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Fekete-Szegö Inequality for a New Class and Its Certain Subclasses of Analytic Functions

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Abstract

We introduce some classes of analytic functions, its subclasses and obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ *for the analytic function* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ *belonging to these classes and subclasses.*

Keywords: *Univalent functions, Starlike functions, Close to convex functions and bounded functions.*

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([7], [8]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$.

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\boldsymbol{S}$, Fekete and Szegö [9] used Löwner's method to prove the following well known result for the class $\boldsymbol{S}$.

Let $f(z) \in \boldsymbol{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if\ \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \leq \mu \leq 1; \\ 4\mu - 3, if \mu \geq 1. \end{cases} \quad (1.2)$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\boldsymbol{S}$ (See Chhichra [1], Babalola [6]).

Let us define some subclasses of $\boldsymbol{S}$.

We denote by S*, the class of univalent starlike functions

$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \boldsymbol{\mathcal{A}}$ and satisfying the condition

$$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \quad (1.3)$$

We denote by $\mathcal{K}$, the class of univalent convex functions $h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \boldsymbol{\mathcal{A}}$ and satisfying the condition

$$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \quad (1.4)$$

A function $f(z) \in \boldsymbol{\mathcal{A}}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \quad (1.5)$$

The class of close to convex functions is denoted by C and was introduced by Kaplan [3] and it was shown by him that all close to convex functions are univalent.

$$S^*(A,B) = \left\{f(z) \in \boldsymbol{\mathcal{A}}; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\} \quad (1.6)$$

$$\mathcal{K}(A,B) = \left\{f(z) \in \boldsymbol{\mathcal{A}}; \frac{\left(zf'(z)\right)'}{f'(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\} \quad (1.7)$$

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}\ (A,B)$ is a subclass of $\mathcal{K}$.

We introduce a new class as $\left\{f(z) \in \mathcal{A}; \frac{z\left\{\left(f'(z)\right)^2+f(z)f''(z)\right\}}{f(z)f'(z)} \prec \frac{1+z}{1-z}; z \in \mathbb{E}\right\}$ and we will denote this class as $S^*(f,f',f'')$.

We will also deal with two subclasses of $S^*(f,f',f'')$ defined as follows:

$$S^*(f,f',f'';A,B) = \left\{f(z) \in \mathcal{A}; \frac{z\left\{\left(f'(z)\right)^2+f(z)f''(z)\right\}}{f(z)f'(z)} \prec \frac{1+Az}{1+Bz}; z \in \mathbb{E}\right\} \tag{1.8}$$

$$S^*(f,f',f'';A,B,\delta) = \left\{f(z) \in \mathcal{A}; \frac{z\left\{\left(f'(z)\right)^2+f(z)f''(z)\right\}}{f(z)f'(z)} \prec \left(\frac{1+Az}{1+Bz}\right)^\delta; z \in \mathbb{E}\right\} \tag{1.9}$$

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination

Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, w(0) = 0, |w(z)| < 1. \tag{1.10}$$

It is known that $|c_1| \leq 1, |c_2| \leq 1 - |c_1|^2$. (1.11)

2 Preliminary Lemmas

For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz}\right)$ so that

$$\frac{1+w(z)}{1-w(z)} = 1 + 2cz + 2z^2 + \cdots. \tag{2.1}$$

3 Main Results

Theorem 3.1: *Let* $f(z) \in S^*(f,f',f'')$, *then*

$|a_3 - \mu a_2^2| \leq$
$$\begin{cases} \frac{19}{36} - \frac{4}{9}\mu, if \mu \leq \frac{5}{8}; & (3.1) \\ \frac{1}{4}, if \ \frac{5}{8} \leq \mu \leq \frac{7}{4}; & (3.2) \\ \frac{4}{9}\mu - \frac{19}{36}, \ \ if \ \mu \geq \ \frac{7}{4}. & (3.3) \end{cases}$$

The results are sharp.

Proof: By definition of $S^*(f, f', f'')$, we have

$$\frac{z\left\{\left(f'(z)\right)^2 + f(z)f''(z)\right\}}{f(z)f'(z)} = \frac{1+w(z)}{1-w(z)}; w(z) \in \mathcal{U}. \quad (3.4)$$

Expanding the series (3.4), we get

$$(1 + 2a_2z + 3a_3z^2 + - - -)^2 + (z + a_2z^2 + a_3z^3 + - - -)(2a_2 + 6a_3z + 12a_4z^2 + - - -) = (1 + a_2z + a_3z^2 + - - -)(1 + 2a_2z + 3a_3z^2 + - - -)(1 + 2c_1z + 2(c_2 + {c_1}^2)z^2 + - - -).$$

$$\{1 + 4a_2z + (6a_3 + 4a_2^2)z^2 + - - -\} + \{2a_2z + (6a_3 + 2a_2^2)z^2 + - - -\} = (1 + 3a_2z + (4a_3 + 2a_2^2)z^2 + - - -)(1 + 2c_1z + 2(c_2 + {c_1}^2)z^2 + - - -).$$

$$1 + 6a_2z + 6(2a_3 + a_2^2)z^2 + - - - = 1 + (3a_2 + 2c_1)z + (4a_3 + 2a_2^2 + 6a_2c_1 + 2c_2 + 2{c_1}^2)z^2 + - - - \quad (3.5)$$

Identifying terms in (3.5), we get

$$a_2 = \frac{2}{3}\ c_1 \quad (3.6)$$

$$a_3 = \frac{1}{4}\ c_2 + \ \frac{19}{36}\ {c_1}^2. \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$a_3 - \mu a_2^2 = \frac{1}{4}c_2 + \left[\frac{19}{36} - \frac{4}{9}\mu\right]c_1^2. \quad (3.8)$$

Taking absolute value, (3.8) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{1}{4}|c_2| + \left|\frac{19}{36} - \frac{4}{9}\mu\right| |c_1^2|. \quad (3.9)$$

Using (1.9) in (3.9), we get

$$|a_3 - \mu a_2^2| \leq \frac{1}{4}(1 - |c_1|^2) + \left|\frac{19}{36} - \frac{4}{9}\mu\right| |c_1|^2 = \frac{1}{4} + \left\{\left|\frac{19}{36} - \frac{4}{9}\mu\right| - \frac{1}{4}\right\} |c_1|^2. \quad (3.10)$$

Case I: $\mu \leq \frac{19}{36}$. (3.10) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} + \left\{\left(\frac{19}{36} - \frac{4}{9}\mu\right) - \frac{1}{4}\right\}|c_1|^2 = \frac{1}{4} + \left\{\frac{5}{18} - \frac{4}{9}\mu\right\}|c_1|^2. \tag{3.11}$$

Subcase I (a): $\mu \leq \frac{5}{8}$. Using (1.9), (3.11) becomes

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} + \left\{\frac{5}{18} - \frac{4}{9}\mu\right\} = \frac{19}{36} - \frac{4}{9}\mu. \tag{3.12}$$

Subcase I (b): $\mu \geq \frac{5}{8}$. We obtain from (3.11)

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} - \left\{\frac{4}{9}\mu - \frac{5}{18}\right\}|c_1|^2 \leq \frac{1}{4}. \tag{3.13}$$

Case II: $\mu \geq \frac{19}{36}$. Preceding as in case I, we get

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} + \left\{\frac{4}{9}\mu - \frac{7}{9}\right\}|c_1|^2. \tag{3.14}$$

Subcase II (a): $\mu \leq \frac{7}{4}$. (3.14)

Takes the form $|a_3 - \mu a_2^2| \leq \frac{1}{4}$. (3.15)

Combining subcase I (b) and subcase II (a), we obtain

$$|a_3 - \mu a_2^2| \leq \frac{1}{4} \; if \; \frac{5}{8} \leq \mu \leq \frac{7}{4}. \tag{3.16}$$

Subcase II (b): $\mu \geq \frac{7}{4}$. Preceding as in subcase I (a), we get

$$|a_3 - \mu a_2^2| \leq \frac{4}{9}\mu - \frac{19}{36}. \tag{3.17}$$

Combining (3.12), (3.16) and (3.17), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by $f_1(z) = \sqrt{2\left\{\frac{1}{1-z} - \log(1-z)\right\}}$.

Extremal function for (3.2) is defined by $f_2(z) = \sqrt{\log\left(\frac{1}{1-z^2}\right)}$.

Theorem 3.2: *Let* $S^*(f, f', f''; A, B)$, *then*

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{(A-B)(5A-14B)}{72} - \frac{(A-B)^2}{9}\mu, if \mu \leq \frac{(5A-14B-9)}{8(A-B)}; & (3.18) \\ \frac{(A-B)}{8}, if\ \frac{(5A-14B-9)}{8(A-B)} \leq \mu \leq \frac{(5A-14B+9)}{8(A-B)}; & (3.19) \\ \frac{(A-B)^2}{9}\mu - \frac{(A-B)(5A-14B)}{72},\ \ if\ \mu \geq \frac{(5A-14B+9)}{8(A-B)}. & (3.20) \end{cases}.$$

The results are sharp.

Proof: By definition of $S^*(f,f^{'},f^{''};A,B)$, we have

$$\frac{z\left\{\left(f^{'}(z)\right)^2+f(z)f^{''}(z)\right\}}{f(z)f^{'}(z)} = \frac{1+Aw(z)}{1+Bw(z)}; w(z) \in \mathcal{U}. \tag{3.21}$$

Expanding the series (3.21), we get

$$1+6a_2z+6(2a_3+a_2^2)z^2+---= 1+\{3a_2+(A-B)c_1\}z+(4a_3+2a_2^2+3a_2(A-B)c_1+(A-B)(c_2-Bc_1^2)z^2+--- \tag{3.22}$$

Identifying terms in (3.22), we get $a_2 = \frac{(A-B)}{3}\ c_1$ (3.23)

And $a_3 = \frac{(A-B)}{8}\ c_2 + \frac{(A-B)(5A-14B)}{72}\ {c_1}^2.$ (3.24)

From (3.23) and (3.24), we obtain

$$a_3-\mu a_2^2 = \frac{(A-B)}{8}\ c_2 + \left[\frac{(A-B)(5A-14B)}{72} - \frac{(A-B)^2}{9}\mu\right]c_1^2. \tag{3.25}$$

Taking absolute value, (3.25) can be rewritten as

$$|a_3-\mu a_2^2| \leq \frac{(A-B)}{8}\ |c_2| + \left|\frac{(A-B)(5A-14B)}{72} - \frac{(A-B)^2}{9}\mu\right| c_1^2. \tag{3.26}$$

Using (1.9) in (3.26), we get

$$|a_3-\mu a_2^2| \leq \frac{(A-B)}{8}(1-|c_1|^2) + \left|\frac{(A-B)(5A-14B)}{72} - \frac{(A-B)^2}{9}\mu\right| |c_1|^2$$

$$= \frac{(A-B)}{8} + \left\{\left|\frac{(A-B)(5A-14B)}{72} - \frac{(A-B)^2}{9}\mu\right| - \frac{(A-B)}{8}\right\}|c_1^2|. \tag{3.27}$$

Case I: $\mu\ \leq \frac{(5A-14B)}{8(A-B)}$. (3.27) can be rewritten as

$$|a_3-\mu a_2^2| \leq \frac{(A-B)}{8} + \left\{\left(\frac{(A-B)(5A-14B)}{72} - \frac{(A-B)^2}{9}\mu\right) - \frac{(A-B)}{8}\right\}|c_1^2|$$

$$= \frac{(A-B)}{8} + \left(\frac{(A-B)(5A-14B-9)}{72} - \frac{(A-B)^2}{9}\mu\right)|c_1^2|. \tag{3.28}$$

Subcase I (a): $\mu \le \frac{(5A-14B-9)}{8(A-B)}$ Using (1.9), (3.28) becomes

$$|a_3 - \mu a_2^2| \le \frac{(A-B)}{8} + \left(\frac{(A-B)(5A-14B-9)}{72} - \frac{(A-B)^2}{9}\mu\right) = \frac{(A-B)(5A-14B)}{72} - \frac{(A-B)^2}{9}\mu. \tag{3.29}$$

Subcase I (b): $\mu \ge \frac{(5A-14B-9)}{8(A-B)}$. We obtain from (3.28)

$$|a_3 - \mu a_2^2| \le \frac{(A-B)}{8} - \left(\frac{(A-B)^2}{9}\mu - \frac{(A-B)(5A-14B-9)}{72}\right)|c_1^2| \le \frac{(A-B)}{8}. \tag{3.30}$$

Case II: $\mu \ge \frac{(5A-14B)}{8(A-B)}$. Preceding as in case I, we get

$$|a_3 - \mu a_2^2| \le \frac{(A-B)}{8} + \left\{\left(\frac{(A-B)^2}{9}\mu - \frac{(A-B)(5A-14B)}{72}\right) - \frac{(A-B)}{8}\right\}|c_1^2|$$

$$\le \frac{(A-B)}{8} + \left(\frac{(A-B)^2}{9}\mu - \frac{(A-B)(5A-14B+9)}{72}\right)|c_1^2|. \tag{3.31}$$

Subcase II (a): $\mu \le \frac{(5A-14B+9)}{8(A-B)}$. (3.31) takes the form

$$|a_3 - \mu a_2^2| \le \frac{(A-B)}{8}. \tag{3.32}$$

Combining subcase I (b) and subcase II (a), we obtain

$$|a_3 - \mu a_2^2| \le \frac{(A-B)}{8} \text{ } if \text{ } \frac{(5A-14B-9)}{8(A-B)} \le \mu \le \frac{(5A-14B+9)}{8(A-B)} \tag{3.33}$$

Subcase II (b): $\mu \ge \frac{(5A-14B+9)}{8(A-B)}$. Preceding as in subcase I (a), we get

$$|a_3 - \mu a_2^2| \le \frac{(A-B)^2}{9}\mu - \frac{(A-B)(5A-14B)}{72}. \tag{3.34}$$

Combining (3.29), (3.33) and (3.34), the theorem is proved.

Extremal function for (3.18) and (3.20) is defined by

$$f_1(z) = \sqrt{\frac{2}{A(A+B)}\left\{(1+Bz)^{\frac{A}{B}}(Az-1)-1\right\}}.$$

Extremal function for (3.19) is defined by $f_2(z) = \sqrt{\frac{2}{(A+B)}\left\{(1+Bz^2)^{\frac{A+B}{2B}} - 1\right\}}$.

Corollary 3.3: Putting $A = 1, B = -1$ in the theorem **3.2**, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{19}{36} - \frac{4}{9}\mu, if \mu \leq \frac{5}{8}; \\ \frac{1}{4}, if\ \frac{5}{8} \leq \mu \leq \frac{7}{4} \\ \frac{4}{9}\mu - \frac{19}{36},\ \ if\ \mu \geq \frac{7}{4}. \end{cases}$$

, which is the result obtained in theorem (3.1).

Theorem 3.4: Let $S^*(f, f^{'}, f^{''}; \delta)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\delta(9\delta^3-14\delta^2-12\delta+36)}{18(3-\delta)(2-\delta)^2} - \frac{4\delta^2}{9(2-\delta)^2}\mu, if \mu \leq \frac{9\delta^4-14\delta^3-21\delta^2+72\delta-36}{8\delta^2(3-\delta)}; & (3.35) \\ \frac{1}{2(3-\delta)}, if\ \frac{9\delta^4-14\delta^3-21\delta^2+72\delta-36}{8\delta^2(3-\delta)} \leq \mu \leq \frac{9\delta^4-14\delta^3-3\delta^2+36}{8\delta^2(3-\delta)}; & (3.36). \\ \frac{4\delta^2}{9(2-\delta)^2}\mu - \frac{\delta(9\delta^3-14\delta^2-12\delta+36)}{18(3-\delta)(2-\delta)^2},\ \ if\ \mu \geq \frac{9\delta^4-14\delta^3-3\delta^2+36}{8\delta^2(3-\delta)}. & (3.37) \end{cases}$$

The results are sharp.

Proof: By definition of $S^*(f, f^{'}, f^{''}; \delta)$, we have

$$\frac{z\left\{\left(f^{'}(z)\right)^2 + f(z)f^{''}(z)\right\}}{f(z)f^{'}(z)} = \left(\frac{1+w(z)}{1-w(z)}\right)^{\delta}; w(z) \in \mathcal{U}. \qquad (3.38)$$

Expanding the series (3.38), we get

$$1 + 6a_2z + 6(2a_3 + a_2^2)z^2 + - - - = \{1 + (3a_2 + 2c_1)z + (4a_3 + 2a_2^2 + 6a_2c_1 + 2c_2 + 2{c_1}^2)z^2 + - - -\}^{\delta}$$
$$= \left\{1 + \delta(3a_2 + 2c_1)z + \delta(4a_3 + \frac{9\delta-5}{2}a_2^2 + 6\delta a_2c_1 + 2c_2 + 2\delta{c_1}^2)z^2 + - - -\right\} \qquad (3.39)$$

Identifying terms in (3.39), we get

$$a_2 = \frac{2\delta}{3(2-\delta)}\ c_1 \qquad (3.40)$$

$$a_3 = \frac{1}{2(3-\delta)}\ c_2 + \frac{\delta(9\delta^3-14\delta^2-12\delta+36)}{18(3-\delta)(2-\delta)^2}\ {c_1}^2. \qquad (3.41)$$

From (3.40) and (3.41), we obtain

$$a_3 - \mu a_2^2 = \frac{1}{2(3-\delta)}\ c_2 + \left[\frac{\delta(9\delta^3-14\delta^2-12\delta+36)}{18(3-\delta)(2-\delta)^2} - \frac{4\delta^2}{9(2-\delta)^2}\mu\right] c_1^2. \qquad (3.42)$$

Taking absolute value, (3.42) can be rewritten as

$$|a_3 - \mu a_2^2| \le \frac{1}{2(3-\delta)}\ |c_2| + \left|\frac{\delta(9\delta^3-14\delta^2-12\delta+36)}{18(3-\delta)(2-\delta)^2} - \frac{4\delta^2}{9(2-\delta)^2}\mu\right| c_1^2. \tag{3.43}$$

Using (1.9) in (3.43), we get

$$|a_3 - \mu a_2^2| \le \frac{1}{2(3-\delta)}(1-|c_1|^2) + \left|\frac{\delta(9\delta^3-14\delta^2-12\delta+36)}{18(3-\delta)(2-\delta)^2} - \frac{4\delta^2}{9(2-\delta)^2}\mu\right| |c_1|^2$$

$$= \frac{1}{2(3-\delta)} + \left\{\left|\frac{\delta(9\delta^3-14\delta^2-12\delta+36)}{18(3-\delta)(2-\delta)^2} - \frac{4\delta^2}{9(2-\delta)^2}\mu\right| - \frac{1}{2(3-\delta)}\right\} |c_1^2|. \tag{3.44}$$

Case I: $\mu \le \frac{(9\delta^3-14\delta^2-12\delta+36)}{8\delta(3-\delta)}$. (3.44) can be rewritten as

$$|a_3 - \mu a_2^2| \le \frac{1}{2(3-\delta)} + \left\{\left(\frac{\delta(9\delta^3-14\delta^2-12\delta+36)}{18(3-\delta)(2-\delta)^2} - \frac{4\delta^2}{9(2-\delta)^2}\mu\right) - \frac{1}{2(3-\delta)}\right\} |c_1^2|$$

$$= \frac{1}{2(3-\delta)} + \left(\frac{9\delta^4-14\delta^3-21\delta^2+72\delta-36}{18(3-\delta)(2-\delta)^2} - \frac{4\delta^2}{9(2-\delta)^2}\mu\right) |c_1^2|. \tag{3.45}$$

Subcase I (a): $\mu \le \frac{9\delta^4-14\delta^3-21\delta^2+72\delta-36}{8\delta^2(3-\delta)}$ Using (1.9), (3.45) becomes

$$|a_3 - \mu a_2^2| \le \frac{1}{2(3-\delta)} + \left(\frac{9\delta^4-14\delta^3-21\delta^2+72\delta-36}{18(3-\delta)(2-\delta)^2} - \frac{4\delta^2}{9(2-\delta)^2}\mu\right)$$

$$= \frac{\delta(9\delta^3-14\delta^2-12\delta+36)}{18(3-\delta)(2-\delta)^2} - \frac{4\delta^2}{9(2-\delta)^2}\mu. \tag{3.46}$$

Subcase I (b): $\mu \ge \frac{9\delta^4-14\delta^3-21\delta^2+72\delta-36}{8\delta^2(3-\delta)}$. We obtain from (3.45)

$$|a_3 - \mu a_2^2| \le \frac{1}{2(3-\delta)} - \left(\frac{4\delta^2}{9(2-\delta)^2}\mu - \frac{9\delta^4-14\delta^3-21\delta^2+72\delta-36}{18(3-\delta)(2-\delta)^2}\right)|c_1^2| \le \frac{1}{2(3-\delta)}. \tag{3.47}$$

Case II: $\mu \ge \frac{(9\delta^3-14\delta^2-12\delta+36)}{8\delta(3-\delta)}$. Preceding as in case I, we get

$$|a_3 - \mu a_2^2| \le \frac{1}{2(3-\delta)} + \left\{\left(\frac{4\delta^2}{9(2-\delta)^2}\mu - \frac{\delta(9\delta^3-14\delta^2-12\delta+36)}{18(3-\delta)(2-\delta)^2}\right) - \frac{1}{2(3-\delta)}\right\} |c_1^2|$$

$$\le \frac{1}{2(3-\delta)} + \left(\frac{4\delta^2}{9(2-\delta)^2}\mu - \frac{9\delta^4-14\delta^3-3\delta^2+36}{18(3-\delta)(2-\delta)^2}\right) |c_1^2|. \tag{3.48}$$

Subcase II (a): $\mu \le \frac{9\delta^4-14\delta^3-3\delta^2+36}{8\delta^2(3-\delta)}$. (3.48) takes the form

$$|a_3 - \mu a_2^2| \le \frac{1}{2(3-\delta)}. \tag{3.49}$$

Combining subcase I (b) and subcase II (a), we obtain

$$|a_3 - \mu a_2^2| \leq \frac{1}{2(3-\delta)} \ if \ \frac{9\delta^4 - 14\delta^3 - 21\delta^2 + 72\delta - 36}{8\delta^2(3-\delta)} \leq \mu \ \leq \frac{9\delta^4 - 14\delta^3 - 3\delta^2 + 36}{8\delta^2(3-\delta)} \qquad (3.50)$$

Subcase II (b): $\mu \ \geq \frac{9\delta^4 - 14\delta^3 - 3\delta^2 + 36}{8\delta^2(3-\delta)}$

Proceeding as in subcase I (a), we get

$$|a_3 - \mu a_2^2| \leq \frac{4\delta^2}{9(2-\delta)^2}\mu - \frac{\delta(9\delta^3 - 14\delta^2 - 12\delta + 36)}{18(3-\delta)(2-\delta)^2}. \qquad (3.51)$$

Combining (3.46), (3.50) and (3.51), the theorem is proved.

Extremal function for (3.35) and (3.37) is defined by

$$f_1(z) = \begin{cases} \sqrt{2\int_0^z e^{2\left\{\frac{1}{\delta-1}\left(\frac{1+z}{1-z}\right)^{\delta-1} + \frac{1}{\delta-3}\left(\frac{1+z}{1-z}\right)^{\delta-3} + \frac{1}{\delta-5}\left(\frac{1+z}{1-z}\right)^{\delta-5} + - - - + \frac{1}{2} log z\right\}} dz}, if \ numerator \ of \ \delta \ is \ even \\ \sqrt{2\int_0^z e^{2\left\{\frac{1}{\delta-1}\left(\frac{1+z}{1-z}\right)^{\delta-1} + \frac{1}{\delta-3}\left(\frac{1+z}{1-z}\right)^{\delta-3} + \frac{1}{\delta-5}\left(\frac{1+z}{1-z}\right)^{\delta-5} + - - - + \frac{1}{2} log \frac{4z}{(1-z)^2}\right\}} dz}, \ if \ numerator \ of \ \delta \ is \ odd \end{cases}.$$

Extremal function for (3.36) is defined by $f_2(z) = \sqrt{\log\left(\frac{1}{1-z^2}\right)}$.

Corollary 3.5: *Putting* $\delta = 1$, *in the theorem* ***3.4***, *we get*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{19}{36} - \frac{4}{9}\mu, if \mu \leq \frac{5}{8}; \\ \frac{1}{4}, if \ \frac{5}{8} \leq \mu \leq \frac{7}{4} \\ \frac{4}{9}\mu - \frac{19}{36}, \ \ if \ \mu \geq \frac{7}{4}. \end{cases}$$

which is the result obtained in theorem (3.1).

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