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Multipoint Algorithms Arising from Optimal in the Sense of Kung-Traub Iterative Procedures for Numerical Solution of Nonlinear Equations

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Abstract

In this paper we will examine self-accelerating in terms of convergence speed and the corresponding index of efficiency in the sense of Ostrowski - Traub of certain standard and most commonly used in practice multipoint iterative methods using several initial approximations for numerical solution of nonlinear equations (method regula falsi, modifications of Euler - Chebyshev method, Halley method, and others) due to optimal in the sense of the Kung-Traub algorithm of order 4 and 8. Some hypothetical iterative procedures generated by algorithms from order of convergence 16 and 32 are also studied (the receipt and publication of which is a matter of time, having in mind the increased interest in such optimal algorithms). The corresponding model theorems for their convergence speed and efficiency index have been formulated and proved.

Keywords: *solving nonlinear equations, order of convergence, optimal algorithm, efficiency index.*

1 Introduction

Traub [25] proposed the concept of efficiency index as a measure for comparing of different methods for solving nonlinear equation

$$f(x) = 0. \tag{1}$$

This index is described by:

$$\tau^{\frac{1}{n}},$$

where τ is the order of convergence and n is the whole number of evaluations per iteration.

Kung and Traub [10] then presented a hypothesis on optimality of root-solvers by giving

$$2^{\frac{n-1}{n}}$$

as the optimal order of convergence.

This means that the Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$k = 0, 1, 2, \ldots$$

by two evaluations per iteration is optimal with 1.414 as the efficiency index.

By taking into account the optimality concept, many authors have tried to build iterative procedures of optimal order of convergence - $\tau = 4$, $\tau = 8$.

The recent results of M. Petkovic [15] and M. Petkovic and L. Petkovic [16], Bi, Wu and Ren [2], Geum and Kim [4], Thurkal and Petkovic [24], Wang and Liu [26], Kou, Wang and Sun [9], Chun and Neta [3], Khattri and Abbasbandy [8], Soleymani [20], Bi, Ren and Wu [1] are presented for optimal multipoint methods for solving nonlinear equations.

M. Petkovic [15] gives a useful detailed review about computational efficiency of many methods in the sense of Kung - Traub hypothesis.

For other nontrivial methods for solving nonlinear equations see, Kyurkchiev and Iliev [11] and monograph by Iliev and Kyurkchiev [7].

In many natural science tasks, from purely physical considerations, for the user of numerical algorithms for solving nonlinear equation (1) is preliminary known system of initial approximations

$$x_1^0, x_2^0, \ldots, x_t^0$$

for the root ξ of equation (1).

As an example, regula falsi methods and modifications of Euler - Chebyshev method and Halley method with a lower order of convergence using two or three initial approximations for the root ξ.

In [13], refined conditions of convergence for the difference analogue of Halley method (using three initial approximations) for solving nonlinear equation are given (see, also [27]).

An efficient modification of a finite - difference analogue of Halley method is proposed in [6].

Naturally arises the task of designing and testing multipoint variants of the classical procedures in the light of the achievements over the past five years important theoretical results related to obtaining optimal in the sense of Kung-Traub algorithms.

In this sense the task of detailed refinement of the self-accelerating multipoint methods using several initial approximations is actual.

2 Main Results

In this paragraph we will begin considerations of the important issue of self-accelerating the most frequently use method it regula falsi with the technique of input optimal in the sense of Kung-Traub algorithms of order $\tau = 4$.

I. The **regula falsi** method given by

$$x_{n+1} = x_n - \frac{f(x_n)(x_{n-1} - x_n)}{f(x_{n-1}) - f(x_n)}, \tag{2}$$

$$n = 0, 1, 2, \ldots$$

requires 2 function evaluations, 2 initial approximations x_{-1}, x_0 and has order of convergence $\tau \approx 1.618$ and efficiency index

$$I = 1.618^{\frac{1}{2}} \approx 1.272.$$

The method given by

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(x_n)}{f'(x_n)}.\frac{f(y_n)}{f(x_n) - 2f(y_n)}, \end{aligned} \tag{3}$$

$$n = 0, 1, 2, \ldots$$

or

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)}{f'(x_n)}.\frac{f\left(x_n - \dfrac{f(x_n)}{f'(x_n)}\right)}{f(x_n) - 2f\left(x_n - \dfrac{f(x_n)}{f'(x_n)}\right)}, \tag{4}$$

$$n = 0, 1, 2, \ldots$$

proposed by Ostrowski in [14] was the first multipoint method of the fourth order.

This method requires 3 functional evaluations and has the efficiency index

$$I = 4^{\frac{1}{3}} \approx 1.587.$$

Its order of convergence is optimal in the sense of the Kung-Traub conjecture (efficiency index is $I = 2^{\frac{2}{3}}$).

We will explicitly mention that there are other algorithms with optimal order of convergence $\tau = 4$. We will explore Ostrowski iteration as the most popular representative of this class of optimal methods.

Here we give a methodological construction of nonstationary algorithms with a raised speed of convergence.

I.1. Let us consider the following nonstationary iterative scheme based on schemes (2) and (4):

$$x_{2n+1} = x_{2n} - \frac{f(x_{2n})(x_{2n} - x_{2n-1})}{f(x_{2n}) - f(x_{2n-1})},$$

$$x_{2n+2} = x_{2n+1} - \frac{f(x_{2n+1})}{f'(x_{2n+1})} - \frac{f(x_{2n+1})}{f'(x_{2n+1})}.\frac{f\left(x_{2n+1} - \dfrac{f(x_{2n+1})}{f'(x_{2n+1})}\right)}{f(x_{2n+1}) - 2f\left(x_{2n+1} - \dfrac{f(x_{2n+1})}{f'(x_{2n+1})}\right)}, \tag{5}$$

$$n = 0, 1, 2, \ldots$$

Let

$$e_i = x_i - \xi, \quad i = -1, 0, 1, \ldots; \;\; c_i(\xi) = \frac{f^{(i)}(\xi)}{f'(\xi)i!}, \;\; i = 2, 3, \ldots,$$

It is well-known that for the error ϵ_i [25] is valid

$$\epsilon_{2n+1} \sim -c_2(\xi)\epsilon_{2n}\epsilon_{2n-1}, \tag{6}$$

and for the procedure (4) [25]:

$$\epsilon_{2n+2} \sim c_2(\xi)[c_2^2(\xi) - c_3(\xi)]e_{2n+1}^4, \tag{7}$$

where $\sim$ denotes the asymptotical equation when $n \to \infty$.

Let

$$\begin{aligned} K &= \max\left\{|c_2(\xi))|, |c_2(\xi)[c_2^2(\xi) - c_3(\xi)]|\right\}, \\ d_{2n-1} &= K^{\frac{1}{2}}|\epsilon_{2n-1}|, \\ d_{2n} &= K|\epsilon_{2n}| \end{aligned}$$

and let $d > 0$, and x_{-1} and x_0 be chosen so that the following inequalities

$$\begin{aligned} d_{-1} &= K^{\frac{1}{2}}|x_{-1} - \xi| \leq d < 1, \\ d_0 &= K|x_0 - \xi| \leq d < 1 \end{aligned}$$

hold true.

From (6) and (7), we have

$$\begin{aligned} d_{2n+1} &= K^{\frac{1}{2}}|\epsilon_{2n+1}| \leq K^{\frac{1}{2}}K|\epsilon_{2n}||\epsilon_{2n-1}| = d_{2n}d_{2n-1}, \\ d_{2n+2} &= K|\epsilon_{2n+2}| \leq KK\epsilon_{2n+1}^4 = \left(K^{\frac{1}{2}}\right)^4\epsilon_{2n+1}^4 = d_{2n+1}^4. \end{aligned} \tag{8}$$

Our results concerning the order of convergence generated by (5) are summarized in the following theorem.

Theorem 2.1 *Assume that the initial approximations x_0, x_{-1} are chosen so that $d_{-1} \leq d < 1$ and $d_0 \leq d < 1$.*

Then for the error of the sequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ determined by (5), we have

$$\begin{aligned} d_{2n-1} &\leq d^{2.5^{n-1}}, \\ d_{2n} &\leq d^{8.5^{n-1}}, \quad n = 1, 2, \ldots \end{aligned} \tag{9}$$

and the order of convergence of the iteration (5) is $\tau = 5$.

Proof. The proof is by induction with respect to the iteration number n. For $n = 0$, from (9), we find

$$\begin{aligned} d_1 &\leq d.d = d^2, \\ d_2 &\leq d_1^4 = (d^2)^4 = d^8. \end{aligned}$$

For $n = 1$, we have

$$\begin{aligned} d_3 &\leq d^{10}, \\ d_4 &\leq d^{40} \end{aligned}$$

and (9) is fulfilled.

Let (9) be fulfilled for $n \leq m$.

For $n = m + 1$, from (8) and (9), we have

$$d_{2(m+1)-1} = d_{2m+1} \leq d_{2m} d_{2m-1} \leq d^{8.5^{m-1}}.d^{2.5^{m-1}} = d^{10.5^{m-1}} = d^{2.5^m},$$

$$d_{2(m+1)} = d_{2m+2} \leq d_{2m+1}^4 < \left(d^{2.5^m}\right)^4 = d^{8.5^m}$$

which completes the induction.

On the other hand,

$$\begin{aligned} d_{2n-1} &= K^{\frac{1}{2}} |\epsilon_{2n-1}|, \\ d_{2n} &= K |\epsilon_{2n}| \end{aligned}$$

and equation (9) can be written as

$$\begin{aligned} |\epsilon_{2n-1}| &\leq K^{-\frac{1}{2}} d^{2.5^{n-1}}, \\ |\epsilon_{2n}| &\leq K^{-1} d^{8.5^{n-1}}, \quad n = 1, 2, \ldots, \end{aligned}$$

and the order of convergence of iteration (5) is equal to 5.
Thus, the theorem is proved.

Remark 1. The new method (5) requires 5 function evaluations, 2 initial approximations x_{-1}, x_0 and has order of convergence $\tau = 5$ and efficiency index

$$I = 5^{\frac{1}{5}} \approx 1.3797.$$

Sharma and Sharma [18] presented the following family of optimal order eight

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(y_n)}{f'(x_n)}.\frac{f(x_n)}{f(x_n) - 2f(y_n)}, \\ x_{n+1} &= z_n - \left(1 + \frac{f(z_n)}{f(x_n)} + \left(\frac{f(z_n)}{f(x_n}\right)^2\right).\frac{f[x_n, y_n]f(z_n)}{f[x_n, z_n]f[y_n, z_n]}. \end{aligned} \tag{10}$$

$$n = 0, 1, 2, \ldots$$

Here, $f[x, y]$ denote the finite difference.

This method requires 4 functional evaluations and has the efficiency index

$$I = 8^{\frac{1}{4}} \approx 1.6817.$$

Its order of convergence is optimal in the sense of the Kung-Traub conjecture ($I = 2^{\frac{3}{4}}$).

I.2. Let us consider the following nonstationary iterative scheme based on schemes (2) and (10):

$$\begin{aligned} x_{2n+1} &= x_{2n} - \frac{f(x_{2n})(x_{2n} - x_{2n-1})}{f(x_{2n}) - f(x_{2n-1})}, \\ y_{2n+1} &= x_{2n+1} - \frac{f(x_{2n+1})}{f'(x_{2n+1})}, \\ z_{2n+1} &= y_{2n+1} - \frac{f(y_{2n+1})}{f'(x_{2n+1})}.\frac{f(x_{2n+1})}{f(x_{2n+1}) - 2f(y_{2n+1})}, \\ x_{2n+2} &= z_{2n+1} - \left(1 + \frac{f(z_{2n+1})}{f(x_{2n+1})} + \left(\frac{f(z_{2n+1})}{f(x_{2n+1}}\right)^2\right).\frac{f[x_{2n+1}, y_{2n+1}]f(z_{2n+1})}{f[x_{2n+1}, z_{2n+1}]f[y_{2n+1}, z_{2n+1}]}, \end{aligned} \tag{11}$$

$$n = 0, 1, 2, \ldots$$

It is known that for the error $\epsilon_{2n+2} = x_{2n+2} - \xi$ [18] is valid

$$\epsilon_{2n+2} \sim A(\xi)e^8_{2n+1}, \tag{12}$$

We will use again the fact that for regula falsi method is satisfied

$$\epsilon_{2n+1} \sim -c_2(\xi)\epsilon_{2n}\epsilon_{2n-1}. \tag{13}$$

Let

$$
\begin{aligned}
K_1 &= \max\{|c_2(\xi))|, |A(\xi)|\}, \\
d_{2n-1} &= K_1^{\frac{1}{4}}|\epsilon_{2n-1}|, \\
d_{2n} &= K_1|\epsilon_{2n}|
\end{aligned}
$$

and let $d > 0$, and x_{-1} and x_0 be chosen so that the following inequalities

$$
\begin{aligned}
d_{-1} &= K_1^{\frac{1}{4}}|x_{-1} - \xi| \leq d < 1, \\
d_0 &= K_1|x_0 - \xi| \leq d < 1
\end{aligned}
$$

hold true.

From (12) and (13), we have

$$
\begin{aligned}
&d_{2n+1} = K_1^{\frac{1}{4}}|\epsilon_{2n+1}| \leq K_1^{\frac{1}{4}}K_1|\epsilon_{2n}||\epsilon_{2n-1}| = d_{2n}d_{2n-1}, \\
&d_{2n+2} = K_1|\epsilon_{2n+2}| \leq K_1K_1\epsilon_{2n+1}^8 = \left(K_1^{\frac{1}{4}}\right)^8 \epsilon_{2n+1}^8 = d_{2n+1}^8.
\end{aligned}
\tag{14}
$$

Our results concerning the order of convergence generated by (11) are summarized in the following theorem.

Theorem 2.2 *Assume that the initial approximations x_0, x_{-1} are chosen so that $d_{-1} \leq d < 1$ and $d_0 \leq d < 1$.*

Then for the error of the sequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ determined by (11), we have

$$
\begin{aligned}
d_{2n-1} &\leq d^{2.9^{n-1}}, \\
d_{2n} &\leq d^{16.9^{n-1}}, \quad n = 1, 2, \ldots
\end{aligned}
\tag{15}
$$

and the order of convergence of the iteration (11) is $\tau = 9$.

Proof. The proof is by induction with respect to the iteration number n.

For $n = 0$, from (14), we find

$$
\begin{aligned}
d_1 &\leq d.d = d^2, \\
d_2 &\leq d_1^8 = (d^2)^8 = d^{16}.
\end{aligned}
$$

For $n = 1$, we have

$$\begin{aligned} d_3 &\leq d^{18}, \\ d_4 &\leq d^{144} \end{aligned}$$

and (15) is fulfilled.

Let (15) be fulfilled for $n \leq m$.

For $n = m + 1$, from (14) and (15), we have

$$d_{2(m+1)-1} = d_{2m+1} \leq d_{2m} d_{2m-1} \leq d^{2.9^{m-1}} . d^{16.9^{m-1}} = d^{18.9^{m-1}} = d^{2.9^m},$$

$$d_{2(m+1)} = d_{2m+2} \leq d_{2m+1}^8 < \left(d^{2.9^m}\right)^8 = d^{16.9^m}$$

which completes the induction.

On the other hand,

$$\begin{aligned} d_{2n-1} &= K_1^{\frac{1}{4}} |\epsilon_{2n-1}|, \\ d_{2n} &= K_1 |\epsilon_{2n}| \end{aligned}$$

and equation (14) can be written as

$$\begin{aligned} |\epsilon_{2n-1}| &\leq K_1^{-\frac{1}{4}} d^{2.9^{n-1}}, \\ |\epsilon_{2n}| &\leq K_1^{-1} d^{16.9^{n-1}}, \quad n = 1, 2, \ldots, \end{aligned}$$

and the order of convergence of iteration (11) is equal to 9.
Thus, the theorem is proved.

Remark 2. The new method (11) requires 6 function evaluations, 2 initial approximations x_{-1}, x_0 and has order of convergence $\tau = 9$ and efficiency index

$$I = 9^{\frac{1}{6}} \approx 1.442.$$

Remark 3. The efficiency index of method (11) is better than index $I \approx 1.412$ by Newton's procedure.

II. We consider the following **modification of Euler - Chebyshev** method [25]:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)}{2f'^3(x_n)} . \frac{f'(x_n) - f'(x_{n-1})}{x_n - x_{n-1}}, \tag{16}$$

$$n = 0, 1, 2, \ldots$$

Here the second derivative is replaced by

$$f''(x_i) \approx \frac{f'(x_i) - f'(x_{i-1})}{x_i - x_{i-1}}.$$

II.1. We will consider the following nonstationary iterative scheme based on schemes (16) and (4):

$$\begin{aligned} x_{2n+1} &= x_{2n} - \frac{f(x_{2n})}{f'(x_{2n})} - \frac{f^2(x_{2n})}{2f'^3(x_{2n})}.\frac{f'(x_{2n}) - f'(x_{2n-1})}{x_{2n} - x_{2n-1}}, \\ x_{2n+2} &= x_{2n+1} - \frac{f(x_{2n+1})}{f'(x_{2n+1})} - \frac{f(x_{2n+1})}{f'(x_{2n+1})}.\frac{f\left(x_{2n+1} - \dfrac{f(x_{2n+1})}{f'(x_{2n+1})}\right)}{f(x_{2n+1}) - 2f\left(x_{2n+1} - \dfrac{f(x_{2n+1})}{f'(x_{2n+1})}\right)}, \end{aligned} \tag{17}$$

$$n = 0, 1, 2, \dots$$

It is known that for the error ϵ_i [25] is valid

$$\epsilon_{2n+1} \sim -\frac{3}{2} c_3(\xi) \epsilon_{2n}^2 \epsilon_{2n-1}, \tag{18}$$

and for the procedure (4) [25]:

$$\epsilon_{2n+2} \sim c_2(\xi)[c_2^2(\xi) - c_3(\xi)] e_{2n+1}^4. \tag{19}$$

Let

$$\begin{aligned} K_2 &= \max\left\{\tfrac{3}{2}|c_3(\xi))|, |c_2(\xi)[c_2^2(\xi) - c_3(\xi)]|\right\}, \\ d_{2n-1} &= K^{\frac{3}{8}}|\epsilon_{2n-1}|, \\ d_{2n} &= K^{\frac{1}{2}}|\epsilon_{2n}| \end{aligned}$$

and let $d > 0$, and x_{-1} and x_0 be chosen so that the following inequalities

$$\begin{aligned} d_{-1} &= K_2^{\frac{3}{8}}|x_{-1} - \xi| \le d < 1, \\ d_0 &= K_2^{\frac{1}{2}}|x_0 - \xi| \le d < 1 \end{aligned}$$

hold true.

From (18) and (19), we have

$$d_{2n+1} = K_2^{\frac{3}{8}}|\epsilon_{2n+1}| \le K_2^{\frac{3}{8}} K_2|\epsilon_{2n}^2||\epsilon_{2n-1}| = K_2^{\frac{3}{8}}|\epsilon_{2n-1}|\left(K_2^{\frac{1}{2}}\epsilon_{2n}^2\right)^2 = d_{2n}^2 d_{2n-1},$$

$$d_{2n+2} = K_2^{\frac{1}{2}}|\epsilon_{2n+2}| \le K_2^{\frac{1}{2}} K_2 \epsilon_{2n+1}^4 = \left(K_2^{\frac{3}{8}}\right)^4 \epsilon_{2n+1}^4 = d_{2n+1}^4. \tag{20}$$

Our results concerning the order of convergence generated by (17) are summarized in the following theorem.

Theorem 2.3 *Assume that the initial approximations x_0, x_{-1} are chosen so that $d_{-1} \le d < 1$ and $d_0 \le d < 1$.*

Then for the error of the sequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ determined by (17), we have

$$\begin{aligned} d_{2n-1} &\le d^{3.9^{n-1}}, \\ d_{2n} &\le d^{12.9^{n-1}}, \quad n = 1, 2, \ldots \end{aligned} \tag{21}$$

and the order of convergence of the iteration (17) is $\tau = 9$.

Proof. The proof is by induction with respect to the iteration number n.

For $n = 0$, from (20), we find

$$\begin{aligned} d_1 &\le d^2.d = d^3, \\ d_2 &\le d_1^4 = (d^3)^4 = d^{12}. \end{aligned}$$

For $n = 1$, we have

$$\begin{aligned} d_3 &\le d^{27}, \\ d_4 &\le d^{108} \end{aligned}$$

and (21) is fulfilled.

Let (21) be fulfilled for $n \le m$.

For $n = m + 1$, from (20) and (21), we have

$$d_{2(m+1)-1} = d_{2m+1} \le d_{2m}^2 d_{2m-1} \le d^{24.9^{m-1}}.d^{3.9^{m-1}} = d^{27.9^{m-1}} = d^{3.9^m},$$

$$d_{2(m+1)} = d_{2m+2} \le d_{2m+1}^4 < \left(d^{3.9^m}\right)^4 = d^{12.9^m}$$

which completes the induction.

On the other hand,

$$\begin{aligned} d_{2n-1} &= K_2^{\frac{3}{8}}|\epsilon_{2n-1}|, \\ d_{2n} &= K_2^{\frac{1}{2}}|\epsilon_{2n}| \end{aligned}$$

and equation (21) can be written as

$$\begin{aligned} |\epsilon_{2n-1}| &\leq K_2^{-\frac{3}{8}} d^{3.9^{n-1}}, \\ |\epsilon_{2n}| &\leq K_2^{-\frac{1}{2}} d^{12.9^{n-1}}, \quad n = 1, 2, \dots, \end{aligned}$$

and the order of convergence of iteration (17) is equal to 9.

Thus, the theorem is proved.

Remark 4. The method (17) requires 6 function evaluations, 2 initial approximations x_{-1}, x_0 and has order of convergence $\tau = 9$ and efficiency index

$$I = 9^{\frac{1}{6}} \approx 1.442.$$

II.2. We will consider the following nonstationary iterative scheme based on schemes (16) and (10):

$$\begin{aligned} x_{2n+1} &= x_{2n} - \frac{f(x_{2n})}{f'(x_{2n})} - \frac{f^2(x_{2n})}{2f'^3(x_{2n})} . \frac{f'(x_{2n}) - f'(x_{2n-1})}{x_{2n} - x_{2n-1}}, \\ y_{2n+1} &= x_{2n+1} - \frac{f(x_{2n+1})}{f'(x_{2n+1})}, \\ z_{2n+1} &= y_{2n+1} - \frac{f(y_{2n+1})}{f'(x_{2n+1})} . \frac{f(x_{2n+1})}{f(x_{2n+1}) - 2f(y_{2n+1})}, \\ x_{2n+2} &= z_{2n+1} - \left(1 + \frac{f(z_{2n+1})}{f(x_{2n+1})} + \left(\frac{f(z_{2n+1})}{f(x_{2n+1}}\right)^2\right) . \frac{f[x_{2n+1}, y_{2n+1}]f(z_{2n+1})}{f[x_{2n+1}, z_{2n+1}]f[y_{2n+1}, z_{2n+1}]}, \end{aligned} \tag{22}$$

$$n = 0, 1, 2, \dots$$

It is known that for the error $\epsilon_{2n+2} = x_{2n+2} - \xi$ [18] is valid

$$\epsilon_{2n+2} \sim A(\xi) e_{2n+1}^8, \tag{23}$$

We will use again the fact that for method (16) is satisfied

$$\epsilon_{2n+1} \sim -\frac{3}{2} c_3(\xi) \epsilon_{2n}^2 \epsilon_{2n-1}, \tag{24}$$

Let

$$\begin{aligned} K_3 &= \max\left\{\tfrac{3}{2}|c_3(\xi)|, |A(\xi)|\right\}, \\ d_{2n-1} &= K_3^{\frac{3}{16}} |\epsilon_{2n-1}|, \\ d_{2n} &= K_3^{\frac{1}{2}} |\epsilon_{2n}| \end{aligned}$$

and let $d > 0$, and x_{-1} and x_0 be chosen so that the following inequalities

$$d_{-1} = K_3^{\frac{3}{16}}|x_{-1} - \xi| \leq d < 1,$$

$$d_0 = K_3^{\frac{1}{2}}|x_0 - \xi| \leq d < 1$$

hold true.

From (23) and (24), we have

$$d_{2n+1} = K_3^{\frac{3}{16}}|\epsilon_{2n+1}| \leq K_3^{\frac{3}{16}}K_3\epsilon_{2n}^2|\epsilon_{2n-1}| = K_3^{\frac{3}{16}}|\epsilon_{2n-1}|\left(K_3^{\frac{1}{2}}\epsilon_{2n}\right)^2 = d_{2n}^2 d_{2n-1},$$

$$d_{2n+2} = K_3^{\frac{1}{2}}|\epsilon_{2n+2}| \leq K_3^{\frac{1}{2}}K_3\epsilon_{2n+1}^8 = \left(K_3^{\frac{3}{16}}\right)^8 \epsilon_{2n+1}^8 = d_{2n+1}^8. \tag{25}$$

Our results concerning the order of convergence generated by (22) are summarized in the following theorem.

Theorem 2.4 *Assume that the initial approximations x_0, x_{-1} are chosen so that $d_{-1} \leq d < 1$ and $d_0 \leq d < 1$.*

Then for the error of the sequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ determined by (22), we have

$$\begin{aligned} d_{2n-1} &\leq d^{3.17^{n-1}}, \\ d_{2n} &\leq d^{24.17^{n-1}}, \quad n = 1, 2, \ldots \end{aligned} \tag{26}$$

and the order of convergence of the iteration (22) is $\tau = 17$.

Proof. The proof is by induction with respect to the iteration number n.

For $n = 0$, from (25), we find

$$d_1 \leq d^2.d = d^3,$$

$$d_2 \leq d_1^8 = (d^3)^8 = d^{24}.$$

For $n = 1$, we have

$$d_3 \leq d^{51},$$

$$d_4 \leq d^{408}$$

and (26) is fulfilled.

Let (26) be fulfilled for $n \leq m$.

For $n = m + 1$, from (25) and (26), we have

$$d_{2(m+1)-1} = d_{2m+1} \leq d_{2m}^2 d_{2m-1} \leq d^{48.17^{m-1}}.d^{3.17^{m-1}} = d^{51.17^{m-1}} = d^{3.17^m},$$

$$d_{2(m+1)} = d_{2m+2} \leq d_{2m+1}^8 < \left(d^{3.17^m}\right)^8 = d^{24.17^m}$$

which completes the induction.

On the other hand,

$$\begin{aligned} d_{2n-1} &= K_3^{\frac{3}{16}} |\epsilon_{2n-1}|, \\ d_{2n} &= K_3^{\frac{1}{2}} |\epsilon_{2n}| \end{aligned}$$

and equation (26) can be written as

$$\begin{aligned} |\epsilon_{2n-1}| &\leq K_3^{-\frac{3}{16}} d^{3.17^{n-1}}, \\ |\epsilon_{2n}| &\leq K_3^{-\frac{1}{2}} d^{24.17^{n-1}}, \quad n = 1, 2, \dots, \end{aligned}$$

and the order of convergence of iteration (22) is equal to 17.

Thus, the theorem is proved.

Remark 5. The method (22) requires 7 function evaluations, 2 initial approximations x_{-1}, x_0 and has order of convergence $\tau = 17$ and efficiency index

$$I = 17^{\frac{1}{7}} \approx 1.4989.$$

III. The following **modification of Euler - Chebyshev** method, where the second derivative is replaced by its approximation by differentiating a two-point Hermite interpolation formula for $f(x)$, the two point being x_i and x_{i-1} is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)\left(2f'(x_n) + f'(x_{n-1}) - 3f(x_n, x_{n-1})\right)}{f'^3(x_n)(x_n - x_{n-1})}, \tag{27}$$

$$n = 0, 1, 2, \dots$$

III.1. We will consider the following nonstationary iterative scheme based on schemes (27) and (4):

$$x_{2n+1} = x_{2n} - \frac{f(x_{2n})}{f'(x_{2n})} - \frac{f^2(x_{2n})\left(2f'(x_{2n}) + f'(x_{2n-1}) - 3f(x_{2n}, x_{2n-1})\right)}{f'^3(x_{2n})(x_{2n} - x_{2n-1})},$$

$$x_{2n+2} = x_{2n+1} - \frac{f(x_{2n+1})}{f'(x_{2n+1})} - \frac{f(x_{2n+1})}{f'(x_{2n+1})}.\frac{f\left(x_{2n+1} - \dfrac{f(x_{2n+1})}{f'(x_{2n+1})}\right)}{f(x_{2n+1}) - 2f\left(x_{2n+1} - \dfrac{f(x_{2n+1})}{f'(x_{2n+1})}\right)}, \tag{28}$$

$$n = 0, 1, 2, \ldots$$

It is known that for the error ϵ_i [17] is valid

$$\epsilon_{2n+1} \sim B(\xi)\epsilon_{2n}^2\epsilon_{2n-1}^2, \tag{29}$$

and for the procedure (4) [25]:

$$\epsilon_{2n+2} \sim c_2(\xi)[c_2^2(\xi) - c_3(\xi)]e_{2n+1}^4. \tag{30}$$

Let

$$\begin{aligned} K_4 &= \max\{|c_2(\xi)[c_2^2(\xi) - c_3(\xi)]|, |B(\xi)|\}, \\ d_{2n-1} &= K_4^{\frac{1}{3}}|\epsilon_{2n-1}|, \\ d_{2n} &= K_4^{\frac{1}{3}}|\epsilon_{2n}| \end{aligned}$$

and let $d > 0$, and x_{-1} and x_0 be chosen so that the following inequalities

$$\begin{aligned} d_{-1} &= K_4^{\frac{1}{3}}|x_{-1} - \xi| \le d < 1, \\ d_0 &= K_4^{\frac{1}{3}}|x_0 - \xi| \le d < 1 \end{aligned}$$

hold true.

From (29) and (30), we have

$$\begin{aligned} d_{2n+1} &= K_4^{\frac{1}{3}}|\epsilon_{2n+1}| \le K_4^{\frac{1}{3}}K_4\epsilon_{2n}^2\epsilon_{2n-1}^2 = \left(K_4^{\frac{1}{3}}\epsilon_{2n-1}\right)^2\left(K_4^{\frac{1}{3}}\epsilon_{2n}\right)^2 = d_{2n}^2 d_{2n-1}^2, \\ d_{2n+2} &= K_4^{\frac{1}{3}}|\epsilon_{2n+2}| \le K_4^{\frac{1}{3}}K_4\epsilon_{2n+1}^4 = \left(K_4^{\frac{1}{3}}\right)^4\epsilon_{2n+1}^4 = d_{2n+1}^4. \end{aligned} \tag{31}$$

Our results concerning the order of convergence generated by (28) are summarized in the following theorem.

Theorem 2.5 *Assume that the initial approximations x_0, x_{-1} are chosen so that $d_{-1} \leq d < 1$ and $d_0 \leq d < 1$.*

Then for the error of the sequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ determined by (28), we have

$$\begin{aligned} d_{2n-1} &\leq d^{4.10^{n-1}}, \\ d_{2n} &\leq d^{16.10^{n-1}}, \quad n = 1, 2, \ldots \end{aligned} \tag{32}$$

and the order of convergence of the iteration (28) is $\tau = 10$.

Proof. The proof is by induction with respect to the iteration number n.

For $n = 0$, from (31), we find

$$\begin{aligned} d_1 &\leq d^2.d^2 = d^4, \\ d_2 &\leq d_1^4 = (d^4)^4 = d^{16}. \end{aligned}$$

For $n = 1$, we have

$$\begin{aligned} d_3 &\leq d^{40}, \\ d_4 &\leq d^{160} \end{aligned}$$

and (32) is fulfilled.

Let (32) be fulfilled for $n \leq m$.

For $n = m + 1$, from (31) and (32), we have

$$d_{2(m+1)-1} = d_{2m+1} \leq d_{2m}^2 d_{2m-1}^2 \leq d^{32.10^{m-1}}.d^{8.10^{m-1}} = d^{40.10^{m-1}} = d^{4.10^m},$$

$$d_{2(m+1)} = d_{2m+2} \leq d_{2m+1}^4 < \left(d^{4.10^m}\right)^4 = d^{16.10^m}$$

which completes the induction.

On the other hand,

$$\begin{aligned} d_{2n-1} &= K_4^{\frac{1}{3}} |\epsilon_{2n-1}|, \\ d_{2n} &= K_4^{\frac{1}{3}} |\epsilon_{2n}| \end{aligned}$$

and equation (32) can be written as

$$\begin{aligned} |\epsilon_{2n-1}| &\leq K_4^{-\frac{1}{3}} d^{4.10^{n-1}}, \\ |\epsilon_{2n}| &\leq K_4^{-\frac{1}{3}} d^{16.10^{n-1}}, \quad n = 1, 2, \ldots, \end{aligned}$$

and the order of convergence of iteration (28) is equal to 10.

Thus, the theorem is proved.

Remark 6. The method (28) requires 7 function evaluations, 2 initial approximations x_{-1}, x_0 and has order of convergence $\tau = 10$ and efficiency index

$$I = 10^{\frac{1}{7}} \approx 1.389.$$

III.2. We will consider the following nonstationary iterative scheme based on schemes (10) and (27):

$$x_{2n+1} = x_{2n} - \frac{f(x_{2n})}{f'(x_{2n})} - \frac{f^2(x_{2n})\,(2f'(x_{2n}) + f'(x_{2n-1}) - 3f(x_{2n}, x_{2n-1}))}{f'^3(x_{2n})(x_{2n} - x_{2n-1})},$$

$$y_{2n+1} = x_{2n+1} - \frac{f(x_{2n+1})}{f'(x_{2n+1})},$$

$$z_{2n+1} = y_{2n+1} - \frac{f(y_{2n+1})}{f'(x_{2n+1})}.\frac{f(x_{2n+1})}{f(x_{2n+1}) - 2f(y_{2n+1})},$$

$$x_{2n+2} = z_{2n+1} - \left(1 + \frac{f(z_{2n+1})}{f(x_{2n+1})} + \left(\frac{f(z_{2n+1})}{f(x_{2n+1}}\right)^2\right).\frac{f[x_{2n+1}, y_{2n+1}]f(z_{2n+1})}{f[x_{2n+1}, z_{2n+1}]f[y_{2n+1}, z_{2n+1}]}, \tag{33}$$

$$n = 0, 1, 2, \dots$$

It is known that for the error $\epsilon_{2n+2} = x_{2n+2} - \xi$ [18] is valid:

$$\epsilon_{2n+2} \sim A(\xi)e^8_{2n+1}, \tag{34}$$

We will use again the fact that for the method (27) is satisfied

$$\epsilon_{2n+1} \sim B(\xi)\epsilon^2_{2n}\epsilon^2_{2n-1}. \tag{35}$$

Let

$$\begin{aligned} K_5 &= \max\{|B(\xi)|, |A(\xi)|\}, \\ d_{2n-1} &= K_5^{\frac{3}{17}}|\epsilon_{2n-1}|, \\ d_{2n} &= K_5^{\frac{7}{17}}|\epsilon_{2n}| \end{aligned}$$

and let $d > 0$, and x_{-1} and x_0 be chosen so that the following inequalities

$$\begin{aligned} d_{-1} &= K_5^{\frac{3}{17}}|x_{-1} - \xi| \le d < 1, \\ d_0 &= K_5^{\frac{7}{17}}|x_0 - \xi| \le d < 1 \end{aligned}$$

hold true.

From (34) and (35), we have

$$d_{2n+1} = K_5^{\frac{3}{17}}|\epsilon_{2n+1}| \leq K_5^{\frac{3}{17}}K_5\epsilon_{2n}^2\epsilon_{2n-1}^2 = \left(K_5^{\frac{3}{17}}\epsilon_{2n-1}\right)^2\left(K_5^{\frac{7}{17}}\epsilon_{2n}\right)^2 = d_{2n}^2d_{2n-1}^2,$$

$$d_{2n+2} = K_5^{\frac{7}{17}}|\epsilon_{2n+2}| \leq K_5^{\frac{7}{17}}K_5\epsilon_{2n+1}^8 = \left(K_5^{\frac{3}{17}}\right)^8\epsilon_{2n+1}^8 = d_{2n+1}^8. \tag{36}$$

Our results concerning the order of convergence generated by (33) are summarized in the following theorem.

Theorem 2.6 *Assume that the initial approximations x_0, x_{-1} are chosen so that $d_{-1} \leq d < 1$ and $d_0 \leq d < 1$.*

Then for the error of the sequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ determined by (33), we have

$$\begin{aligned} d_{2n-1} &\leq d^{4.18^{n-1}}, \\ d_{2n} &\leq d^{32.18^{n-1}}, \quad n = 1, 2, \ldots \end{aligned} \tag{37}$$

and the order of convergence of the iteration (33) is $\tau = 18$.

Proof. The proof is by induction with respect to the iteration number n.

For $n = 0$, from (36), we find

$$\begin{aligned} d_1 &\leq d^2.d^2 = d^4, \\ d_2 &\leq d_1^8 = (d^4)^8 = d^{32}. \end{aligned}$$

For $n = 1$, we have

$$\begin{aligned} d_3 &\leq d^{72}, \\ d_4 &\leq d^{576} \end{aligned}$$

and (37) is fulfilled.

Let (37) be fulfilled for $n \leq m$.

For $n = m + 1$, from (36) and (37), we have

$$d_{2(m+1)-1} = d_{2m+1} \leq d_{2m}^2d_{2m-1}^2 \leq d^{64.18^{m-1}}.d^{8.18^{m-1}} = d^{72.18^{m-1}} = d^{4.18^m},$$

$$d_{2(m+1)} = d_{2m+2} \leq d_{2m+1}^8 < \left(d^{4.18^m}\right)^8 = d^{32.18^m}$$

which completes the induction.

On the other hand,

$$d_{2n-1} = K_5^{\frac{3}{17}}|\epsilon_{2n-1}|,$$

$$d_{2n} = K_5^{\frac{7}{17}}|\epsilon_{2n}|$$

and equation (37) can be written as

$$|\epsilon_{2n-1}| \leq K_5^{-\frac{3}{17}} d^{4.18^{n-1}},$$

$$|\epsilon_{2n}| \leq K_5^{-\frac{7}{17}} d^{32.18^{n-1}}, \quad n = 1, 2, \ldots,$$

and the order of convergence of iteration (33) is equal to 18.

Thus, the theorem is proved.

Remark 7. The method (33) requires 8 function evaluations, 2 initial approximations x_{-1}, x_0 and has order of convergence $\tau = 18$ and efficiency index

$$I = 18^{\frac{1}{8}} \approx 1.435.$$

From the Table 1 which is given below, the user of the most common practice multipoint iterative methods using several initial approximations for numerical solution of nonlinear equations can be oriented to the self-accelerating with the help of optimal in the sense of Kung-Traub algorithms from order 4 and 8 in terms of convergence speed and efficiency index.

Table 1

method	order of convergence τ	efficiency index I
(5)	5	1.3797
(11)	9	1.442
(17)	9	1.442
(22)	17	1.4989
(28)	10	1.389
(33)	18	1.435

We will pose the following problem:

Problem. *Let us construct an iteration procedure (with memory) with order of convergence $\tau = 33$ and efficiency index - better than $2^{\frac{1}{2}} \approx 1.414$ of Newton's method using*:

a) *a system of two initial approximations x_{-1} and x_0*;

b) *information about f and f'*;

In [12] Li, Mu, Ma and Wang presented a modification of Newton's method with higher-order of convergence.

The modification of Newton's method is based on known King's fourth - order method. The new method requires three-step per iteration.

Analysis of convergence demonstrates that the order of convergence is 16.

If the initial point x_0 is sufficiently close to simple root x^*, then the method [12] defined by

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)},\\
z_n &= y_n - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)}.\frac{f(y_n)}{f'(x_n)},\\
x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)} - \frac{f\left(z_n - \dfrac{f(z_n)}{f'(z_n)}\right)}{f'(z_n)}.\frac{2f(z_n) - f\left(z_n - \dfrac{f(z_n)}{f'(z_n)}\right)}{2f(z_n) - 5f\left(z_n - \dfrac{f(z_n)}{f'(z_n)}\right)},
\end{aligned}
\tag{38}
$$

$$n = 0, 1, 2, \dots$$

has sixteenth - order of convergence.

Remark 8. This method requires four evaluations of the function, namely,

$$f(x_n),\ \ f(y_n),\ \ f(z_n),\ \ f\left(z_n - \frac{f(z_n)}{f'(z_n)}\right)$$

and two evaluations of first derivatives $f'(x_n)$, $f'(z_n)$ and is not optimal in the sense of Kung - Traub.

Here we give a methodological construction of nonstationary algorithms with a raised speed of convergence.

For solving this task it is appropriate to use the following iterative nonstationary algorithm for solving the nonlinear equation $f(x) = 0$:

$$
\begin{aligned}
x_{2n+1} &= x_{2n} - \frac{f(x_{2n})}{f'(x_{2n})} - \frac{f^2(x_{2n})}{2f'^3(x_{2n})}.\frac{f'(x_{2n}) - f'(x_{2n-1})}{x_{2n} - x_{2n-1}}, \\
y_{2n+1} &= x_{2n+1} - \frac{f(x_{2n+1})}{f'(x_{2n+1})}, \\
z_{2n+1} &= y_{2n+1} - \frac{2f(x_{2n+1}) - f(y_{2n+1})}{2f(x_{2n+1}) - 5f(y_{2n+1})}.\frac{f(y_{2n+1})}{f'(x_{2n+1})}, \\
x_{2n+2} &= z_{2n+1} - \frac{f(z_{2n+1})}{f'(z_{2n+1})} - \frac{f\left(z_{2n+1} - \frac{f(z_{2n+1})}{f'(z_{2n+1})}\right)}{f'(z_{2n+1})} \times \\
&\times \quad \frac{2f(z_{2n+1}) - f\left(z_{2n+1} - \frac{f(z_{2n+1})}{f'(z_{2n+1})}\right)}{2f(z_{2n+1}) - 5f\left(z_{2n+1} - \frac{f(z_{2n+1})}{f'(z_{2n+1})}\right)}, \\
& n = 0, 1, 2, \ldots .
\end{aligned} \tag{39}
$$

Here $\{x_{2n+1}\}$ is generated by (38), $\{x_{2n+2}\}$ based on the algorithm (16).

It is known that for the error ϵ_i [25] is valid

$$\epsilon_{2n+1} \sim -\frac{3}{2} c_3(\xi) \epsilon_{2n}^2 \epsilon_{2n-1}, \tag{40}$$

and for the procedure (38) [12]:

$$\epsilon_{2n+2} \sim -c_2(\xi)^5 c_3(\xi)^5 e_{2n+1}^{16}, \tag{41}$$

where $\sim$ denotes the asymptotical equation when $n \to \infty$.

Let

$$
\begin{aligned}
K_6 &= \max\left\{\tfrac{3}{2}|c_3(\xi)|, |c_2(\xi)^5 c_3(\xi)^5|\right\}, \\
d_{2n-1} &= K_6^{\frac{3}{32}} |\epsilon_{2n-1}|, \\
d_{2n} &= K_6^{\frac{1}{2}} |\epsilon_{2n}|
\end{aligned}
$$

and let $d > 0$, and x_{-1} and x_0 be chosen so that the following inequalities

$$
\begin{aligned}
d_{-1} &= K_6^{\frac{3}{32}} |x_{-1} - \xi| \leq d < 1, \\
d_0 &= K_6^{\frac{1}{2}} |x_0 - \xi| \leq d < 1
\end{aligned}
$$

hold true.

From (40) and (41), we have

$$d_{2n+1} = K_6^{\frac{3}{32}}|\epsilon_{2n+1}| \le K_6^{\frac{3}{32}} K_6 \epsilon_{2n}^2 |\epsilon_{2n-1}| = \left(K_6^{\frac{3}{32}}|\epsilon_{2n-1}|\right)\left(K_6^{\frac{1}{2}}\epsilon_{2n}\right)^2 = d_{2n}^2 d_{2n-1},$$

$$d_{2n+2} = K_6^{\frac{1}{2}}|\epsilon_{2n+2}| \le K_6^{\frac{1}{2}} K_6 \epsilon_{2n+1}^{16} = \left(K_6^{\frac{3}{32}}\epsilon_{2n+1}\right)^{16} = d_{2n+1}^{16}. \tag{42}$$

Our results concerning the order of convergence generated by (39) are summarized in the following theorem.

Theorem 2.7 *Assume that the initial approximations x_0, x_{-1} are chosen so that $d_{-1} \le d < 1$ and $d_0 \le d < 1$.*

Then for the error of the sequences $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{x_{2n+2}\}_{n=0}^{\infty}$ determined by (39), we have

$$\begin{aligned} d_{2n-1} &\le d^{3.33^{n-1}}, \\ d_{2n} &\le d^{48.33^{n-1}}, \quad n = 1, 2, \dots \end{aligned} \tag{43}$$

and the order of convergence of the iteration (39) is $\tau = 33$.

Proof. The proof is by induction with respect to the iteration number n.

For $n = 0$, from (42), we find

$$\begin{aligned} d_1 &\le d^2.d = d^3, \\ d_2 &\le d_1^{16} \le (d^3)^{16} = d^{48}. \end{aligned}$$

For $n = 1$, we have

$$\begin{aligned} d_3 &\le d_2^2 d_1 \le \left(d^{48}\right)^2 d^3 = d^{99}, \\ d_4 &\le d_3^{16} \le \left(d^{99}\right)^{16} = d^{1584} \end{aligned}$$

and (43) is fulfilled.

Let (43) be fulfilled for $n \le m$.

For $n = m + 1$, from (42) and (43), we have

$$d_{2(m+1)-1} = d_{2m+1} \le d_{2m}^2 d_{2m-1} \le d^{96.33^{m-1}+3.33^{m-1}} = d^{3.33.33^{m-1}} = d^{3.33^m},$$

$$d_{2(m+1)} = d_{2m+2} \le d_{2m+1}^{16} < \left(d^{3.33^m}\right)^{16} = d^{48.33^m}$$

which completes the induction.

On the other hand,

$$\begin{aligned} d_{2n-1} &= K_6^{\frac{3}{32}} |\epsilon_{2n-1}|, \\ d_{2n} &= K_6^{\frac{1}{2}} |\epsilon_{2n}| \end{aligned}$$

and equation (43) can be written as

$$\begin{aligned} |\epsilon_{2n-1}| &\leq K_6^{-\frac{3}{32}} d^{3.33^{n-1}}, \\ |\epsilon_{2n}| &\leq K_6^{-\frac{1}{2}} d^{48.33^{n-1}}, \quad n = 1, 2, \ldots, \end{aligned}$$

and the order of convergence of iteration (39) is equal to 33.

Thus, the theorem is proved.

Remark 9. The method (39) requires 9 function evaluations, 2 initial approximations x_{-1}, x_0 and has order of convergence $\tau = 33$ and efficiency index

$$I = 33^{\frac{1}{9}} \approx 1.4747$$

which is better than $2^{\frac{1}{2}} \approx 1.414$ of Newton's method.

3 Concluding Remarks

As we have previously noted, an iterative procedure (38) with order of convergence $\tau = 16$ is not optimal in the sense of Kung - Traub, because it uses six calculations of functions.

Let us assume for a moment that iteration algorithm (B) with order of convergence $\tau = 16$ using five functional calculations i.e. optimal in the sense of Kung - Traub was found.

To examine the issue related to self-accelerating and the corresponding efficiency index of the base method - the modification of Euler - Chebishev method - (16), with already familiar "put in" of the optimal method (B).

As a result, we get nonstationary algorithm, which use two initial approximations x_0 and x_{-1}. For brevity we denote it $(16) - (B)$.

It is not difficult to comply, that the new iterative scheme $(16) - (B)$ will have order of convergence $\tau = 33$, and it consumes eight calculations of functions and it has an efficiency index

$$I = 33^{\frac{1}{8}} \approx 1.5481.$$

It is a matter of time, having in mind the massive research in the area of numerical methods over the last five years, the design of algorithms with optimal order of convergence $\tau = 32$ and $\tau = 64$ using six respectively seven calculations of functions.

Let us denote by (C) the algorithm with this order of convergence $\tau = 32$.

Naturally arises the task of testing the combined procedure $(16) - (C)$.

The following model theorem will be valid

Theorem A. With the same symbols in this paper and imposed requirements for initial approximations x_0, x_{-1}, the order of convergence of the model iteration $(16) - (C)$ is satisfied

$$\begin{aligned} d_{2n-1} &\leq d^{3.65^{n-1}}, \\ d_{2n} &\leq d^{96.65^{n-1}}, \quad n = 1, 2, \ldots \end{aligned} \tag{44}$$

Remark 10. The method $(16) - (C)$ requires 9 function evaluations, 2 initial approximations x_{-1}, x_0 and has order of convergence $\tau = 65$ and efficiency index

$$I = 65^{\frac{1}{9}} \approx 1.5901.$$

Remark 11. The reader may account how self-accelerating are to the order of convergence and efficiency index methods of type $(2) - (C)$ and $(27) - (C)$ using two initial approximations.

Denote now with (D) this optimal in the sense of Kung - Traub algorithm with order of convergence $\tau = 64$ using 7 calculations of functions.

We will examine the combined procedure $(16) - D)$.

The following model theorem is valid

Theorem 3.1 *With the same symbols in this paper and imposed requirements for the initial approximations* x_0, x_{-1}, *the order of convergence of the model iteration* $(16) - (D)$ *is satisfied*

$$\begin{aligned} d_{2n-1} &\leq d^{3.129^{n-1}}, \\ d_{2n} &\leq d^{192.129^{n-1}}, \quad n = 1, 2, \ldots \end{aligned} \tag{45}$$

Remark 12. The method $(16) - (D)$ requires 10 function evaluations, 2 initial approximations x_{-1}, x_0 and has order of convergence $\tau = 129$ and efficiency index

$$I = 129^{\frac{1}{10}} \approx 1.6257.$$

We are now able to offer the following Table 2 for possible self-accelerating, as the basis on multipoint method (16) with the "put in" of newly found optimal in the sense of Kung-Traub algorithms from order 16, 32 and 64 in terms of convergence speed and efficiency index.

Table 2

method	order of convergence τ	efficiency index I
(16)-(B)	33	1.5481
(16)-(C)	65	1.5901
(16)-(D)	129	1.6257

Remark 13. Let us denote by τ_1 the order of convergence of the corresponding optimal algorithm of order 4, 8, 16, 32, 64, and with τ_2 - the order of convergence of the basic multipoint iteration algorithm, as an example (16), generated by the optimal in the sense of Kung - Traub algorithm.

A detailed analysis (see Theorem 2.3, Theorem 2.4, Theorem A, Theorem B) gives us reason to conclude that

$$\tau_2 = 2\tau_1 + 1.$$

Remark 14. Of course, based on the analysis of the data in Table 1 and Table 2, the user of such multipoint iterative schemes should consider for himself "what is the price that he is willing to pay" to achieve speed and efficiency of the used nonstationary algorithm.

Remark 15. The results obtained in this article can be successfully used to refine the self-accelerating of multipoint iterations using three initial approximations $x_{-2},\ x_{-1},\ x_0$.

IV. Consider the following finite-difference analogue of Halley method

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_n, x_{n-1}] + f[x_n, x_{n-1}, x_{n-2}](x_n - x_{n-1})}, \tag{46}$$

$$n = 0, 1, 2, \ldots$$

which requires three initial approximations $x_{-2},\ x_{-1},\ x_0$.

IV.1. We will explore the issue of self acceleration of this procedure, in combination with optimal algorithm in the sense of Kung - Traub, as an example with order of convergence 4 - (4):

$$\begin{aligned} x_{2n+1} &= x_{2n} - \frac{f(x_{2n})}{f[x_{2n}, x_{2n-1}] + f[x_{2n}, x_{2n-1}, x_{2n-2}](x_{2n} - x_{2n-1})}, \\ x_{2n+2} &= x_{2n+1} - \frac{f(x_{2n+1})}{f'(x_{2n+1})} - \frac{f(x_{2n+1})}{f'(x_{2n+1})} . \frac{f\left(x_{2n+1} - \frac{f(x_{2n+1})}{f'(x_{2n+1})}\right)}{f(x_{2n+1}) - 2f\left(x_{2n+1} - \frac{f(x_{2n+1})}{f'(x_{2n+1})}\right)}, \end{aligned} \tag{47}$$

$$n = 0, 1, 2, \dots$$

It is known that for the error $\epsilon_i = x_i - \xi, \quad i = -2, -1, 0, 1, 2 \dots$; [25] is valid

$$\epsilon_{2n+1} \sim L(\xi)\epsilon_{2n}\epsilon_{2n-1}\epsilon_{2n-2}, \tag{48}$$

$$\epsilon_{2n+2} \sim M(\xi)\epsilon_{2n+1}^4. \tag{49}$$

Let

$$\begin{aligned} K_7 &= \max\left\{|L(\xi))|, |M(\xi)|\right\}, \\ d_{2n-1} &= K_7^{\frac{3}{8}}|\epsilon_{2n-1}|, \\ d_{2n} &= K_7^{\frac{1}{2}}|\epsilon_{2n}|, \\ d_{2n-2} &= K_7^{\frac{1}{2}}|\epsilon_{2n-2}|, \end{aligned}$$

and let $d > 0$, and x_{-2}, x_{-1} and x_0 be chosen so that the following inequalities

$$\begin{aligned} d_{-2} &= K_7^{\frac{1}{2}}|x_{-2} - \xi| \leq d < 1, \\ d_{-1} &= K_7^{\frac{3}{8}}|x_{-1} - \xi| \leq d < 1, \\ d_0 &= K_7^{\frac{1}{2}}|x_0 - \xi| \leq d < 1 \end{aligned}$$

hold true.

From (48) and (49), we have

$$
\begin{aligned}
d_{2n+1} &= K_7^{\frac{3}{8}}|\epsilon_{2n+1}| \leq K_7^{\frac{3}{8}}K_7|\epsilon_{2n}||\epsilon_{2n-1}||\epsilon_{2n-2}| = K_7^{\frac{3}{8}}|\epsilon_{2n-1}|K_7^{\frac{1}{2}}|\epsilon_{2n}|K_7^{\frac{1}{2}}|\epsilon_{2n-2}| = \\
&= d_{2n}d_{2n-1}d_{2n-2}, \\
d_{2n+2} &= K_7^{\frac{1}{2}}|\epsilon_{2n+2}| \leq K_7^{\frac{1}{2}}K_7\epsilon_{2n+1}^4 = \left(K_7^{\frac{3}{8}}\epsilon_{2n+1}\right)^4 = d_{2n+1}^4.
\end{aligned}
\tag{50}
$$

Evidently, from (50), we find

$$
\begin{aligned}
& d_1 \leq d^3,\ d_2 \leq d^{12},\ d_3 \leq d^{16},\ d_4 \leq d^{64}, \\
& d_5 \leq d^{92},\ d_6 \leq d^{368},\ d_7 \leq d^{524},\ d_8 \leq d^{2096}, \\
& d_9 \leq d^{2988},\ d_{10} \leq d^{11952}.
\end{aligned}
$$

Our results concerning the order of convergence generated by (47) are summarized in the following theorem.

Theorem 3.2 *Assume that the initial approximations x_0, x_{-1}, x_{-2} are chosen so that $d_{-2} \leq d,\ d_{-1} \leq d < 1$ and $d_0 \leq d < 1$.*

Then for the error of the sequences $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{x_{2n+2}\}_{n=0}^{\infty}$ determined by (47), we have

$$
\begin{aligned}
d_{2n-1} &\leq d^{\tau_{2n-1}}, \\
d_{2n} &\leq d^{\tau_{2n}},
\end{aligned}
\tag{51}
$$

where

$$\tau_{m+6} = 4\tau_{m+4} + 9\tau_{m+2} + 4\tau_m, \tag{52}$$

$$m = 0, 1, 2, \ldots$$

and the order of convergence of the iteration (47) is

$$\tau = \frac{5 + \sqrt{41}}{2}.$$

.

Proof. It is well known that for the recursion:

$$\gamma_{i+1} = \sum_{j=1}^{n} A_j \gamma_{i-j+1}, \quad i = n-1, n-2, \ldots,$$

(for any initial conditions) corresponds to the characteristic polynomial:

$$\rho^n = \sum_{j=1}^{n} A_j \rho^{n-j}.$$

In our case, for the recursion

$$\tau_{m+6} = 4\tau_{m+4} + 9\tau_{m+2} + 4\tau_m,$$

characteristic polynomial is of the type

$$\rho^3 - 4\rho^2 - 9\rho - 4 = 0. \tag{53}$$

Equation (53) has the roots:

$$\rho_1 = \frac{5+\sqrt{41}}{2}, \quad \rho_2 = \frac{5-\sqrt{41}}{2}, \quad \rho_3 = -1.$$

From the general iterative theory [25], (see, also [5]) it follows that the order of convergence of the iteration procedure, defined by (47) is given by the only real root of equation (53) with magnitude greater than 1.

On the other hand,

$$\begin{aligned} |\epsilon_{2n+1}| &\leq K_7^{-\frac{3}{8}} d_{2n+1}, \\ |\epsilon_{2n+2}| &\leq K_7^{-\frac{1}{2}} d_{2n+2}, \end{aligned}$$

and consequently we can conclude that the order of convergence of iteration (47) is

$$\tau = \frac{5+\sqrt{41}}{2} \approx 5.70156...$$

Thus, the theorem is proven.

Remark 16. Of course precise analysis can be made following the cited monographs [25], [5].

Suffice it to seek representation for the values τ_i as a linear combination of the roots of characteristic equation ρ_1, ρ_2 and ρ_3.

As an example,

$$\tau_{2n+1} = c_1\rho_1^n + c_2\rho_2^n + c_3\rho_3^n$$

with condition

$$\tau_1 = 3, \;\; \tau_3 = 16, \;\; \tau_5 = 92.$$

The solution of the system

$$\begin{aligned}
&c_1 + c_2 + c_3 = 3 \\
&c_1 \frac{5+\sqrt{41}}{2} + c_2 \frac{5-\sqrt{41}}{2} - c_3 = 16 \\
&c_1 \left(\frac{5+\sqrt{41}}{2}\right)^2 + c_2 \left(\frac{5-\sqrt{41}}{2}\right)^2 + c_3 = 92
\end{aligned}$$

is the following:

$$\begin{aligned}
c_1 &= \frac{1}{82}(123 + 17\sqrt{41}) \\
c_2 &= \frac{1}{82}(123 - 17\sqrt{41}) \\
c_3 &= 0.
\end{aligned}$$

This lead to the following inequality

$$d_{2n} \leq d^{\frac{1}{82}(41+19\sqrt{41})\rho_1^n + \frac{1}{82}(41-19\sqrt{41})\rho_2^n}, \tag{54}$$

The proof is by induction with respect to the iteration number n.
Let (54) be fulfilled for $n \leq m$.
For $n = m + 1$, we have

$$d_{2(m+1)} = d_{2m+2} \leq d_{2m+1}^4 \leq d^{\frac{4}{82}(123+17\sqrt{41})\rho_1^m + \frac{4}{82}(123-17\sqrt{41})\rho_2^m}.$$

Using the equalities

$$123 \pm 17\sqrt{41} = \frac{1}{4}(41 \pm 19\sqrt{41}).\frac{1}{2}(5 \pm \sqrt{41})$$

we have

$$d_{2(m+1)} \leq d^{\frac{1}{82}(41+19\sqrt{41})\rho_1^{m+1} + \frac{1}{82}(41-19\sqrt{41})\rho_2^{m+1}}$$

which completes the induction.

Evidently, $-1 < \rho_2 < 0$, so that, asymptotically,

$$d^{\frac{1}{82}(41-19\sqrt{41})\rho_2^m} \approx 1$$

and

$$|\epsilon_{2n}| \approx K_7^{-\frac{1}{2}} d^{\frac{1}{82}(41+19\sqrt{41})\rho_1^n}.$$

Remark 17. The method (47) requires 6 function evaluations, 3 initial approximations x_{-2}, x_{-1} x_0 and has order of convergence $\tau = 5.70156$ and efficiency index

$$I = 5.70156^{\frac{1}{6}} \approx 1.3365.$$

IV.2. We will consider the following nonstationary iterative scheme based on the scheme (46) in combination with an optimal algorithm in the sense of Kung - Traub with order of convergence 8 - (10) [18]:

$$\begin{aligned}
x_{2n+1} &= x_{2n} - \frac{f(x_{2n})}{f[x_{2n}, x_{2n-1}] + f[x_{2n}, x_{2n-1}, x_{2n-2}](x_{2n} - x_{2n-1})}, \\
y_{2n+1} &= x_{2n+1} - \frac{f(x_{2n+1})}{f'(x_{2n+1})}, \\
z_{2n+1} &= y_{2n+1} - \frac{f(y_{2n+1})}{f'(x_{2n+1})} . \frac{f(x_{2n+1})}{f(x_{2n+1}) - 2f(y_{2n+1})}, \\
x_{2n+2} &= z_{2n+1} - \left(1 + \frac{f(z_{2n+1})}{f(x_{2n+1})} + \left(\frac{f(z_{2n+1})}{f(x_{2n+1}}\right)^2\right) . \frac{f[x_{2n+1}, y_{2n+1}]f(z_{2n+1})}{f[x_{2n+1}, z_{2n+1}]f[y_{2n+1}, z_{2n+1}]},
\end{aligned} \tag{55}$$

$$n = 0, 1, 2, \ldots$$

It is known that for the error $\epsilon_i = x_i - \xi, \quad i = -2, -1, 0, 1, 2 \ldots$; [25], [18] is valid

$$\epsilon_{2n+1} \sim L(\xi)\epsilon_{2n}\epsilon_{2n-1}\epsilon_{2n-2}, \tag{56}$$

$$\epsilon_{2n+2} \sim A(\xi)\epsilon_{2n+1}^8. \tag{57}$$

Let

$$\begin{aligned}
K_8 &= \max\{|L(\xi))|, |A(\xi)|\}, \\
d_{2n-1} &= K_8^{\frac{3}{16}}|\epsilon_{2n-1}|, \\
d_{2n} &= K_8^{\frac{1}{2}}|\epsilon_{2n}|, \\
d_{2n-2} &= K_8^{\frac{1}{2}}|\epsilon_{2n-2}|,
\end{aligned}$$

and let $d > 0$, and x_{-2}, x_{-1} and x_0 be chosen so that the following inequalities

$$\begin{aligned} d_{-2} &= K_8^{\frac{1}{2}}|x_{-2} - \xi| \le d < 1, \\ d_{-1} &= K_8^{\frac{3}{16}}|x_{-1} - \xi| \le d < 1, \\ d_0 &= K_8^{\frac{1}{2}}|x_0 - \xi| \le d < 1 \end{aligned}$$

hold true.

From (56) and (57), we have

$$\begin{aligned} d_{2n+1} &= K_8^{\frac{3}{16}}|\epsilon_{2n+1}| \le K_8^{\frac{3}{16}} K_8|\epsilon_{2n}||\epsilon_{2n-1}||\epsilon_{2n-2}| = K_8^{\frac{3}{16}}|\epsilon_{2n-1}|K_8^{\frac{1}{2}}|\epsilon_{2n}|K_8^{\frac{1}{2}}|\epsilon_{2n-2}| = \\ &= d_{2n}d_{2n-1}d_{2n-2}, \\ d_{2n+2} &= K_8^{\frac{1}{2}}|\epsilon_{2n+2}| \le K_8^{\frac{1}{2}} K_8 \epsilon_{2n+1}^8 = \left(K_8^{\frac{3}{16}}\epsilon_{2n+1}\right)^8 = d_{2n+1}^8. \end{aligned} \tag{58}$$

Evidently, from (58), we find

$$\begin{aligned} &d_1 \le d^3,\ d_2 \le d^{24},\ d_3 \le d^{28},\ d_4 \le d^{224}, \\ &d_5 \le d^{276},\ d_6 \le d^{2208},\ d_7 \le d^{2708},\ d_8 \le d^{21664}, \\ &d_9 \le d^{26580},\ d_{10} \le d^{212640}. \end{aligned}$$

Our results concerning the order of convergence generated by (55) are summarized in the following theorem.

Theorem 3.3 *Assume that the initial approximations x_0, x_{-1}, x_{-2} are chosen so that $d_{-2} \le d$, $d_{-1} \le d < 1$ and $d_0 \le d < 1$.*

Then for the error of the sequences $\{x_{2n+1}\}_{n=0}^{\infty}$ and $\{x_{2n+2}\}_{n=0}^{\infty}$ determined by (55), we have

$$\begin{aligned} d_{2n-1} &\le d^{\tau_{2n-1}}, \\ d_{2n} &\le d^{\tau_{2n}}, \end{aligned} \tag{59}$$

where

$$\tau_{m+6} = 8\tau_{m+4} + 17\tau_{m+2} + 8\tau_m, \tag{60}$$

$$m = 0, 1, 2, \ldots$$

and the order of convergence of the iteration (55) is

$$\tau = \frac{9 + \sqrt{113}}{2}.$$

.

Proof. The detailed proof of this theorem will not be given because the reasoning follows the statement of the proof of Theorem 3.2.

We note only that in this case the recursion

$$\tau_{m+6} = 8\tau_{m+4} + 17\tau_{m+2} + 8\tau_m,$$

satisfies the characteristic polynomial

$$\rho^3 - 8\rho^2 - 17\rho - 8 = 0. \tag{61}$$

Equation (61) has the roots:

$$\rho_1 = \frac{9 + \sqrt{113}}{2}, \quad \rho_2 = \frac{9 - \sqrt{113}}{2}, \quad \rho_3 = -1.$$

From the general iterative theory [25], (see, also [5]) it follows that the order of convergence of the iteration procedure, defined by (55) is given by the only positive root of equation (61).

On the other hand,

$$|\epsilon_{2n+1}| \leq K_8^{-\frac{3}{16}} d_{2n+1},$$

$$|\epsilon_{2n+2}| \leq K_8^{-\frac{1}{2}} d_{2n+2},$$

and consequently we can conclude that the order of convergence of iteration (55) is

$$\tau = \frac{9 + \sqrt{113}}{2} \approx 5.78702...$$

Thus, the theorem is proved.

Remark 18. The method (55) requires 7 function evaluations, 3 initial approximations x_{-2}, x_{-1} x_0 and has order of convergence $\tau = 5.78702$ and efficiency index

$$I = 5.78702^{\frac{1}{7}} \approx 1.285.$$

Remark 19. The received during last year numerical methods which have high index of efficiency by F. Soleymani and his coauthors [19], [21], [22], [23] can be successfully used for generation of new iterative methods for solving nonlinear equations using several initial approximations by proposed in our paper techniques.

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