{2}\ \ \text{'H}_2)^{-1}\|\dot{x}_0^1(t) - \dot{x}_0^2(t)\|$$

$$+ \frac{T}{2}\,\text{'H}_1\,(\,1 - \frac{T}{2}\ \ \text{'H}_2)^{-1}\ [\,F_1\ F_2\ (1 - \frac{T^2}{4}\ \text{'H}_2)\ \|x_0^1(t) - x_0^2(t)\| + \frac{T^2}{4}\,\text{'H}_2\, F_1\, F_2\, \|\dot{x}_0^1(t) - \dot{x}_0^2(t)\|]$$

And hence

$$\|\dot{x}^0(t,x_0^1) - \dot{x}^0(t,x_0^2)\| \leq \frac{T}{2}\,\text{'H}_1 F_1\, F_2\ \ \|x_0^1(t) - x_0^2(t)\| + \ F_1\,(1 - \frac{T}{2}\ \ \text{'H}_1)\ [1 + + \frac{T^3}{8}\ F_1\, F_2\ \ \text{'H}_1\ \ \text{'H}_2]\ \|\dot{x}_0^1(t) - \dot{x}_0^2(t)\|] \quad \dots (44)$$

So, substituting the inequalities (43) and (44) in (38), we get the inequality (37).

VI Existence and Uniqueness Periodic Solution

In this section, we prove the existence and uniqueness theorem for the problem (1) by using Banach fixed point theorem [9].

Theorem 6: *Let the vector functions* $f(t,x,\dot{x},y,\dot{y},z)$ *and* $g(t,x,\dot{x},w,\dot{w})$ *on the (1) are defined and continuous on the domain (2) and satisfies assumptions and all conditions of theorem1, then the problem (1) has a unique periodic continuous solution on the domain (2).*

Proof: Let *(C [0,T]* $, \|.\|$ *)* be a Banach space and T^* be a mapping on *C [0,T]* as follows:

$$T^* x(t,x_0) = x_0 + \int_0^t [\, Lf(s, x(s,x_0), \dot{x}(s,x_0), Ax(s,x_0), A\dot{x}(s,x_0), \int_0^{h(s)} g(\tau, x(\tau,x_0), \dot{x}(s,x_0), Bx(\tau,x_0),$$

$$B\dot{x}(\tau,x_0))d\tau]ds$$
$$-\frac{1}{T}\int_t^T [\,Lf(s,x(s,x_0)\,,\dot{x}(s,x_0),Ax(s,x_0)A\dot{x}(s,x_0)\int_0^{h(s)} g(\tau,x(\tau,x_0),\dot{x}(\tau,x_0)\;,Bx(\tau,x_0)$$
$$,B\dot{x}(\tau,x_0))d\tau]ds$$

and

$$T^*\dot{x}(t,x_0)$$
$$=\dot{x}_0$$
$$+\int_0^t f(s,x(s,x_0),\dot{x}(s,x_0),Ax(s,x_0),A\dot{x}(s,x_0),\int_0^{h(s)} g(\tau,x(\tau,x_0),\dot{x}(s,x_0),Bx(\tau,x_0),$$

$$B\dot{x}(\tau,x_0))d\tau]ds$$
$$-\frac{1}{T}\int_t^T [\,f(s,x(s,x_0),,\dot{x}(s,x_0),Ax(s,x_0)A\dot{x}(s,x_0)\,,\int_0^{h(s)} g(\tau,x(\tau,x_0),\dot{x}(\tau,x_0)\,,Bx(\tau,x_0),$$
$$B\dot{x}(\tau,x_0))d\tau]ds$$

Since

$$\int_0^{h(s)} g(\tau,x(\tau,x_0),\dot{x}(s,x_0),Bx(\tau,x_0),B\dot{x}(\tau,x_0))d\tau]ds$$
$$-\frac{1}{T}\int_t^T [\,Lf(s,x(s,x_0),,\dot{x}(s,x_0),Ax(s,x_0)A\dot{x}(s,x_0)$$

$$,\int_0^{h(s)} g(\tau,x(\tau,x_0),\dot{x}(\tau,x_0)\,,Bx(\tau,x_0),B\dot{x}(\tau,x_0))d\tau]ds$$

is continuous on the same domain (2) and also

$$\int_0^t [\,Lf(s,x(s,x_0),\dot{x}(s,x_0),Ax(s,x_0),A\dot{x}(s,x_0),$$

$$\int_0^{h(s)} g(\tau,x(\tau,x_0),\dot{x}(s,x_0),Bx(\tau,x_0),B\dot{x}(\tau,x_0))d\tau]ds$$

$$-\frac{1}{T}\int_t^T [\,Lf(s,x(s,x_0)\,,\dot{x}(s,x_0),Ax(s,x_0)A\dot{x}(s,x_0),$$

$$\int_0^{h(s)} g(\tau,x(\tau,x_0),\dot{x}(\tau,x_0)\,,Bx(\tau,x_0),B\dot{x}(\tau,x_0))d\tau]ds$$

is continuous on the same domain .

There fore T^**: C [0,T]* $\longrightarrow$ *C [0,T]*

Now, we shall to prove that T^* is a contraction mapping on *C [0,T]*.

Let $x(t, x_0)$, $z(t, x_0)$ be a vector functions on *C [0,T]* , then

$\| T^* x(t,x_0) - T^* z(t,x_0) \|$

$\leq \max_{t\in[0,T]} \{ \int_0^t | [Lf(s,x(s,x_0),\dot{x}(s,x_0),Ax(s,x_0),A\dot{x}(s,x_0),$

$\int_0^{h(s)} g(\tau,x(\tau,x_0),\dot{x}(s,x_0),Bx(\tau,x_0),$

$B\dot{x}(\tau,x_0))d\tau]ds$
$-\frac{1}{T}\int_t^T [Lf(s,x(s,x_0),,\dot{x}(s,x_0),Ax(s,x_0),A\dot{x}(s,x_0) , \int_0^{h(s)} g(\tau,x(\tau,x_0),\dot{x}(\tau,x_0)$

$,Bx(\tau,x_0),B\dot{x}(\tau,x_0))d\tau]ds$
$-\int_0^t [Lf(s,z(s,x_0),\dot{z}(s,x_0),Az(s,x_0),A\dot{z}(s,x_0), \int_0^{h(s)} g(\tau,z(\tau,x_0),$

$\dot{z}(s,x_0),,Bz(\tau,x_0),B\dot{z}(\tau,x_0))d\tau]ds$

$$-\frac{1}{T}\int_t^T [Lf(s,z(s,x_0),\dot{z}(s,x_0),Az(s,x_0)A\dot{z}(s,x_0)$$

$$,\int_0^{h(s)} g(\tau,z(\tau,x_0),\dot{z}(\tau,x_0) ,Bz(\tau,x_0),B\dot{z}(\tau,x_0))d\tau]ds \,|\, \}$$

$\leq \frac{H_1 T^2}{4} \|x(t,x_0) - r(t,x_0)\| + \frac{H_2 T^2}{4} \|\dot{x}(t,x_0) - \dot{r}(t,x_0)\|$ … (45)

and also by the same way

$\| T^* \dot{x}(t,x_0) - T^* \dot{z}(t,x_0) \|$

$\leq \max_{t\in[0,T]} \{ \int_0^t | [f(s,x(s,x_0),\dot{x}(s,x_0),Ax(s,x_0),A\dot{x}(s,x_0),$

$\int_0^{h(s)} g(\tau,x(\tau,x_0),\dot{x}(s,x_0),Bx(\tau,x_0),$

$B\dot{x}(\tau,x_0))d\tau]ds$
$-\frac{1}{T}\int_t^T [f(s,x(s,x_0),,\dot{x}(s,x_0),Ax(s,x_0),A\dot{x}(s,x_0) , \int_0^{h(s)} g(\tau,x(\tau,x_0),\dot{x}(\tau,x_0)$

$$, Bx(\tau, x_0), B\dot{x}(\tau, x_0))d\tau]ds$$

$$- \int_0^t [\, f(s, z(s, x_0), \dot{z}(s, x_0), Az(s, x_0), A\dot{z}(s, x_0), \int_0^{h(s)} g(\tau, z(\tau, x_0),$$

$$\dot{z}(s, x_0),, Bz(\tau, x_0), B\dot{z}(\tau, x_0))d\tau]ds - \frac{1}{T}\int_t^T [\, f(s, z(s, x_0), \dot{z}(s, x_0), Az(s, x_0)A\dot{z}(s, x_0),$$

$$, \int_0^{h(s)} g(\tau, z(\tau, x_0), \dot{z}(\tau, x_0)\, , Bz(\tau, x_0), B\dot{z}(\tau, x_0))d\tau]ds \mid \}$$

$$\leq \frac{Ή_1 T}{2} \; \|x(t, x_0) - r(t, x_0)\| + \frac{Ή_1 T}{2} \; \|\dot{x}(t, x_0) - \dot{r}(t, x_0)\| \qquad \dots (46)$$

Rewrite (45) and (46) in a vector form:

$$\begin{pmatrix} \|T^* x(t, x_0) - T^* z(t, x_0)\| \\ \|T^* \dot{x}(t, x_0) - T^* \dot{z}(t, x_0)\| \end{pmatrix} \leq \begin{pmatrix} \frac{Ή_1 T^2}{4} & \frac{Ή_2 T^2}{4} \\ \frac{Ή_1 T}{2} & \frac{Ή_2 T}{2} \end{pmatrix} \begin{pmatrix} \|x(t, x_0) - z(t, x_0)\| \\ \|\dot{x}(t, x_0) - \dot{z}(t, x_0)\| \end{pmatrix}$$

By the condition $\lambda_{max}\, \Lambda < 1$. Then T^* is a contraction mapping and hence by Banach fixed point theorem then there exists a fixed point $x(t, x_0)$ in *C [0,T]*.

Such that

$T^* x(t, x_0) = x(t, x_0)$

Therefore

$$x(t, x_0) = x_0 + \int_0^t [Lf(s, x(s, x_0), \dot{x}(s, x_0), Ax(s, x_0), A\dot{x}(s, x_0), \int_0^{h(s)} g(\tau, x(\tau, x_0)$$

$$, \dot{x}(s, x_0), Bx(\tau, x_0), B\dot{x}(\tau, x_0))d\tau]ds - \frac{1}{T}\int_t^T [\, Lf(s, x(s, x_0),, \dot{x}(s, x_0), Ax(s, x_0)$$

$$, A\dot{x}(s, x_0)\, , \int_0^{h(s)} g(\tau, x(\tau, x_0), \dot{x}(\tau, x_0)\, , Bx(\tau, x_0), B\dot{x}(\tau, x_0))d\tau]ds$$

is a unique periodic continuous solution of the problem (1).

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