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Eigenvalue Problem for Discontinuous Dirac System with Eigenparameter in a Transmission Condition

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Abstract

We deal with an eigenvalue problem for discontinuous Dirac system which includes an eigenvalue parameter in a transmission condition. We investigate asymptotic behavior of eigenvalues and eigenfunctions of this Dirac system.

Keywords: *Asymptotic approximation, discontinuous Dirac system, eigenvalue problem.*

1 Introduction

The basic and comprehensive results about Dirac operators were given in [12]. The oscillation property and asymptotic formulas for the eigenvalues were given in [10] and the computation of the eigenvalues of Dirac systems using a regularized sampling method was investigated in [5]. They are examples for continuous Dirac systems (see also [4, 17]). Direct and inverse problems for Dirac operators with discontinuities inside an interval were investigated in [3], the computation of the eigenvalues of discontinuous Dirac system using Hermite interpolation technique was given in [15]. In these discontinuous works, neither the boundary conditions nor the discontinuity condition depend on the spectral parameter. Inverse spectral problems and uniqueness theorems of the inverse problems for the same Dirac operator with eigenvalue dependent boundary and jump conditions were studied in [11, 18], respectively. Also, in [8] the authors investigated asymptotic behavior of eigenvalues and constructed Green's function for a Dirac system which has two points of discontinuity and contains an eigenparameter in a boundary and two transmission conditions.

We consider the following discontinuous Dirac system

$$\left.\begin{aligned} u_2'(x)-r_1(x)u_1(x)=\lambda u_1(x)\\ u_1'(x)+r_2(x)u_2(x)=-\lambda u_2(x)\end{aligned}\right\},\ x\in I, \tag{1}$$

$$B_1(u):=u_1(-1)\sin\alpha+u_2(-1)\cos\alpha=0, \tag{2}$$

$$B_2(u):=u_1(1)\sin\beta+u_2(1)\cos\beta=0, \tag{3}$$

and transmission conditions

$$T_1(u):=u_1(0^-)-u_1(0^+)=0, \tag{4}$$

$$T_2(u):=u_2(0^-)-u_2(0^+)+\lambda\delta u_1(0^-)=0, \tag{5}$$

where $I:=[-1,0)\cup(0,1]$; λ complex spectral parameter; $u(x)=\begin{pmatrix}u_1(x)\\ u_2(x)\end{pmatrix}$; the real valued function $r_1(.)$ and $r_2(.)$ are continuous in $[-1,0)$ and $(0,1]$ and have finite limits $r_1(0^\pm):=\lim_{x\to 0^\pm} r_1(x)$, $r_2(0^\pm):=\lim_{x\to 0^\pm} r_2(x)$; $\delta\in\mathbb{R}$ and $\delta>0$; $\alpha,\beta\in[0,\pi)$.

Defined by equations (4) and (5) are called transmission (or jump; discontinuity) conditions. Boundary-value problems with transmission conditions inside the interval often appear in mathematical physics, mechanics, electronics, geophysics and other branches of natural sciences (see [13, 16]).

In this paper, we investigate asymptotic behavior of eigenvalues and vector-valued eigenfunctions of a discontinuous Dirac system which contains an eigenparameter in a transmission condition with the same eigenparameter in the equations. For studying these asymptotic behaviors, we derive the integral equations similarly to the discontinuous Sturm Liouville problem (see [1, 2, 6]) and solve these integral equations by the method of successive approximations. Finally, we obtain asymptotic formulas for eigenvalues depending on whether $Q_1(0)$ and $Q_2(1)$ are zero or not, where

$$Q_1(0)=\frac{1}{2}\int_{-1}^{0}\left(r_1(y)+r_2(y)\right)dy,\ Q_2(1)=\frac{1}{2}\int_{0}^{1}\left(r_1(y)+r_2(y)\right)dy.$$

2 Spectral Properties

Let $L:=L_2(-1,0)\oplus L_2(0,1)$, then we define the Hilbert space $H:=L\oplus\mathbb{C}$ with an inner product

$$\langle u(.),v(.)\rangle_H:=\int_{-1}^{1}\boldsymbol{u}^{\tau}(x)\overline{\boldsymbol{v}}(x)dx+\frac{1}{\delta}h\overline{k}, \tag{6}$$

where τ denotes the matrix transpose, $u(x)=\begin{pmatrix}\boldsymbol{u}(x)\\ h\end{pmatrix}$, $v(x)=\begin{pmatrix}\boldsymbol{v}(x)\\ k\end{pmatrix}\in H$; $\boldsymbol{u}(x)=\begin{pmatrix}u_1(x)\\ u_2(x)\end{pmatrix}$, $\boldsymbol{v}(x)=\begin{pmatrix}v_1(x)\\ v_2(x)\end{pmatrix}\in L,\ u_i(.),v_i(.)\in L_2(-1,1)\ (i=1,2);\ h,k\in\mathbb{C}$.

Equation (1) can be written as

$$\ell(u):=B\boldsymbol{u}'(x)-\Omega(x)\boldsymbol{u}(x)=\lambda\boldsymbol{u}(x), \tag{7}$$

where $B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, $\Omega(x)=\begin{pmatrix}r_1(x) & 0\\ 0 & r_2(x)\end{pmatrix}$, $\boldsymbol{u}(x)=\begin{pmatrix}u_1(x)\\ u_2(x)\end{pmatrix}$.

For functions $\boldsymbol{u}(x)$, which is defined on I and has finite limit $u(0^{\pm}):=\lim_{x\to 0^{\pm}}\boldsymbol{u}(x)$, by $u_{(1)}(x)$ and $u_{(2)}(x)$ we denote the functions

$$u_{(1)}(x)=\begin{cases}u(x), & x\in[-1,0),\\ u(0^-), & x=0,\end{cases}\qquad u_{(2)}(x)=\begin{cases}u(x), & x\in(0,1],\\ u(0^+), & x=0,\end{cases}$$

which are defined on $I_1:=[-1,0]$ and $I_2:=[0,1]$, respectively.

Lemma 2.1 *The eigenvalues of the problem (1)-(5) are real.*

Proof: Assume the contrary that λ_0 is a nonreal eigenvalue of the problem (1)-(5). Let $\boldsymbol{u}(x)=\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$ be a corresponding (non-trivial) eigenfunction. By equation (1), for $x\in I$, we have

$$\frac{d}{dx}\left\{u_1(x)\overline{u_2}(x)-\overline{u_1}(x)u_2(x)\right\}=\left(\overline{\lambda}_0-\lambda_0\right)\left\{\left|u_1(x)\right|^2+\left|u_2(x)\right|^2\right\}.$$

Integrating the above equation through $[-1,0)$ and $(0,1]$, we obtain, respectively,

$$\begin{aligned}\left(\overline{\lambda}_0-\lambda_0\right)\left(\int_{-1}^{0}\left(\left|u_1(x)\right|^2+\left|u_2(x)\right|^2\right)dx\right)&=u_1\left(0^-\right)\overline{u_2}\left(0^-\right)-\overline{u_1}\left(0^-\right)u_2\left(0^-\right)\\&\quad-\left(u_1(-1)\overline{u_2}(-1)-\overline{u_1}(-1)u_2(-1)\right),\end{aligned}\tag{8}$$

$$\begin{aligned}\left(\overline{\lambda}_0-\lambda_0\right)\left(\int_{0}^{1}\left(\left|u_1(x)\right|^2+\left|u_2(x)\right|^2\right)dx\right)&=u_1(1)\overline{u_2}(1)-\overline{u_1}(1)u_2(1)\\&\quad-\left(u_1\left(0^+\right)\overline{u_2}\left(0^+\right)-\overline{u_1}\left(0^+\right)u_2\left(0^+\right)\right).\end{aligned}\tag{9}$$

Then from the boundary conditions (2), (3) and the transmission conditions (4), (5), we have, respectively,

$$u_1(-1)\overline{u_2}(-1)-\overline{u_1}(-1)u_2(-1)=0,\ u_1(1)\overline{u_2}(1)-\overline{u_1}(1)u_2(1)=0,$$

and

$$u_1\left(0^+\right)\overline{u_2}\left(0^+\right)-\overline{u_1}\left(0^+\right)u_2\left(0^+\right)=u_1\left(0^-\right)\overline{u_2}\left(0^-\right)-\overline{u_1}\left(0^-\right)u_2\left(0^-\right)+\delta\left(\overline{\lambda}_0-\lambda_0\right)u_1\left(0^-\right)\overline{u_1}\left(0^-\right)$$

Since $\lambda_0\neq\overline{\lambda}_0$ and $\delta>0$, it follows that

$$\int_{-1}^{1}\left(\left|u_1(x)\right|^2+\left|u_2(x)\right|^2\right)dx+\delta\left|u_1\left(0^-\right)\right|^2=0.\tag{10}$$

Then $u_i(x)=0$ $(i=1,2)$ and this is a contradiction. Consequently, λ_0 must be real. □

Lemma 2.2 *Two eigenfunctions* $u(x,\lambda)=\begin{pmatrix} u_1(x,\lambda) \\ u_2(x,\lambda) \end{pmatrix}$ *and* $v(x,\mu)=\begin{pmatrix} v_1(x,\mu) \\ v_2(x,\mu) \end{pmatrix}$ *corresponding to different eigenvalues* λ *and* μ, *are orthogonal, i.e.,*

$$\int_{-1}^{1} u^{\tau}(x,\lambda)v(x,\mu)dx+\frac{1}{\delta}hk=0$$

Now we will construct a special fundamental system of solutions of equation (1). By virtue of Theorem 1.1 in [12], we shall define two solutions of equation (1)

$$\phi(x,\lambda)=\begin{pmatrix}\phi_1(.,\lambda)\\ \phi_2(.,\lambda)\end{pmatrix}, \chi(x,\lambda)=\begin{pmatrix}\chi_1(.,\lambda)\\ \chi_2(.,\lambda)\end{pmatrix}$$

where

$$\phi_1(x,\lambda)=\begin{cases}\phi_{11}(x,\lambda), & x\in[-1,0),\\ \phi_{12}(x,\lambda), & x\in(0,1],\end{cases} \quad \phi_2(x,\lambda)=\begin{cases}\phi_{21}(x,\lambda), & x\in[-1,0),\\ \phi_{22}(x,\lambda), & x\in(0,1],\end{cases} \tag{11}$$

$$\chi_1(x,\lambda)=\begin{cases}\chi_{11}(x,\lambda), & x\in[-1,0),\\ \chi_{12}(x,\lambda), & x\in(0,1],\end{cases} \quad \chi_2(x,\lambda)=\begin{cases}\chi_{21}(x,\lambda), & x\in[-1,0),\\ \chi_{22}(x,\lambda), & x\in(0,1],\end{cases} \tag{12}$$

as follows: Let $\boldsymbol{u}=\begin{pmatrix}\phi_{11}(x,\lambda)\\ \phi_{21}(x,\lambda)\end{pmatrix}$ be the solution of equation (1) on $[-1,0]$, which satisfies the initial condition (2);

$$\begin{pmatrix}u_1(-1)\\ u_2(-1)\end{pmatrix}=\begin{pmatrix}cos\,\alpha\\ -sin\,\alpha\end{pmatrix}. \tag{13}$$

After defining this solution, we define the solution $\boldsymbol{u}=\begin{pmatrix}\phi_{12}(x,\lambda)\\ \phi_{22}(x,\lambda)\end{pmatrix}$ of equation (1) on $[0,1]$ by means of the solution $\phi_1(x,\lambda)$ and by the transmission conditions (4) and (5);

$$\begin{pmatrix}u_1(0)\\ u_2(0)\end{pmatrix}=\begin{pmatrix}\phi_{11}(0^-,\lambda)\\ \phi_{21}(0^-,\lambda)+\lambda\delta\phi_{11}(0^-,\lambda)\end{pmatrix}. \tag{14}$$

Hence $\phi(x,\lambda)$ satisfies equation (1) on I, the boundary condition (2) and the transmission conditions (4) and (5).

Similarly, first we define the solution $\boldsymbol{u}=\begin{pmatrix}\chi_{12}(x,\lambda)\\ \chi_{22}(x,\lambda)\end{pmatrix}$ of equation (1) on $[0,1]$ by the initial condition (3);

$$\begin{pmatrix}u_1(1)\\ u_2(1)\end{pmatrix}=\begin{pmatrix}cos\,\beta\\ -sin\,\beta\end{pmatrix}. \tag{15}$$

After defining this solution, we define the solution $\boldsymbol{u}=\begin{pmatrix}\chi_{11}(x,\lambda)\\ \chi_{21}(x,\lambda)\end{pmatrix}$ of equation (1) on $[-1,0]$ by the transmission conditions (4) and (5);

$$\begin{pmatrix}u_1(0)\\ u_2(0)\end{pmatrix}=\begin{pmatrix}\chi_{12}\left(0^+,\lambda\right)\\ \chi_{22}\left(0^+,\lambda\right)-\lambda\delta\chi_{12}\left(0^+,\lambda\right)\end{pmatrix}. \tag{16}$$

Hence $\chi(x,\lambda)$ satisfies equation (1) on I, the boundary condition (3) and the transmission conditions (4) and (5).

Let $W(\phi,\chi)(x,\lambda)$ denote the Wronskian of $\phi(.,\lambda)$ and $\chi(.,\lambda)$. Since the Wronskian $W(\phi,\chi)(x,\lambda)$ is independent on variable $x\in I_i\ (i=1,2)$ and $\phi(.,\lambda)$ and $\chi(.,\lambda)$ are entire functions of parameter λ for each $x\in I_i\ (i=1,2)$, then the function

$$\omega_i(\lambda):=W(\phi,\chi)(x,\lambda)=\phi_{1i}(x,\lambda)\chi_{2i}(x,\lambda)-\phi_{2i}(x,\lambda)\chi_{1i}(x,\lambda), \tag{17}$$

are entire function of parameter λ. After a short calculation, we see that $\omega_1(\lambda)=\omega_2(\lambda)$ for each $\lambda\in\mathbb{C}$. The zeros of the functions $\omega_1(\lambda)$ and $\omega_2(\lambda)$ coincide. Then, we may take into consideration the characteristic function $\omega(\lambda)$ as

$$\omega(\lambda):=\omega_1(\lambda)=\omega_2(\lambda). \tag{18}$$

Lemma 2.3 *All eigenvalues of the problem (1)-(5) are just zeros of the function* $\omega(\lambda)$. *Moreover, every zero of* $\omega(\lambda)$ *has multiplicity one.*

Proof: Since the function $\phi_1(.,\lambda)$ and $\phi_2(.,\lambda)$ satisfy the boundary condition (2) and the transmission conditions (4) and (5) to find the eigenvalues of the problem

(1)-(5), we have to insert the functions $\phi_1(.,\lambda)$ and $\phi_2(.,\lambda)$ in the boundary condition (3) and find the roots of this equation.

By equation (1) we obtain for $\lambda,\mu\in\mathbb{C}$, $\lambda\neq\mu$,

$$\frac{d}{dx}\{\phi_1(x,\lambda)\phi_2(x,\mu)-\phi_1(x,\mu)\phi_2(x,\lambda)\}=(\mu-\lambda)\{\phi_1(x,\lambda)\phi_1(x,\mu)+\phi_2(x,\lambda)\phi_2(x,\mu)\}$$

Integrating the above equation through $[-1,0)$ and $(0,1]$, and taking into account the initial conditions (13), (14) and (16), we obtain

$$\begin{aligned}&\phi_{12}(1,\lambda)\phi_{22}(1,\mu)-\phi_{12}(1,\mu)\phi_{22}(1,\lambda)=(\mu-\lambda)\Big\{\delta\phi_{11}(0,\lambda)\phi_{11}(0,\mu)\\&+\int_{-1}^{0}\left(\phi_{11}(x,\lambda)\phi_{11}(x,\mu)+\phi_{21}(x,\lambda)\phi_{21}(x,\mu)\right)dx\\&+\int_{0}^{1}\left(\phi_{12}(x,\lambda)\phi_{12}(x,\mu)+\phi_{22}(x,\lambda)\phi_{22}(x,\mu)\right)dx\Big\}.\end{aligned}\tag{19}$$

Dividing both sides of equality (19) by $(\mu-\lambda)$ and by letting $\mu\to\lambda$, we arrive to the relation

$$\begin{aligned}\phi_{22}(1,\lambda)\frac{\partial\phi_{12}(1,\lambda)}{\partial\lambda}-\phi_{12}(1,\lambda)\frac{\partial\phi_{22}(1,\lambda)}{\partial\lambda}&=\delta\left|\phi_{11}(0,\lambda)\right|^2+\int_{-1}^{0}\left(\left|\phi_{11}(x,\lambda)\right|^2+\left|\phi_{21}(x,\lambda)\right|^2\right)dx\\&+\int_{0}^{1}\left(\left|\phi_{12}(x,\lambda)\right|^2+\left|\phi_{22}(x,\lambda)\right|^2\right)dx.\end{aligned}\tag{20}$$

We show that equation

$$\omega(\lambda)=\phi_{12}(1,\lambda)\sin\beta+\phi_{22}(1,\lambda)\cos\beta=0,\tag{21}$$

has only simple roots. Assume the converse, that is, equation (21) has a double root λ_0. Then the following two equations hold

$$\phi_{12}(1,\lambda_0)\sin\beta+\phi_{22}(1,\lambda_0)\cos\beta=0\ ,\tag{22}$$

$$\frac{\partial\phi_{12}(1,\lambda_0)}{\partial\lambda}\sin\beta+\frac{\partial\phi_{22}(1,\lambda_0)}{\partial\lambda}\cos\beta=0.\tag{23}$$

The equation (22) and (23) imply that

$$\phi_{22}(1,\lambda_0)\frac{\partial\phi_{12}(1,\lambda_0)}{\partial\lambda}-\phi_{12}(1,\lambda_0)\frac{\partial\phi_{22}(1,\lambda_0)}{\partial\lambda}=0. \tag{24}$$

Combining (24) and (20), with $\lambda=\lambda_0$, we obtain

$$\int_{-1}^{1}\left(\left|\phi_1(x,\lambda_0)\right|^2+\left|\phi_2(x,\lambda_0)\right|^2\right)dx+\delta\left|\phi_{11}(0,\lambda_0)\right|^2=0. \tag{25}$$

It follows that $\phi_1(x,\lambda_0)=\phi_2(x,\lambda_0)=0$, which is imposible. This proves the lemma. □

3 Asymptotic Approximate Formulas

Now we derive asymptotic formulas for the eigenvalues and for the vector-valued eigenfunctions of the problem (1)-(5). Similarly to the Sturm-Liouville problem, it follows that there exist infinitely many eigenvalues. We essentially used the integral equations for the fundamental system of solutions of the Sturm-Liouville equation (see [1, 2, 6, 7, 9, 12, 14]). In the case of the Dirac system, the explicit form of the vector-valued function $\phi(.,\lambda)$ from equalities (13) and (14), we obtain the following integral equations:

$$\begin{aligned}\phi_{11}(x,\lambda)={}&cos\left(\lambda(x+1)-\alpha\right)-\int_{-1}^{x}sin\left(\lambda(x-y)\right)r_1(y)\phi_{11}(y,\lambda)\,dy\\&-\int_{-1}^{x}cos\left(\lambda(x-y)\right)r_2(y)\phi_{21}(y,\lambda)\,dy,\end{aligned} \tag{26}$$

$$\begin{aligned}\phi_{21}(x,\lambda)={}&sin\left(\lambda(x+1)-\alpha\right)+\int_{-1}^{x}cos\left(\lambda(x-y)\right)r_1(y)\phi_{11}(y,\lambda)\,dy\\&-\int_{-1}^{x}sin\left(\lambda(x-y)\right)r_2(y)\phi_{21}(y,\lambda)\,dy,\end{aligned} \tag{27}$$

$$\begin{aligned}\phi_{12}(x,\lambda)={}&\phi_{11}\left(0^-,\lambda\right)cos(\lambda x)-\left(\phi_{21}\left(0^-,\lambda\right)+\lambda\delta\phi_{11}\left(0^-,\lambda\right)\right)sin(\lambda x)\\&-\int_{0}^{x}sin\left(\lambda(x-y)\right)r_1(y)\phi_{12}(y,\lambda)\,dy-\int_{0}^{x}cos\left(\lambda(x-y)\right)r_2(y)\phi_{22}(y,\lambda)\,dy,\end{aligned} \tag{28}$$

$$\phi_{22}(x,\lambda)=\phi_{11}\left(0^-,\lambda\right)sin(\lambda x)+\left(\phi_{21}\left(0^-,\lambda\right)+\lambda\delta\phi_{11}\left(0^-,\lambda\right)\right)cos(\lambda x)$$
$$+\int_0^x cos\left(\lambda(x-y)\right)r_1(y)\phi_{12}(y,\lambda)\,dy-\int_0^x sin\left(\lambda(x-y)\right)r_2(y)\phi_{22}(y,\lambda)\,dy, \tag{29}$$

for the components of the vector-valued function $\phi(.,\lambda)$.

Lemma 3.1 *For* $|\lambda|\to\infty$, *the solution* $\phi(.,\lambda)$ *has the following asymptotic representation uniformly with respect to* $x,\ x\in I$,

$$\phi_{11}(x,\lambda)=cos\left(\lambda(x+1)-\alpha\right)-Q_1(x)sin\left(\lambda(x+1)-\alpha\right)+O\left(\frac{1}{\lambda}\right), \tag{30}$$

$$\phi_{21}(x,\lambda)=sin\left(\lambda(x+1)-\alpha\right)+Q_1(x)cos\left(\lambda(x+1)-\alpha\right)+O\left(\frac{1}{\lambda}\right), \tag{31}$$

$$\begin{aligned}\phi_{12}(x,\lambda)=&\lambda\delta\left(Q_1(0)sin(\lambda-\alpha)-cos(\lambda-\alpha)\right)\left(sin(\lambda x)+Q_2(x)cos(\lambda x)\right)\\&+\left(1-Q_1(0)Q_2(x)\right)cos\left(\lambda(x+1)-\alpha\right)-\left(Q_1(0)+Q_2(x)\right)sin\left(\lambda(x+1)-\alpha\right)\\&-\delta R_1(x)\left(Q_1(0)sin(\lambda-\alpha)-cos(\lambda-\alpha)\right)sin(\lambda x)+O\left(\frac{1}{\lambda}\right),\end{aligned} \tag{32}$$

$$\begin{aligned}\phi_{22}(x,\lambda)=&\lambda\delta\left(Q_1(0)sin(\lambda-\alpha)-cos(\lambda-\alpha)\right)\left(Q_2(x)sin(\lambda x)-cos(\lambda x)\right)\\&+\left(1-Q_1(0)Q_2(x)\right)sin\left(\lambda(x+1)-\alpha\right)+\left(Q_1(0)+Q_2(x)\right)cos\left(\lambda(x+1)-\alpha\right)\\&-\delta R_2(x)\left(Q_1(0)sin(\lambda-\alpha)-cos(\lambda-\alpha)\right)cos(\lambda x)+O\left(\frac{1}{\lambda}\right),\end{aligned} \tag{33}$$

where

$$\begin{aligned}Q_1(x)&=\frac{1}{2}\int_{-1}^{x}\left(r_1(y)+r_2(y)\right)dy,\\Q_2(x)&=\frac{1}{2}\int_{0}^{x}\left(r_1(y)+r_2(y)\right)dy,\end{aligned} \tag{34}$$

$$\begin{aligned}R_1(x)&=\frac{1}{4}\left(r_1(x)-r_2(x)+r_1(0)-r_2(0)\right),\\R_2(x)&=\frac{1}{4}\left(r_1(x)-r_2(x)-r_1(0)+r_2(0)\right).\end{aligned} \tag{35}$$

Proof: These asymptotic formulas are obtained by solving the systems (26)-(29) by the method of successive approximations. □

The characteristic function of the problem (1)-(5) is defined by

$$\omega(\lambda)=\phi_{12}(1,\lambda)\sin\beta+\phi_{22}(1,\lambda)\cos\beta. \tag{36}$$

The function $\omega(\lambda)$ is an analytic functions in λ and its zeros are precisely eigenvalues of the problem (1)-(5).

Theorem 3.1 *The characteristic function $\omega(\lambda)$ has the following asymptotic representation:*

$$\begin{aligned}\omega(\lambda)=&\lambda\delta\big(Q_1(0)\sin(\lambda-\alpha)-\cos(\lambda-\alpha)\big)\big(Q_2(1)\sin(\lambda+\beta)-\cos(\lambda+\beta)\big)\\&+\big(1-Q_1(0)Q_2(1)\big)\sin(2\lambda-\alpha+\beta)+\big(Q_1(0)+Q_2(1)\big)\cos(2\lambda-\alpha+\beta)\\&-\delta\big(Q_1(0)\sin(\lambda-\alpha)-\cos(\lambda-\alpha)\big)\Bigg(\frac{r_2(0)-r_1(0)}{4}\cos(\lambda+\beta)\\&+\frac{r_1(1)-r_2(1)}{4}\cos(\lambda-\beta)\Bigg)+O\left(\frac{1}{\lambda}\right).\end{aligned} \tag{37}$$

Proof: By substituting the obtained asymptotic formulas for $\phi_{12}(.,\lambda)$ and $\phi_{22}(.,\lambda)$ (equations (32) and (33)) in the definitions of (36), then (37) can be obtained. □

Theorem 3.2 *The eigenvalues λ_n of the problem (1)-(5) have the following asymptotic representation:*

Case 1. If $Q_1(0)\neq 0,\ Q_2(1)\neq 0,$ then

$$\begin{aligned}\tilde{\lambda}&=\left(n+\frac{1}{2}\right)\pi+\alpha+\operatorname{arccot}\big(Q_1(0)\big)+O\left(\frac{1}{n}\right),\\\tilde{\tilde{\lambda}}&=\left(n+\frac{1}{2}\right)\pi-\beta+\operatorname{arccot}\big(Q_2(1)\big)+O\left(\frac{1}{n}\right),\end{aligned} \tag{38}$$

Case 2. If $Q_1(0)=0,\ Q_2(1)\neq 0,$ then

$$\begin{aligned}\tilde{\lambda}&=\left(n+\frac{1}{2}\right)\pi+\alpha+O\left(\frac{1}{n}\right),\\\tilde{\tilde{\lambda}}&=\left(n+\frac{1}{2}\right)\pi-\beta+\operatorname{arccot}\big(Q_2(1)\big)+O\left(\frac{1}{n}\right),\end{aligned} \tag{39}$$

Case 3. If $Q_1(0) \neq 0,\ Q_2(1) = 0,$ then

$$\begin{aligned}\tilde{\lambda} &= \left(n + \frac{1}{2}\right)\pi + \alpha + \operatorname{arccot}\left(Q_1(0)\right) + O\left(\frac{1}{n}\right),\\ \tilde{\tilde{\lambda}} &= \left(n + \frac{1}{2}\right)\pi - \beta + O\left(\frac{1}{n}\right),\end{aligned} \tag{40}$$

Case 4. If $Q_1(0) = 0,\ Q_2(1) = 0,$ then

$$\begin{aligned}\tilde{\lambda} &= \left(n + \frac{1}{2}\right)\pi + \alpha + O\left(\frac{1}{n}\right),\\ \tilde{\tilde{\lambda}} &= \left(n + \frac{1}{2}\right)\pi - \beta + O\left(\frac{1}{n}\right),\end{aligned} \tag{41}$$

where $Q_1(0) = \frac{1}{2}\int_{-1}^{0}\left(r_1(y) + r_2(y)\right)dy,\ Q_2(1) = \frac{1}{2}\int_{0}^{1}\left(r_1(y) + r_2(y)\right)dy.$

Proof: The asymptotic formula (37) can be written in the form

$$\omega(\lambda) = \lambda\delta\left(Q_1(0)\,sin(\lambda - \alpha) - cos(\lambda - \alpha)\right)\left(Q_2(1)\,sin(\lambda + \beta) - cos(\lambda + \beta)\right) + O(1). \tag{42}$$

And from equation (42) denoting

$$w_1(\lambda) = \lambda\delta\left(Q_1(0)\,sin(\lambda - \alpha) - cos(\lambda - \alpha)\right)\left(Q_2(1)\,sin(\lambda + \beta) - cos(\lambda + \beta)\right)$$

and

$$w_2(\lambda) = O(1).$$

According to [1-4], see also [11, 18], the proof is completed by using well-known Rouche's theorem. □

By putting asymtotic formulas (38)-(41) for eigenvalues λ_n in the asymptotic formulas (30)-(33) for the solutions $\phi(.,\lambda)$, the asymptotic formulae of the eigen-vector-functions $\phi(.,\lambda_n)$ can be obtained.

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