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Maximal Subgroups of the Semigroup B_X (D) Defined by Semilattices of the Class $\Sigma_3^{(X, 8)}$

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Abstract

By the symbol $\Sigma_3^{(X, 8)}$ we denote the class of all X- semilattices of unions whose every element is isomorphic to an X- semilattice of form D = {Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, D}, where

$Z_7 \subset Z_5 \subset Z_3 \subset Z_1 \subset \breve{D}$, $Z_7 \subset Z_6 \subset Z_4 \subset Z_2 \subset \breve{D}$, $Z_7 \subset Z_5 \subset Z_4 \subset Z_1 \subset \breve{D}$, $Z_7 \subset Z_5 \subset Z_4 \subset Z_2 \subset \breve{D}$,
$Z_7 \subset Z_6 \subset Z_4 \subset Z_1 \subset \breve{D}$,
$Z_i \setminus Z_j \neq \varnothing$, $(i, j) \in \{(5,6),(6,5),(3,6),(6,3),(4,3),(3,4),(2,3),(3,2),(2,1),(1,2)\}$.

In the given paper we give a full description maximal subgroups of the complete semigroups of binary elations defined by semilattices of the class $\Sigma_3^{(X, 8)}$.

Keywords: *Semilattice, Semigroup, Binary Relation, Idempotent Element.*

1 Introduction

Let X be an arbitrary nonempty set, D be a X-semilattice of unions, i.e. a nonempty set of subsets of the set X that is closed with respect to the set-theoretic operations of unification of elements from D, f be an arbitrary mapping from X into D. To each such a mapping f there corresponds a binary relation α_f on the set X that satisfies the condition $\alpha_f = \bigcup_{x\in X}(\{x\}\times f(x))$. The set of all such α_f ($f: X\to D$) is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by a X-semilattice of unions D (see [1, Item 2.1, p. 34]).

By $\varnothing$ we denote an empty binary relation or empty subset of the set X. The condition $(x,y)\in\alpha$ will be written in the form $x\alpha y$. Further let $x,y\in X$, $Y\subseteq X$, $\alpha\in B_X(D)$, $T\in D$, $\varnothing\neq D'\subseteq D$ and $t\in \breve{D}=\bigcup_{Y\in D} Y$. Then by symbols we denote the following sets:

$$
\begin{aligned}
&y\alpha=\{x\in X \mid y\alpha x\},\ Y\alpha=\bigcup_{y\in Y} y\alpha,\ V(D,\alpha)=\{Y\alpha \mid Y\in D\},\\
&X^{*}=\{T \mid \varnothing\neq T\subseteq X\},\ D'_t=\{Z'\in D' \mid t\in Z'\},\ Y_T^{\alpha}=\{x\in X \mid x\alpha=T\},\\
&D'_T=\{Z'\in D' \mid T\subseteq Z'\},\ \ddot{D}'_T=\{Z'\in D' \mid Z'\subseteq T\},\ l(D',T)=\cup(D'\setminus D'_T).
\end{aligned}
\tag{1.1}
$$

Under symbol $\wedge(D, D_t)$ we mean an exact lower bound of the set D_t in the semilattice D.

Definition 1.1: *Let $\varepsilon\in B_X(D)$. If $\varepsilon\circ\varepsilon=\varepsilon$ or $\alpha\circ\varepsilon=\alpha$ for any $\alpha\in B_X(D)$, then ε is called an idempotent element or called right unit of the semigroup $B_X(D)$ respectively (see [1], [2], [3]).*

Definition 1.2: *We say that a complete $X-$semilattice of unions D is an $XI-$semilattice of unions if it satisfies the following two conditions:*

a) $\wedge(D,D_t)\in D$ for any $t\in\breve{D}$;

b) $Z=\bigcup_{t\in Z}\wedge(D,D_t)$ for any nonempty element Z of D (see [1, Definition 1.14.2], [2, Definition 1.14.2] or [6]).

Definition 1.3: *The one-to-one mapping φ between the complete $X-$semilattices of unions D' and D'' is called a complete isomorphism if the condition $\varphi(\cup D_1)=\bigcup_{T'\in D_1}\varphi(T')$ is fulfilled for each nonempty subset D_1 of the semilattice D' (see [1, Definition 6.2.3], [2, Definition 6.2.3] or [5]).*

Definition 1.4: *We say that a nonempty element T is a nonlimiting element of the set D' if $T \setminus l(D',T) \neq \varnothing$ and a nonempty element T is a limiting element of the set D' if $T \setminus l(D',T) = \varnothing$ (see [1, Definition 1.13.1 and 1.13.2] or [2, Definition 1.13.1 and 1.13.2]).*

Theorem 1.1: *Let X be a finite set and $D(\alpha)$ be the set of all those elements T of the semilattice $Q = V(D,\alpha) \setminus \{\varnothing\}$ which are nonlimiting elements of the set $\ddot{Q}_T$. A binary relation α having a quasinormal representation $\alpha = \bigcup_{T \in V(D,\alpha)} \left(Y_T^\alpha \times T\right)$ is an idempotent element of this semigroup iff*

a) $V(D,\alpha)$ is complete XI – semilattise of unions;

b) $\bigcup_{T' \in \ddot{D}(\alpha)_T} Y_{T'}^\alpha \supseteq T$ for any $T \in D(\alpha)$;

c) $Y_T^\alpha \cap T \neq \varnothing$ for any nonlimiting element of the set $\ddot{D}(\alpha)_T$ (see [1, Theorem 6.3.9], [2, Theorem 6.3.9] or [5]).

Theorem 1.2: *Let $D = \{\breve{D}, Z_1, Z_2, ..., Z_{m-1}\}$ be some finite X-semilattice of unions and $C(D) = \{P_0, P_1, P_2, ..., P_{m-1}\}$ be the family of sets of pairwise nonintersecting subsets of the set X. If φ is a mapping of the semilattice D on the family of sets C(D) which satisfies the condition $\varphi(\breve{D}) = P_0$ and $\varphi(Z_i) = P_i$ for any $i = 1, 2, ..., m-1$ and $\hat{D}_Z = D \setminus \{T \in D \mid Z \subseteq T\}$, then the following equalities are valid:*

$$\breve{D} = P_0 \cup P_1 \cup P_2 \cup ... \cup P_{m-1}, \; Z_i = P_0 \cup \bigcup_{T \in \hat{D}_{Z_i}} \varphi(T). \tag{1.2}$$

In the sequel these equalities will be called formal.

It is proved that if the elements of the semilattice D are represented in the form (1.2), then among the parameters P_i $(i = 0, 1, 2, ..., m-1)$ there exist such parameters that cannot be empty sets. Such sets P_i $(0 < i \leq m-1)$ are called basis sources, whereas sets P_j $(0 \leq j \leq m-1)$ which can be empty sets too are called completeness sources.

The number the basis sources we denote by symbol δ.

It is proved that under the mapping φ the number of covering elements of the pre-image of a basis source is always equal to one, while under the mapping φ the number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one (see [1, Item 11.4], [2, Item 11.4] or [3]).

Denote by the symbol $G_X(D,\varepsilon)$ a maximal subgroup of the semigroup B_X (D) whose unit is an idempotent binary relation ε of the semigroup B_X (D).

Theorem 1.3: *For any idempotent element* $\varepsilon \in B_X(D)$, *the group* $G_X(D,\varepsilon)$ *is antiisomorphic to the group of all complete automorphism of the semilattice* $V(D,\varepsilon)$ *(see [1, Theorem 7.4.2], [2, Theorem 7.4.2] or [4]).*

2 Results

Let X and $\Sigma_3(X,8)$ be respectively an any nonempty set and a class intreisomorphic X-semilattices of unions where every element is isomorphic to some X-semilattice of unions $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$, that satisfying the conditions.

$$\begin{array}{l}
Z_7 \subset Z_6 \subset Z_4 \subset Z_2 \subset \breve{D},\ \ Z_7 \subset Z_6 \subset Z_4 \subset Z_1 \subset \breve{D},\\
Z_7 \subset Z_5 \subset Z_4 \subset Z_2 \subset \breve{D},\ \ Z_7 \subset Z_5 \subset Z_4 \subset Z_1 \subset \breve{D},\\
Z_7 \subset Z_5 \subset Z_3 \subset Z_1 \subset \breve{D};\\
Z_1 \setminus Z_2 \neq \varnothing,\ \ Z_2 \setminus Z_1 \neq \varnothing,\ \ Z_3 \setminus Z_2 \neq \varnothing,\ \ Z_2 \setminus Z_3 \neq \varnothing,\\
Z_3 \setminus Z_4 \neq \varnothing,\ \ Z_4 \setminus Z_3 \neq \varnothing,\ \ Z_3 \setminus Z_6 \neq \varnothing,\ \ Z_6 \setminus Z_3 \neq \varnothing,\\
Z_5 \setminus Z_6 \neq \varnothing,\ \ Z_6 \setminus Z_5 \neq \varnothing;
\end{array} \tag{2.1}$$

The semilattice satisfying the conditions (2.1) is shown in Figure 1.

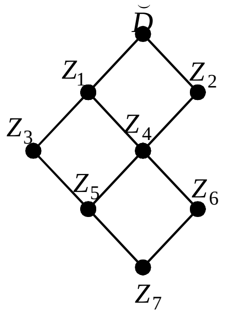


Fig. 1

Lemma 2.1: *Let* $D \in \Sigma_3(X,8)$. *Then the following sets exhaust all subsemilattices of the semilattice* $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$.

1)$\{Z_7\},\{Z_6\},\{Z_5\},\{Z_4\},\{Z_3\},\{Z_2\},\{Z_1\},\{\breve{D}\}$. (see diagram 1 of the figure 2);

$$\begin{array}{l}
2)\{Z_7,Z_1\},\{Z_7,Z_2\},\{Z_7,Z_3\},\{Z_7,Z_4\},\{Z_7,Z_5\},\{Z_7,Z_6\},\{Z_7,\breve{D}\},\\
\ \ \{Z_6,Z_4\},\{Z_6,Z_2\},\{Z_6,Z_1\},\{Z_6,\breve{D}\},\{Z_5,Z_4\},\{Z_5,Z_3\},\{Z_5,Z_2\},\\
\ \ \{Z_5,Z_1\},\{Z_5,\breve{D}\},\{Z_4,Z_2\},\{Z_4,Z_1\},\{Z_4,\breve{D}\},\{Z_3,Z_1\},\{Z_3,\breve{D}\},\\
\ \ \{Z_2,\breve{D}\},\{Z_1,\breve{D}\}.
\end{array}$$

(see diagram 2 of the figure 2);

3) $\{Z_7,Z_6,Z_4\},\{Z_7,Z_5,Z_2\},\{Z_7,Z_6,Z_1\},\{Z_7,Z_6,\breve{D}\},\{Z_7,Z_5,Z_4\},$
$\{Z_7,Z_5,Z_3\},\{Z_7,Z_5,Z_2\},\{Z_7,Z_5,Z_1\},\{Z_7,Z_5,\breve{D}\},\{Z_7,Z_4,Z_2\},$
$\{Z_7,Z_4,Z_1\},\{Z_7,Z_4,\breve{D}\},\{Z_7,Z_3,Z_1\},\{Z_7,Z_3,\breve{D}\},\{Z_7,Z_2,\breve{D}\},$
$\{Z_7,Z_1,\breve{D}\},\{Z_6,Z_4,Z_2\},\{Z_6,Z_4,Z_1\},\{Z_6,Z_4,\breve{D}\},\{Z_6,Z_2,\breve{D}\},$
$\{Z_6,Z_1,\breve{D}\},\{Z_5,Z_4,Z_2\},\{Z_5,Z_4,Z_1\},\{Z_5,Z_4,\breve{D}\},\{Z_5,Z_3,Z_1\},$
$\{Z_5,Z_3,\breve{D}\},\{Z_5,Z_2,\breve{D}\},\{Z_5,Z_1,\breve{D}\},\{Z_4,Z_2,\breve{D}\},\{Z_4,Z_1,\breve{D}\},$
$\{Z_3,Z_1,\breve{D}\}.$
(see diagram 3 of the figure 2);

4) $\{Z_7,Z_6,Z_4,Z_2\},\{Z_7,Z_6,Z_4,Z_1\},\{Z_7,Z_6,Z_4,\breve{D}\},\{Z_7,Z_6,Z_2,\breve{D}\},$
$\{Z_7,Z_6,Z_1,\breve{D}\}\{Z_7,Z_5,Z_4,Z_2\},\{Z_7,Z_5,Z_4,Z_1\},\{Z_7,Z_5,Z_4,\breve{D}\},$
$\{Z_7,Z_5,Z_3,Z_1\},\{Z_7,Z_5,Z_3,\breve{D}\},\{Z_7,Z_5,Z_2,\breve{D}\},\{Z_7,Z_5,Z_1,\breve{D}\},$
$\{Z_7,Z_4,Z_2,\breve{D}\},\{Z_7,Z_4,Z_1,\breve{D}\},\{Z_7,Z_3,Z_1,\breve{D}\},\{Z_5,Z_4,Z_2,\breve{D}\},$
$\{Z_5,Z_3,Z_1,\breve{D}\},\{Z_5,Z_4,Z_1,\breve{D}\},\{Z_6,Z_4,Z_2,\breve{D}\},\{Z_6,Z_4,Z_1,\breve{D}\}.$
(see diagram 4 of the figure 2);

5) $\{Z_7,Z_6,Z_4,Z_2,\breve{D}\},\{Z_7,Z_6,Z_4,Z_1,\breve{D}\},\{Z_7,Z_5,Z_4,Z_1,\breve{D}\},\{Z_7,Z_5,Z_4,Z_2,\breve{D}\},$
$\{Z_7,Z_5,Z_3,Z_1,\breve{D}\}.$
(see diagram 5 of the figure 2);

6) $\{Z_7,Z_4,Z_3,Z_1\},\{Z_7,Z_2,Z_1,\breve{D}\},\{Z_7,Z_6,Z_5,Z_4\},\{Z_4,Z_2,Z_1,\breve{D}\},$
$\{Z_6,Z_2,Z_1,\breve{D}\},\{Z_7,Z_6,Z_3,Z_1\},\{Z_7,Z_3,Z_2,\breve{D}\},\{Z_5,Z_3,Z_2,\breve{D}\}.$
(see diagram 6 of the figure 2);

7) $\{Z_7,Z_6,Z_2,Z_1,\breve{D}\},\{Z_7,Z_5,Z_4,Z_3,Z_1\},\{Z_7,Z_5,Z_3,Z_2,\breve{D}\},$
$\{Z_7,Z_5,Z_2,Z_1,\breve{D}\},\{Z_7,Z_4,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_2,Z_1,\breve{D}\},$
$\{Z_5,Z_4,Z_2,Z_1,\breve{D}\}.$
(see diagram 7 of the figure 2);

8) $\{Z_7,Z_6,Z_4,Z_2,Z_1,\breve{D}\};\{Z_7,Z_5,Z_4,Z_2,Z_1,\breve{D}\}.$ (see diagram 8 of the figure 2);

9) $\{Z_7,Z_5,Z_4,Z_3,Z_1,\breve{D}\}.$(see diagram 9 of the figure 2);

10) $\{Z_7,Z_6,Z_5,Z_4,Z_2\},\{Z_7,Z_6,Z_5,Z_4,Z_1\},\{Z_7,Z_6,Z_5,Z_4,\breve{D}\},$
$\{Z_7,Z_6,Z_3,Z_1,\breve{D}\},\{Z_5,Z_4,Z_3,Z_1,\breve{D}\},\{Z_7,Z_4,Z_3,Z_1,\breve{D}\},$
(see diagram 10 of the figure 2);

11) $\{Z_7,Z_6,Z_5,Z_4,Z_2,\breve{D}\},\{Z_7,Z_6,Z_5,Z_4,Z_1,\breve{D}\}.$ (see diagram 11 of the figure 2);

12) $\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_1\},\{Z_7,Z_6,Z_3,Z_2,Z_1,\breve{D}\},\{Z_7,Z_4,Z_3,Z_2,Z_1,\breve{D}\},\{Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$.
(see diagram 12 of the figure 2);

13) $\{Z_7,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$. (see diagram 13 of the figure 2);

14) $\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_1,\breve{D}\}$, (see diagram 14 of the figure 2);

15) $\{Z_7,Z_6,Z_5,Z_4,Z_2,Z_1,\breve{D}\}$, (see diagram 15 of the figure 2);

16) $\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$. (see diagram 16 of the figure 2);

17) $\{Z_3,Z_2,\breve{D}\},\{Z_2,Z_1,\breve{D}\}.\{Z_6,Z_5,Z_4\},\{Z_6,Z_3,Z_1\},\{Z_4,Z_3,Z_1\}$.
(see diagram 17 of the figure 2);

18) $\{Z_6,Z_5,Z_4,\breve{D}\},\{Z_6,Z_5,Z_4,Z_2\},\{Z_6,Z_5,Z_4,Z_1\},\{Z_4,Z_3,Z_1,\breve{D}\}$,
$\{Z_6,Z_3,Z_1,\breve{D}\}$.
(see diagram 18 of the figure 2);

19) $\{Z_3,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_3,Z_1\}$. (see diagram 19 of the figure 2);

20) $\{Z_6,Z_5,Z_4,Z_2,\breve{D}\},\{Z_6,Z_5,Z_4,Z_1,\breve{D}\}$, (see diagram 20 of the figure 2);

21) $\{Z_6,Z_4,Z_3,Z_1,\breve{D}\}$, (see diagram 21 of the figure 2);

22) $\{Z_7,Z_6,Z_4,Z_3,Z_1\},\{Z_5,Z_3,Z_2,Z_1,\breve{D}\},\{Z_7,Z_3,Z_2,Z_1,\breve{D}\}$,
(see diagram 22 of the figure 2);

23) $\{Z_6,Z_5,Z_4,Z_3,Z_1\},\{Z_6,Z_3,Z_2,Z_1,\breve{D}\},\{Z_4,Z_3,Z_2,Z_1,\breve{D}\}$,
(see diagram 23 of the figure 2);

24) $\{Z_6,Z_5,Z_4,Z_3,Z_1,\breve{D}\}$, (see diagram 24 of the figure 2);

25) $\{Z_6,Z_5,Z_4,Z_2,Z_1,\breve{D}\}$, (see diagram 25 of the figure 2);

26) $\{Z_6,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$, (see diagram 26 of the figure 2);

27) $\{Z_7,Z_5,Z_3,Z_2,Z_1,\breve{D}\}$; (see diagram 27 of the figure 2);

28) $\{Z_7,Z_6,Z_4,Z_3,Z_1,\breve{D}\}$, (see diagram 28 of the figure 2);

29) $\{Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, (see diagram 29 of the figure 2);

30) $\{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, (see diagram 30 of the figure 2);

Proof: It is ease to see that, the sets $\{Z_7\},\{Z_6\},\{Z_5\},\{Z_4\},\{Z_3\},\{Z_2\},\{Z_1\},\{\breve{D}\}$ are subsemilattices of the semilattice D.

The number all subsets of the semilattise *D*, every set of which contains two elements, is equal to $C_8^2 = 28$. In this case *X*- subsemilattices of the semilattice *D* are the following sets:

$\{Z_7,Z_1\},\{Z_7,Z_2\},\{Z_7,Z_3\},\{Z_7,Z_4\},\{Z_7,Z_5\},\{Z_7,Z_6\},\{Z_7,\breve{D}\},\{Z_5,\breve{D}\},\{Z_6,Z_4\},\{Z_6,Z_2\},\{Z_6,Z_1\},$
$\{Z_6,\breve{D}\},\{Z_3,\breve{D}\},\{Z_5,Z_4\},\{Z_5,Z_3\},\{Z_5,Z_2\},\{Z_5,Z_1\},\{Z_4,Z_2\},\{Z_4,Z_1\},\{Z_4,\breve{D}\},\{Z_3,Z_1\},\{Z_2,\breve{D}\},\{Z_1,\breve{D}\}$

Remainder 5 subsets of the semilattice *D*, whose every element contains two elements is not an *X*-subsemilattice.

The number all subsets of the semilattise *D*, every set of which contains three elements, is equal to $C_8^3 = 56$. In this case *X*-subsemilattices of the semilattice *D* are the following sets:

$\{Z_7,Z_6,Z_4\},\{Z_7,Z_5,Z_2\},\{Z_7,Z_6,Z_1\},\{Z_7,Z_6,\breve{D}\},\{Z_7,Z_5,Z_4\},\{Z_7,Z_5,Z_3\},\{Z_7,Z_5,Z_2\},\{Z_7,Z_5,Z_1\},$
$\{Z_7,Z_5,\breve{D}\},\{Z_7,Z_4,Z_2\},\{Z_7,Z_4,Z_1\},\{Z_7,Z_4,\breve{D}\},\{Z_7,Z_3,Z_1\},\{Z_7,Z_3,\breve{D}\},\{Z_7,Z_2,\breve{D}\},\{Z_7,Z_1,\breve{D}\},$
$\{Z_6,Z_5,Z_4\},\{Z_6,Z_4,Z_2\},\{Z_6,Z_4,Z_1\},\{Z_6,Z_4,\breve{D}\},\{Z_6,Z_3,Z_1\},\{Z_6,Z_2,\breve{D}\},\{Z_6,Z_1,\breve{D}\},\{Z_5,Z_4,Z_2\},$
$\{Z_5,Z_4,Z_1\},\{Z_5,Z_4,\breve{D}\},\{Z_5,Z_3,Z_1\},\{Z_5,Z_3,\breve{D}\},\{Z_5,Z_2,\breve{D}\},\{Z_5,Z_1,\breve{D}\},\{Z_4,Z_3,Z_1\},\{Z_4,Z_2,\breve{D}\},$
$\{Z_4,Z_1,\breve{D}\},\{Z_3,Z_2,\breve{D}\},\{Z_3,Z_1,\breve{D}\},\{Z_2,Z_1,\breve{D}\}$

Remainder 20 subsets of the semilattice *D*, whose every element contains three elements is not an *X*- subsemilattice.

The number all subsets of the semilattise *D*, every set of which contains four elements, is equal to $C_8^4 = 70$. In this case *X*- subsemilattices of the semilattice *D* are the following sets:

$\{Z_7,Z_6,Z_5,Z_4\},\{Z_7,Z_6,Z_4,Z_2\},\{Z_7,Z_6,Z_4,Z_1\},\{Z_7,Z_5,Z_4,Z_2\},\{Z_7,Z_6,Z_4,\breve{D}\},\{Z_7,Z_6,Z_3,\breve{D}\},\{Z_7,Z_6,Z_3,Z_1\},$
$\{Z_7,Z_5,Z_3,Z_1\},\{Z_7,Z_6,Z_2,\breve{D}\},\{Z_7,Z_6,Z_1,\breve{D}\},\{Z_7,Z_5,Z_4,Z_1\},\{Z_7,Z_5,Z_4,\breve{D}\},\{Z_7,Z_5,Z_3,\breve{D}\},\{Z_7,Z_5,Z_2,\breve{D}\},$
$\{Z_7,Z_5,Z_1,\breve{D}\},\{Z_7,Z_4,Z_3,Z_1\},\{Z_7,Z_4,Z_2,\breve{D}\},\{Z_7,Z_4,Z_1,\breve{D}\},\{Z_7,Z_3,Z_2,\breve{D}\},\{Z_7,Z_3,Z_1,\breve{D}\},\{Z_7,Z_2,Z_1,\breve{D}\},$
$\{Z_6,Z_2,Z_1,\breve{D}\},\{Z_5,Z_4,Z_3,Z_1\},\{Z_5,Z_4,Z_2,\breve{D}\},\{Z_6,Z_5,Z_4,Z_2\},\{Z_6,Z_5,Z_4,\breve{D}\},\{Z_6,Z_5,Z_4,Z_1\},\{Z_6,Z_4,Z_3,Z_1\},$
$\{Z_6,Z_5,Z_3,Z_1\},\{Z_6,Z_4,Z_2,\breve{D}\},\{Z_6,Z_4,Z_1,\breve{D}\},\{Z_6,Z_3,Z_1,\breve{D}\},\{Z_4,Z_3,Z_1,\breve{D}\},\{Z_4,Z_2,Z_1,\breve{D}\},\{Z_5,Z_3,Z_2,\breve{D}\},$
$\{Z_5,Z_4,Z_1,\breve{D}\},\{Z_5,Z_3,Z_1,\breve{D}\},$

Remainder 33 subsets of the semilattice *D*, whose every element contains four elements is not an *X*- subsemilattice.

The number all subsets of the semilattise *D*, every set of which contains five elements, is equal to $C_8^5 = 56$. In this case *X*- subsemilattices of the semilattice *D* are the following sets:

$$\{Z_7, Z_6, Z_5, Z_4, Z_2\}, \{Z_7, Z_6, Z_5, Z_4, Z_1\}, \{Z_7, Z_6, Z_5, Z_4, \breve{D}\}, \{Z_7, Z_6, Z_4, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_4, Z_2, \breve{D}\}, \{Z_7, Z_6, Z_3, Z_1, \breve{D}\},$$
$$\{Z_7, Z_6, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_5, Z_4, Z_3, Z_1\}, \{Z_7, Z_5, Z_4, Z_2, \breve{D}\}, \{Z_7, Z_5, Z_4, Z_1, \breve{D}\}, \{Z_7, Z_5, Z_3, Z_2, \breve{D}\}, \{Z_7, Z_5, Z_3, Z_1, \breve{D}\},$$
$$\{Z_7, Z_5, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_4, Z_3, Z_1, \breve{D}\}, \{Z_7, Z_4, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_6, Z_5, Z_4, Z_3, Z_1\}, \{Z_6, Z_5, Z_4, Z_2, \breve{D}\},$$
$$\{Z_6, Z_5, Z_4, Z_1, \breve{D}\}, \{Z_6, Z_4, Z_3, Z_1, \breve{D}\}, \{Z_6, Z_4, Z_2, Z_1, \breve{D}\}, \{Z_6, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_5, Z_4, Z_3, Z_1, \breve{D}\}, \{Z_5, Z_4, Z_2, Z_1, \breve{D}\},$$
$$\{Z_5, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_4, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_4, Z_3, Z_1\},$$

Remainder 29 subsets of the semilattice *D*, whose every element contains five elements is not an *X*- subsemilattice.

The number all subsets of the semilattise *D*, every set of which contains six elements, is equal to $C_8^6 = 28$. In this case *X*- subsemilattices of the semilattice *D* are the following sets:

$$\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_1\}, \{Z_7, Z_6, Z_5, Z_4, Z_2, \breve{D}\}, \{Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \{Z_7, Z_6, Z_5, Z_4, Z_1, \breve{D}\}, \{Z_7, Z_5, Z_4, Z_2, Z_1, \breve{D}\},$$
$$\{Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_4, Z_3, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_4, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_5, Z_4, Z_3, Z_1, \breve{D}\},$$
$$\{Z_7, Z_5, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_4, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}, \{Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_5, Z_4, Z_3, \breve{D}\},$$

Remainder 13 subsets of the semilattice D, whose every element contains six elements is not an X - subsemilattice.

The number all subsets of the semilattise D, every set of which contains seven elements, is equal to $C_8^7 = 8$. In this case X - subsemilattices of the semilattice D are the following sets:

$$\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\},$$
$$\{Z_7, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}, \{Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\},$$

Remainder 3 subsets of the semilattice D, whose every element contains seven elements is not an X - subsemilattice.

The number all subsets of the semilattise D, every set of which contains eight elements, is equal to $C_8^8 = 1$. This set is $\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$.

The Lemma is proved.

From the proven lemma it follows that diagrams shown in fig. 2, exhaust all diagrams of subsemilattices of the semilattice D .

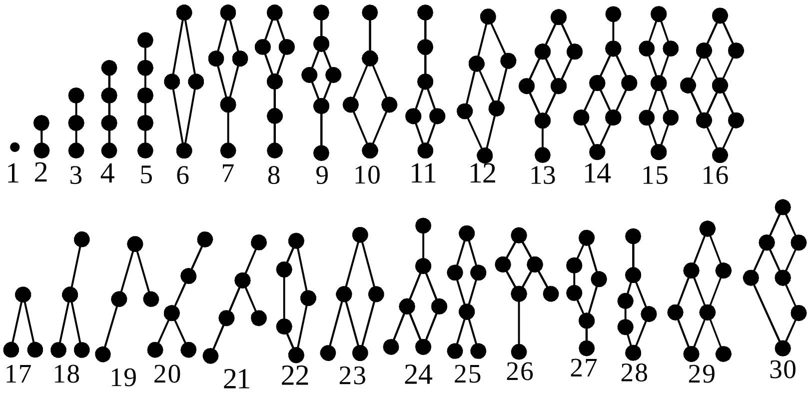


Figure 2

Lemma 2.2: *Let* $D \in \Sigma_3(X,8)$ *and* $Z_7 \neq \varnothing$. *Then any subsemilattices of the semilattice* D *having diagram* $17-30$ *of the figure 2 are never* $XI-$*semilattice.*

Proof: Remark, that the all subsemilattices of semilattice D which has diagrams of form $17-30$ are never $XI-$semilattices. For example we consider the semilattice which has the diagram of the form 30 of the figure 3 (see diagram figure 30).

Let $D' = \{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$ and $C(D') = \{P_0, P_1, P_2, P_3, P_4, P_5, P_6\}$ is a family sets,

where

$P_0, P_1, P_2, P_3, P_4, P_5, P_6$ are pairwise disjoint subsets of the set X and $\varphi = \begin{pmatrix} \breve{D} & Z_1 & Z_2 & Z_3 & Z_4 & Z_6 & Z_7 \\ P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{pmatrix}$ is a mapping of the semilattice D' onto the family sets $C(D')$. Then for the formal equalities of the semilattice D' we have a form:

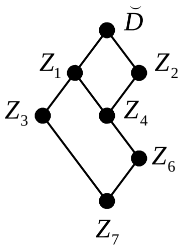


Figure 3

$$
\begin{aligned}
\breve{D} &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_6 \cup P_7,\\
Z_1 &= P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_6 \cup P_7,\\
Z_2 &= P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_6 \cup P_7,\\
Z_3 &= P_0 \cup P_2 \cup P_4 \cup P_6 \cup P_7,\\
Z_4 &= P_0 \cup P_3 \cup P_6 \cup P_7,\\
Z_6 &= P_0 \cup P_3 \cup P_7\\
Z_7 &= P_0
\end{aligned}
$$

Here the elements P_1, P_2, P_3, P_6 are basis sources, the element P_0, P_4, P_7 is sources of completenes of the semilattice D'. Therefore $|X| \geq 3$ and, that

$$D'_t=\begin{cases} D', \ if \ \ t\in P_0, \\ \{Z_2,\breve{D}\}, \ if \ \ t\in P_1 \\ \{Z_3,Z_1,\breve{D}\}, \ if \ \ t\in P_2, \\ \{Z_6,Z_4,Z_2,Z_1,\breve{D}\}, \ if \ \ t\in P_3, \\ \{Z_3,Z_2,Z_1,\breve{D}\}, \ if \ \ t\in P_4, \\ \{Z_4,Z_3,Z_2,Z_1,\breve{D}\}, \ if \ \ t\in P_5, \\ \{Z_6,Z_4,Z_3,Z_2,Z_1,\breve{D}\}, \ if \ \ t\in P_6, \end{cases} \qquad \Lambda(D',D'_t)=\begin{cases} Z_7, \ if \ \ t\in P_0 \\ Z_2, \ if \ \ t\in P_1 \\ Z_3, \ if \ \ t\in P_2 \\ Z_6, \ if \ \ t\in P_3 \\ Z_7, \ if \ \ t\in P_4 \\ Z_7, \ if \ \ t\in P_5 \\ Z_7, \ if \ \ t\in P_6 \end{cases}$$

We have $D'^{\wedge}=\{Z_7,Z_6,Z_3,Z_2\}$ and $\Lambda(D',D'_t)\in D'$ for all $t\in\breve{D}$. But element Z_4 is not union of some elements of the set $D'^{\wedge}$. So, from the Definition 1.2 follows that semilattice D' which has diagram 41 of the figure 3 never is XI – semilattice.

In the same manner it can be proved that any subsemilattice of the semilattice D having diagrams 17-30 are never an XI – semilattice.

Lemma is proved.

Lemma 2.3: Let $D\in\Sigma_3(X,8)$ and $Z_7\neq\varnothing$. Then the following sets are all *XI*-subsemilattices of the given semilattice *D*:

1) $\{Z_7\},\{Z_6\},\{Z_5\},\{Z_4\},\{Z_3\},\{Z_2\},\{Z_1\},\{\breve{D}\}$; (see diagram 1 of the figure 2);

2) $\{Z_7,Z_6\},\{Z_7,Z_5\},\{Z_7,Z_4\},\{Z_7,Z_3\},\{Z_7,Z_2\},\{Z_7,Z_1\},\{Z_7,\breve{D}\},\{Z_6,Z_4\},\{Z_6,Z_2\},$
$\{Z_6,Z_1\},\{Z_6,\breve{D}\},\{Z_5,Z_4\},\{Z_5,Z_3\},\{Z_5,Z_2\},\{Z_5,Z_1\},\{Z_5,\breve{D}\},\{Z_4,Z_2\},\{Z_4,Z_1\},$
$\{Z_4,\breve{D}\},\{Z_3,Z_1\},\{Z_3,\breve{D}\},\{Z_2,\breve{D}\},\{Z_1,\breve{D}\}$;
(see diagram 2 of the figure 2);

3) $\{Z_7,Z_6,Z_4\},\{Z_7,Z_6,Z_2\},\{Z_7,Z_6,Z_1\},\{Z_7,Z_6,\breve{D}\},\{Z_7,Z_5,Z_4\},\{Z_7,Z_5,Z_3\},\{Z_7,Z_5,Z_2\},\{Z_7,Z_5,Z_1\},$
$\{Z_7,Z_5,\breve{D}\},\{Z_7,Z_4,Z_2\},\{Z_7,Z_4,Z_1\},\{Z_7,Z_4,D\},\{Z_7,Z_3,Z_1\},\{Z_7,Z_3,\breve{D}\},\{Z_7,Z_2,\breve{D}\},\{Z_7,Z_1,\breve{D}\},$
$\{Z_6,Z_4,Z_2\},\{Z_6,Z_4,Z_1\},\{Z_6,Z_4,\breve{D}\},\{Z_6,Z_2,\breve{D}\},\{Z_6,Z_1,\breve{D}\},\{Z_5,Z_4,Z_2\},\{Z_5,Z_4,Z_1\},\{Z_5,Z_4,\breve{D}\},$
$\{Z_5,Z_3,Z_1\},\{Z_5,Z_3,\breve{D}\},\{Z_5,Z_2,\breve{D}\},\{Z_5,Z_1,\breve{D}\},\{Z_4,Z_2,\breve{D}\},\{Z_4,Z_1,\breve{D}\},\{Z_3,Z_1,\breve{D}\}$;
(see diagram 3 of the figure 2);

4) $\{Z_7,Z_6,Z_4,Z_2\},\{Z_7,Z_6,Z_4,Z_1\},\{Z_7,Z_6,Z_4,\breve{D}\},\{Z_7,Z_6,Z_2,D\},\{Z_7,Z_6,Z_1,\breve{D}\},$
$\{Z_7,Z_5,Z_4,Z_2\},\{Z_7,Z_5,Z_4,Z_1\},\{Z_7,Z_5,Z_4,\breve{D}\}$ $\{Z_7,Z_5,Z_3,Z_1\},\{Z_7,Z_5,Z_3,\breve{D}\},$
$\{Z_7,Z_5,Z_2,\breve{D}\},\{Z_7,Z_5,Z_1,\breve{D}\},\{Z_7,Z_4,Z_2,D\},\{Z_7,Z_4,Z_1,\breve{D}\},\{Z_7,Z_3,Z_1,\breve{D}\}$
$\{Z_6,Z_4,Z_2,\breve{D}\},\{Z_6,Z_4,Z_1,\breve{D}\},\{Z_5,Z_4,Z_2,\breve{D}\},\{Z_5,Z_4,Z_1,\breve{D}\},\{Z_5,Z_3,Z_1,\breve{D}\}$;
(see diagram 4 of the figure 2);

5) $\{Z_7,Z_6,Z_4,Z_2,\breve{D}\},\{Z_7,Z_6,Z_4,Z_1,\breve{D}\},\{Z_7,Z_5,Z_4,Z_2,\breve{D}\},\{Z_7,Z_5,Z_4,Z_1,\breve{D}\},\{Z_7,Z_5,Z_3,Z_1,\breve{D}\},$
(see diagram 5 of the figure 2);

6) $\{Z_7,Z_6,Z_5,Z_4\},\{Z_7,Z_6,Z_3,Z_1\},\{Z_7,Z_4,Z_3,Z_1\},\{Z_7,Z_3,Z_2,\breve{D}\},\{Z_7,Z_2,Z_1,\breve{D}\},$
$\{Z_6,Z_2,Z_1,\breve{D}\},\{Z_5,Z_4,Z_3,Z_1\},\{Z_5,Z_3,Z_2,\breve{D}\},\{Z_5,Z_2,Z_1,\breve{D}\},\{Z_4,Z_2,Z_1,\breve{D}\};$
(see diagram 6 of the figure 2);

7) $\{Z_7,Z_6,Z_2,Z_1,\breve{D}\},\{Z_7,Z_5,Z_4,Z_3,Z_1,\},\{Z_7,Z_5,Z_3,Z_2,\breve{D}\},\{Z_7,Z_5,Z_2,Z_1,\breve{D}\},$
$\{Z_7,Z_4,Z_2,Z_1,\breve{D}\},\{Z_6,Z_4,Z_2,Z_1,\breve{D}\}\{Z_5,Z_4,Z_2,Z_1,\breve{D}\};$
(see diagram 7 of the figure 2);

8) $\{Z_7,Z_6,Z_4,Z_2,Z_1,\breve{D}\},\{Z_7,Z_5,Z_4,Z_2,Z_1,\breve{D}\};$ (see diagram 8 of the figure 2);

9) $\{Z_7,Z_5,Z_4,Z_3,Z_1,\breve{D}\};$ (see diagram 9 of the figure 2);

10) $\{Z_7,Z_6,Z_5,Z_4,Z_2\},\{Z_7,Z_6,Z_5,Z_4,Z_1\},\{Z_7,Z_6,Z_5,Z_4,\breve{D}\},\{Z_7,Z_4,Z_3,Z_1,\breve{D}\},$
$\{Z_7,Z_6,Z_3,Z_1,\breve{D}\},\{Z_5,Z_4,Z_3,Z_1,\breve{D}\};$
(see diagram 10 of the figure 2);

11) $\{Z_7,Z_6,Z_5,Z_4,Z_2,\breve{D}\},\{Z_7,Z_6,Z_5,Z_4,Z_1,\breve{D}\};$ (see diagram 11 of the figure 2);

12) $\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_1\},\{Z_7,Z_6,Z_3,Z_2,Z_1,\breve{D}\},\{Z_7,Z_4,Z_3,Z_2,Z_1,\breve{D}\},\{Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\};$
(see diagram 12 of the figure 2);

13) $\{Z_7,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\};$ (see diagram 13 of the figure 2);

14) $\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_1,\breve{D}\};$ (see diagram 14 of the figure 2);

15) $\{Z_7,Z_6,Z_5,Z_4,Z_2,Z_1,\breve{D}\};$ (see diagram 15 of the figure 2);

16) $\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\};$ (see diagram 16 of the figure 2);

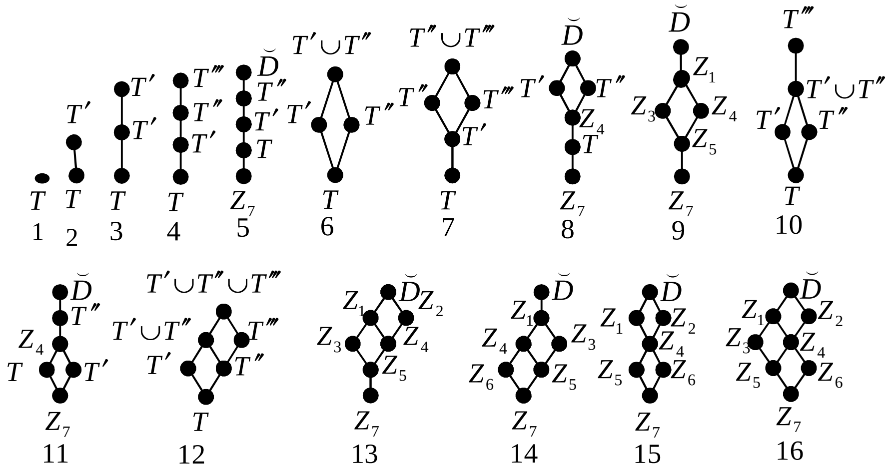
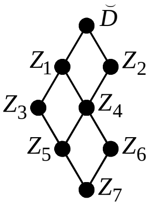


Figure 4

Proof: The statements $1)$, $2)$, $3)$, $4)$ 5) immediately follows from the Theorems [1] 11.6.1, [2] 11.6.1, the statements $6)$, $7)$,$8)$ $9)$ $10)$ $11)$ immediately follows from the Theorems [1] 11.6.3, [2] 11.6.3 the statement $12)$ immediately follows from the Theorems [1] 11.7.2 and the statements $15)$ immediately follows from the Theorems [1] 13.11.1.

Now we will proof the statement $16)$. Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$ where



$$
\begin{aligned}
&Z_7\subset Z_5\subset Z_3\subset Z_1\subset \breve{D},\ Z_7\subset Z_6\subset Z_4\subset Z_2\subset \breve{D},\\
&Z_7\subset Z_5\subset Z_4\subset Z_1\subset \breve{D},\ Z_7\subset Z_5\subset Z_4\subset Z_2\subset \breve{D},\\
&Z_7\subset Z_6\subset Z_4\subset Z_1\subset \breve{D},\ Z_5\setminus Z_6\neq\varnothing,\ Z_6\setminus Z_5\neq\varnothing,\\
&Z_4\setminus Z_3\neq\varnothing,\ Z_3\setminus Z_4\neq\varnothing, Z_2\setminus Z_1\neq\varnothing,\ Z_1\setminus Z_2\neq\varnothing,\\
&Z_6\cup Z_5=Z_4,\ Z_4\cup Z_3=Z_1, Z_2\cup Z_1=\breve{D}
\end{aligned}
$$

Let $C(D)=\{P_0,P_1,P_2,P_3,P_4,P_5,P_6,P_7\}$ is a family sets, where $P_0,P_1,P_2,P_3,P_4,P_5,P_6,P_7$ are pairwise disjoint subsets of the set X and $\varphi=\begin{pmatrix}\breve{D} & Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & Z_6 & Z_7\\ P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7\end{pmatrix}$ is a mapping of the semilattice D onto the family sets $C(D)$. Then for the formal equalities of the semilattice D we have a form:

$$
\begin{aligned}
\breve{D} &= P_0\cup P_1\cup P_2\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7,\\
Z_1 &= P_0\cup P_2\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7,\\
Z_2 &= P_0\cup P_1\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7,\\
Z_3 &= P_0\cup P_2\cup P_4\cup P_5\cup P_6\cup P_7,\\
Z_4 &= P_0\cup P_3\cup P_5\cup P_6\cup P_7,\\
Z_5 &= P_0\cup P_6\cup P_7,\\
Z_6 &= P_0\cup P_3\cup P_5\cup P_7,\\
Z_7 &= P_0.
\end{aligned}
\tag{2.5}
$$

Of the formal equalities (2.5) follows, that

$$
D_t'=\begin{cases}
\breve{D},\ if\ \ t\in P_0,\\
\{Z_2,\breve{D}\},\ if\ \ t\in P_1,\\
\{Z_3,Z_1,\breve{D}\},\ if\ \ t\in P_2,\\
\{Z_6,Z_4,Z_2,Z_1,\breve{D}\},\ if\ \ t\in P_3,\\
\{Z_3,Z_2,Z_1,\breve{D}\},\ if\ \ t\in P_4,\\
\{Z_6,Z_4,Z_3,Z_2,Z_1,\breve{D}\},\ if\ \ t\in P_5,\\
\{Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\},\ if\ \ t\in P_6,\\
\{Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\},\ if\ \ t\in P_7,
\end{cases}
\qquad
\wedge(D,D_t)=\begin{cases}
Z_7,\ if\ \ t\in P_0,\\
Z_2,\ if\ \ t\in P_1,\\
Z_3,\ if\ \ t\in P_2,\\
Z_6,\ if\ \ t\in P_3,\\
Z_5,\ if\ \ t\in P_4,\\
Z_7,\ if\ \ t\in P_5,\\
Z_5,\ if\ \ t\in P_6,\\
Z_7,\ if\ \ t\in P_7,
\end{cases}
$$

We have $D^{\wedge}=\{Z_7,Z_6,Z_5,Z_3,Z_2\}$, $\wedge(D,D_t)\in D$ for all t and $Z_4=Z_6\cup Z_5$, $Z_1=Z_6\cup Z_3$, $\breve{D}=Z_3\cup Z_2$. So, from the Definition 1.2 follows that semilattice D which has diagram of the figure 4 is $XI-$semilattice.

In the same manner it can be proved that any subsemilattice of the semilattice D having diagrams 13 and 14 are an XI – semilattice.

Lemma is proved

Corollary 2.1: *Let $D \in \Sigma_3(X,8)$ and $Z_7 = \varnothing$. Then the following sets are all XI – subsemilattices of the given semilattice D:*

1) $\{\varnothing\}$ (see diagram 1 of the figure 2);

2) $\{\varnothing, Z_6\}, \{\varnothing, Z_5\}, \{\varnothing, Z_4\}, \{\varnothing, Z_3\}, \{\varnothing, Z_2\}, \{\varnothing, Z_1\}, \{\varnothing, \breve{D}\}$
(see diagram 2 of the figure 2);

3) $\{\varnothing, Z_6, Z_4\}, \{\varnothing, Z_6, Z_2\}, \{\varnothing, Z_6, Z_1\}, \{\varnothing, Z_6, \breve{D}\}, \{\varnothing, Z_5, Z_4\}, \{\varnothing, Z_5, Z_3\}, \{\varnothing, Z_5, Z_2\}, \{\varnothing, Z_5, Z_1\}$,
$\{\varnothing, Z_5, \breve{D}\}, \{\varnothing, Z_4, Z_2\}, \{\varnothing, Z_4, Z_1\}, \{\varnothing, Z_4, D\}, \{\varnothing, Z_3, Z_1\}, \{\varnothing, Z_3, \breve{D}\}, \{\varnothing, Z_2, \breve{D}\}, \{\varnothing, Z_1, \breve{D}\}$,
(see diagram 3 of the figure 2);

4) $\{\varnothing, Z_6, Z_4, Z_2\}, \{\varnothing, Z_6, Z_4, Z_1\}, \{\varnothing, Z_6, Z_4, \breve{D}\}, \{\varnothing, Z_6, Z_2, D\}, \{\varnothing, Z_6, Z_1, \breve{D}\}$,
$\{\varnothing, Z_5, Z_4, Z_2\}, \{\varnothing, Z_5, Z_4, Z_1\}, \{\varnothing, Z_5, Z_4, \breve{D}\}$ $\{\varnothing, Z_5, Z_3, Z_1\}, \{\varnothing, Z_5, Z_3, \breve{D}\}$,
$\{\varnothing, Z_5, Z_2, \breve{D}\}, \{\varnothing, Z_5, Z_1, \breve{D}\}, \{\varnothing, Z_4, Z_2, D\}, \{\varnothing, Z_4, Z_1, \breve{D}\}, \{\varnothing, Z_3, Z_1, \breve{D}\}$
(see diagram 4 of the figure 2);

5) $\{\varnothing, Z_6, Z_4, Z_2, \breve{D}\}, \{\varnothing, Z_6, Z_4, Z_1, \breve{D}\}, \{\varnothing, Z_5, Z_4, Z_2, \breve{D}\}, \{\varnothing, Z_5, Z_4, Z_1, \breve{D}\}, \{\varnothing, Z_5, Z_3, Z_1, \breve{D}\}$,
(see diagram 5 of the figure 2);

6) $\{\varnothing, Z_6, Z_5, Z_4\}, \{\varnothing, Z_6, Z_3, Z_1\}, \{\varnothing, Z_4, Z_3, Z_1\}, \{\varnothing, Z_3, Z_2, \breve{D}\}, \{\varnothing, Z_2, Z_1, \breve{D}\}$,
(see diagram 6 of the figure 2);

7) $\{\varnothing, Z_6, Z_2, Z_1, \breve{D}\}, \{\varnothing, Z_5, Z_4, Z_3, Z_1,\}, \{\varnothing, Z_5, Z_3, Z_2, \breve{D}\}, \{\varnothing, Z_5, Z_2, Z_1, \breve{D}\}, \{\varnothing, Z_4, Z_2, Z_1, \breve{D}\}$
(see diagram 7 of the figure 2);

8) $\{\varnothing, Z_6, Z_4, Z_2, Z_1, \breve{D}\}, \{\varnothing, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$; (see diagram 8 of the figure 2);

9) $\{\varnothing, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$; (see diagram 9 of the figure 2);

10) $\{\varnothing, Z_6, Z_5, Z_4, Z_2\}, \{\varnothing, Z_6, Z_5, Z_4, Z_1\}, \{\varnothing, Z_6, Z_5, Z_4, \breve{D}\}, \{\varnothing, Z_4, Z_3, Z_1, \breve{D}\}, \{\varnothing, Z_6, Z_3, Z_1, \breve{D}\}$
(see diagram 10 of the figure 2);

11) $\{\varnothing, Z_6, Z_5, Z_4, Z_2, \breve{D}\}, \{\varnothing, Z_6, Z_5, Z_4, Z_1, \breve{D}\}$; (see diagram 11 of the figure 2);

12) $\{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_1\}, \{\varnothing, Z_6, Z_3, Z_2, Z_1, \breve{D}\}, \{\varnothing, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$

(see diagram 12 of the figure 2);

13) $\{\varnothing, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$; (see diagram 13 of the figure 2);

14) $\{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_1, \breve{D}\}$; (see diagram 14 of the figure 2);

15) $\{\varnothing, Z_6, Z_5, Z_4, Z_2, Z_1, \breve{D}\}$; (see diagram 15 of the figure 2);

16) $\{\varnothing, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$; (see diagram 16 of the figure 2);

Proof: This corollary immediately follows from the lemmas 2.2.

The corollary is proved.

Theorem 2.1: *Let* $D \in \Sigma_3(X,8)$, $Z_7 \neq \varnothing$ *and* $\alpha \in B_X(D)$. *Binary relation* α *is an idempotent relation of the semigroup* $B_X(D)$ *iff binary relation* α *satisfies only one conditions of the following conditions:*

1) $\alpha = X \times T$, where $T \in D$;

2) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T')$, where $T, T' \in D$, $T \subset T'$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\varnothing\}$, and satisfies the conditions: $Y_T^\alpha \supseteq T$, $Y_{T'}^\alpha \cap T' \neq \varnothing$;

3) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'')$, where $T, T', T'' \in D$, $T \subset T' \subset T''$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$, and satisfies the conditions: $Y_T^\alpha \supseteq T$, $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_{T''}^\alpha \cap T'' \neq \varnothing$;

4) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T'''}^\alpha \times T''')$, where $T, T', T'', T''' \in D$, $T \subset T' \subset T'' \subset T'''$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_{T'''}^\alpha \notin \{\varnothing\}$, and satisfies the conditions: $Y_T^\alpha \supseteq T$, $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T''}^\alpha \supseteq T''$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_{T''}^\alpha \cap T'' \neq \varnothing$, $Y_{T'''}^\alpha \cap T''' \neq \varnothing$;

5) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_0^\alpha \times \breve{D})$, where $Z_7 \subset T \subset T' \subset T'' \subset \breve{D}$, $Y_7^\alpha, Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_0^\alpha \notin \{\varnothing\}$, and satisfies the conditions: $Y_7^\alpha \supseteq Z_7$, $Y_7^\alpha \cup Y_T^\alpha \supseteq T$, $Y_7^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_7^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T''}^\alpha \supseteq T''$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_{T''}^\alpha \cap T'' \neq \varnothing$, $Y_0^\alpha \cap \breve{D} \neq \varnothing$;

6) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T' \cup T''}^\alpha \times (T' \cup T''))$, where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$, $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_T^\alpha \cup Y_{T''}^\alpha \supseteq T''$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_{T''}^\alpha \cap T'' \neq \varnothing$;

7) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'''}^\alpha\times T'''\right)\cup\left(Y_{T''\cup T'''}^\alpha\times\left(T''\cup T'''\right)\right)$, where, $T\subset T'\subset T''$, $T\subset T'\subset T'''$, $T''\setminus T'''\neq\varnothing$, $T'''\setminus T''\neq\varnothing$ $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_{T'''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\supseteq T$ $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_T^\alpha\cup Y_{T'}^\alpha\cup Y_{T'''}^\alpha\supseteq T'''$ $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$, $Y_{T'''}^\alpha\cap T'''\neq\varnothing$.

8) $\alpha=\left(Y_7^\alpha\times Z_7\right)\cup\left(Y_T^\alpha\times T\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_2^\alpha\times Z_2\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $T\in\{Z_6,Z_5\}$, $Y_7^\alpha, Y_T^\alpha, Y_4^\alpha, Y_2^\alpha, Y_1^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq Z_7$, $Y_7^\alpha\cup Y_T^\alpha\supseteq T$, $Y_7^\alpha\cup Y_T^\alpha\cup Y_4^\alpha\supseteq Z_4$, $Y_7^\alpha\cup Y_T^\alpha\cup Y_4^\alpha\cup Y_2^\alpha\supseteq Z_2$, $Y_7^\alpha\cup Y_T^\alpha\cup Y_4^\alpha\cup Y_1^\alpha\supseteq Z_1$, $Y_T^\alpha\cap T\neq\varnothing$, $Y_4^\alpha\cap Z_4\neq\varnothing$ $Y_2^\alpha\cap Z_2\neq\varnothing$, $Y_1^\alpha\cap Z_1\neq\varnothing$;

9) $\alpha=\left(Y_7^\alpha\times Z_7\right)\cup\left(Y_5^\alpha\times Z_5\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_3^\alpha\times Z_3\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $Z_7\subset Z_5\subset Z_3$, $Z_7\subset Z_5\subset Z_4$, $Z_3\setminus Z_4\neq\varnothing$, $Z_4\setminus Z_3\neq\varnothing$ $Y_7^\alpha, Y_5^\alpha, Y_4^\alpha, Y_3^\alpha, Y_1^\alpha, Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq Z_7$, $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_3^\alpha\supseteq Z_3$ $Y_7^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\supseteq Z_4$, $Y_5^\alpha\cap Z_5\neq\varnothing$, $Y_3^\alpha\cap Z_3\neq\varnothing$, $Y_4^\alpha\cap Z_4\neq\varnothing$, $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

10) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'\cup T''}^\alpha\times\left(T'\cup T''\right)\right)\cup\left(Y_{T'''}^\alpha\times T'''\right)$, where $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$, $T'\cup T''\subset T'''$ $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_{T'''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_{T''}^\alpha\supseteq T''$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$, $Y_{T'''}^\alpha\cap T'''\neq\varnothing$;

11) $\alpha=\left(Y_7^\alpha\times Z_7\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_5^\alpha\times Z_5\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_T^\alpha\times T\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $T\in\{Z_2,Z_1\}$, $Y_7^\alpha, Y_6^\alpha, Y_5^\alpha, Y_T^\alpha, Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions:, $Y_7^\alpha\cup Y_6^\alpha\supseteq Z_6$, $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$ $Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_T^\alpha\supseteq T$, $Y_6^\alpha\cap Z_6\neq\varnothing$, $Y_5^\alpha\cap Z_5\neq\varnothing$ $Y_T^\alpha\cap T\neq\varnothing$, $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

12) $\alpha=\left(Y_T^\alpha\times T\right)\cup\left(Y_{T'}^\alpha\times T'\right)\cup\left(Y_{T''}^\alpha\times T''\right)\cup\left(Y_{T'\cup T''}^\alpha\times\left(T'\cup T''\right)\right)\cup\left(Y_{T'''}^\alpha\times T'''\right)\cup\left(Y_{T'\cup T''\cup T'''}^\alpha\times\left(T'\cup T''\cup T'''\right)\right)$, where $T\subset T'$, $T\subset T''$, $T'\setminus T''\neq\varnothing$, $T''\setminus T'\neq\varnothing$, $T''\subset T'''$, $\left(T'\cup T''\right)\setminus T'''\neq\varnothing$, $T'''\setminus\left(T'\cup T''\right)\neq\varnothing$, , $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_{T'''}^\alpha, Y_{T'\cup T''\cup T'''}^\alpha\notin\{\varnothing\}$ and satisfies the conditions:, $Y_T^\alpha\cup Y_{T'}^\alpha\supseteq T'$, $Y_T^\alpha\cup Y_{T''}^\alpha\supseteq T''$ $Y_T^\alpha\cup Y_{T''}^\alpha\cup Y_{T'''}^\alpha\supseteq T'''$, $Y_{T'}^\alpha\cap T'\neq\varnothing$, $Y_{T''}^\alpha\cap T''\neq\varnothing$ $Y_{T'''}^\alpha\cap T'''\neq\varnothing$;

13) $\alpha=\left(Y_7^\alpha\times Z_7\right)\cup\left(Y_5^\alpha\times Z_5\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_3^\alpha\times Z_3\right)\cup\left(Y_2^\alpha\times Z_2\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $Z_5\subset Z_3$, $Z_5\subset Z_4$, $Z_3\setminus Z_4\neq\varnothing$, $Z_4\setminus Z_3\neq\varnothing$, $Z_4\subset Z_2$, $Z_1\setminus Z_2\neq\varnothing$, $Z_2\setminus Z_1\neq\varnothing$, $Y_7^\alpha, Y_5^\alpha, Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq Z_7$, $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$ $Y_7^\alpha\cup Y_5^\alpha\cup Y_3^\alpha\supseteq Z_3$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\supseteq Z_4$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_1^\alpha\supseteq Z_1$, $Y_5^\alpha\cap Z_5\neq\varnothing$, $Y_3^\alpha\cap Z_3\neq\varnothing$, $Y_4^\alpha\cap Z_4\neq\varnothing$ $Y_1^\alpha\cap Z_1\neq\varnothing$;

14) $\alpha=\left(Y_7^\alpha\times Z_7\right)\cup\left(Y_6^\alpha\times Z_6\right)\cup\left(Y_5^\alpha\times Z_5\right)\cup\left(Y_4^\alpha\times Z_4\right)\cup\left(Y_3^\alpha\times Z_3\right)\cup\left(Y_1^\alpha\times Z_1\right)\cup\left(Y_0^\alpha\times\breve{D}\right)$, where $Z_6\subset Z_4$, $Y_7^\alpha, Y_6^\alpha, Y_5^\alpha, Y_4^\alpha, Y_3^\alpha, Y_1^\alpha, Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$, $Y_7^\alpha\cup Y_6^\alpha\supseteq Z_6$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_3^\alpha\supseteq Z_3$, $Y_5^\alpha\cap Z_5\neq\varnothing$, $Y_6^\alpha\cap Z_6\neq\varnothing$, $Y_3^\alpha\cap Z_3\neq\varnothing$ $Y_0^\alpha\cap\breve{D}\neq\varnothing$;

15) $\alpha=(Y_7^\alpha\times Z_7)\cup(Y_6^\alpha\times Z_6)\cup(Y_5^\alpha\times Z_5)\cup(Y_4^\alpha\times Z_4)\cup(Y_2^\alpha\times Z_2)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$, where $Y_7^\alpha,Y_6^\alpha,Y_5^\alpha,Y_4^\alpha,Y_2^\alpha,Y_1^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\cup Y_6^\alpha\supseteq Z_6$, $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$, $Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_2^\alpha\supseteq Z_2$, $Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_1^\alpha\supseteq Z_1$ $Y_6^\alpha\cap Z_6\neq\varnothing$, $Y_5^\alpha\cap Z_5\neq\varnothing$, $Y_2^\alpha\cap Z_2\neq\varnothing$ $Y_1^\alpha\cap Z_1\neq\varnothing$;

16) $\alpha=(Y_7^\alpha\times Z_7)\cup(Y_6^\alpha\times Z_6)\cup(Y_5^\alpha\times Z)\cup(Y_4^\alpha\times Z_4)\cup(Y_3^\alpha\times Z_3)\cup(Y_2^\alpha\times Z_2)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$, where, $Y_7^\alpha,Y_6^\alpha,Y_5^\alpha,Y_4^\alpha,Y_3^\alpha,Y_2^\alpha,Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha\supseteq Z_7$ $Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5$, $Y_7^\alpha\cup Y_6^\alpha\supseteq Z_6$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_3^\alpha\supseteq Z_3$, $Y_7^\alpha\cup Y_5^\alpha\cup Y_6^\alpha\cup Y_4^\alpha\cup Y_2^\alpha\supseteq Z_2$, $Y_5^\alpha\cap Z_5\neq\varnothing$, $Y_6^\alpha\cap Z_6\neq\varnothing$, $Y_3^\alpha\cap Z_3\neq\varnothing$, $Y_2^\alpha\cap Z_2\neq\varnothing$;

Proof: The statements 1), 2), 3), 4) and 5) immediately follows from the Corollary [1] 13.1.1, [2] 13.1.1, the statements 6) -11) immediately follows from the Corollary [1] 13.3.1, [2] 13.3.1, the statement 15) immediately follows from the Theorem [1] 13.7.2, [2] 13.7.2.

Now we will prove statement 16. $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}$ to begin with, we note that D is an $XI-$semilattice of unions(see lemma 2.2) then It is easy to see, that the set $D(\alpha)=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1\}$ is a generating set of the semilattice D. Then the following equalities are hold:

$$\ddot{D}(\alpha)_{Z_7}=\{Z_7\},\ \ddot{D}(\alpha)_{Z_6}=\{Z_7,Z_6\},\ \ddot{D}(\alpha)_{Z_5}=\{Z_7,Z_5\},\ \ddot{D}(\alpha)_{Z_4}=\{Z_7,Z_6,Z_5,Z_4\},$$
$$\ddot{D}(\alpha)_{Z_3}=\{Z_7,Z_5,Z_3\},\ \ddot{D}(\alpha)_{Z_2}=\{Z_7,Z_6,Z_5,Z_4,Z_2\},\ \ddot{D}(\alpha)_{Z_1}=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_1\}.$$

By statement $b)$ of the Theorem 1.3 follows that the following conditions are true:

$$Y_7^\alpha\supseteq Z_7,\ Y_7^\alpha\cup Y_6^\alpha\supseteq Z_6,\ Y_7^\alpha\cup Y_5^\alpha\supseteq Z_5,\ Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\supseteq Z_4$$
$$Y_7^\alpha\cup Y_5^\alpha\cup Y_3^\alpha\supseteq Z_3,\ Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_2^\alpha\supseteq Z_2,$$
$$Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_3^\alpha\cup Y_1^\alpha\supseteq Z_1,$$

For last conditions we have:

$$Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha=(Y_7^\alpha\cup Y_6^\alpha)\cup(Y_7^\alpha\cup Y_5^\alpha)\cup Y_4^\alpha\supseteq$$
$$\supseteq T_6\cup T_5\cup Y_4^\alpha=Z_6\cup Z_5\cup Y_4^\alpha=Z_4\cup Y_4^\alpha\supseteq Z_4,$$
$$Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_3^\alpha\cup Y_1^\alpha=(Y_7^\alpha\cup Y_5^\alpha\cup Y_3^\alpha)\cup(Y_7^\alpha\cup Y_6^\alpha)\cup Y_4^\alpha\cup Y_1^\alpha\supseteq$$
$$\supseteq Z_3\cup T_6\cup Y_4^\alpha\cup Y_1^\alpha=Z_3\cup Z_6\cup Y_4^\alpha\cup Y_1^\alpha=Z_1\cup Y_4^\alpha\cup Y_1^\alpha\supseteq Z_1.$$

Since φ is isomorphism. Further, it is to see, that the following equality are true:

$$l\left(\ddot{D}_{Z_6},Z_6\right)=\cup\left(\ddot{D}_{Z_6}\setminus\{Z_6\}\right)=Z_7,\ Z_6\setminus l\left(\ddot{D}_{Z_6},Z_6\right)=Z_6\setminus Z_7\neq\varnothing;$$
$$l\left(\ddot{D}_{Z_5},Z_5\right)=\cup\left(\ddot{D}_{Z_5}\setminus\{Z_5\}\right)=Z_7,\ Z_5\setminus l\left(\ddot{D}_{Z_5},Z_5\right)=Z_5\setminus Z_7\neq\varnothing;$$
$$l\left(\ddot{D}_{Z_4},Z_4\right)=\cup\left(\ddot{D}_{Z_4}\setminus\{Z_4\}\right)=\cup\{Z_7,Z_6,Z_5\}=Z_4,\ Z_4\setminus l\left(\ddot{D}_{Z_4},Z_4\right)=Z_4\setminus Z_4=\varnothing;$$
$$l\left(\ddot{D}_{Z_3},Z_3\right)=\cup\left(\ddot{D}_{Z_3}\setminus\{Z_3\}\right)=\cup\{Z_7,Z_5\}=Z_5,\ Z_3\setminus l\left(\ddot{D}_{Z_3},Z_3\right)=Z_3\setminus Z_5\neq\varnothing;$$

$$l(\ddot{D}_{Z_2}, Z_2) = \cup(\ddot{D}_{Z_2} \setminus \{Z_2\}) = \cup\{Z_7, Z_6, Z_5, Z_4\} = Z_4,\ Z_2 \setminus l(\ddot{D}_{Z_2}, Z_2) = Z_2 \setminus Z_4 \neq \varnothing;$$
$$l(\ddot{D}_{Z_1}, Z_1) = \cup(\ddot{D}_{Z_1} \setminus \{Z_1\}) = \cup\{Z_7, Z_6, Z_5, Z_4, Z_3\} = Z_1,\ Z_1 \setminus l(\ddot{D}_{Z_1}, Z_1) = Z_1 \setminus Z_1 = \varnothing;$$

We have the elements Z_6, Z_5, Z_3 and Z_2 are nonlimiting elements of the sets $\ddot{D}(\alpha)_{z_6}$, $\ddot{D}(\alpha)_{Z_5}$, $\ddot{D}(\alpha)_{Z_3}$ and $\ddot{D}(\alpha)_{Z_2}$ respectively. By statement $c)$ of the Theorem 1.1.3 it follows, that the conditions $Y_6^\alpha \cap Z_6 \neq \varnothing$, $Y_5^\alpha \cap Z_5 \neq \varnothing$, $Y_3^\alpha \cap Z_3 \neq \varnothing$ and $Y_2^\alpha \cap Z_2 \neq \varnothing$ are hold.

Therefore the following conditions are hold:

$$\begin{aligned} &Y_7^\alpha \supseteq Z_7,\ Y_7^\alpha \cup Y_6^\alpha \supseteq Z_6,\ Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5,\\ &Y_7^\alpha \cup Y_5^\alpha \cup Y_3^\alpha \supseteq Z_3,\ Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_2^\alpha \supseteq Z_2,\\ &Y_6^\alpha \cap Z_6 \neq \varnothing,\ Y_5^\alpha \cap Z_5 \neq \varnothing,\ Y_3^\alpha \cap Z_3 \neq \varnothing,\ Y_2^\alpha \cap Z_2 \neq \varnothing. \end{aligned} \tag{2.11}$$

In the same manner it can be proved that any subsemilattice of the semilattice D having diagrams 13 and 14.

Theorem is proved.

Corollary 2.2: *Let $D \in \Sigma_3(X,8)$, $Z_7 = \varnothing$ and $\alpha \in B_X(D)$. Binary relation α is an idempotent relation of the semigroup $B_X(D)$ iff binary relation α satisfies only one conditions of the following conditions:*

1) $\alpha = \varnothing$;

2) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_T^\alpha \times T)$, where $\varnothing \neq T \in D$, $Y_T^\alpha \notin \{\varnothing\}$, and satisfies the conditions: $Y_T^\alpha \supseteq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$;

3) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T')$, where $\varnothing \neq T \subset T' \in D$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\varnothing\}$, and satisfies the conditions: $Y_7^\alpha \supseteq \varnothing$, $Y_7^\alpha \cup Y_T^\alpha \supseteq T$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$;

4) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'')$, where $\varnothing \neq T \subset T' \subset T'' \in D$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$, and satisfies the conditions: $Y_7^\alpha \supseteq \varnothing$, $Y_7^\alpha \cup Y_T^\alpha \supseteq T$, $Y_7^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_{T''}^\alpha \cap T'' \neq \varnothing$;

5) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_0^\alpha \times \breve{D})$, where $\varnothing \neq T \subset T' \subset T'' \subset \breve{D}$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_0^\alpha \notin \{\varnothing\}$, and satisfies the conditions: $Y_7^\alpha \supseteq \varnothing$, $Y_7^\alpha \cup Y_T^\alpha \supseteq T$, $Y_7^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_7^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T''}^\alpha \supseteq T''$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_{T''}^\alpha \cap T'' \neq \varnothing$, $Y_0^\alpha \cap \breve{D} \neq \varnothing$;

6) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T'))$, where

$\varnothing \neq T, T' \in D$, $T' \setminus T \neq \varnothing$, $T \setminus T' \neq \varnothing$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \cup Y_T^\alpha \supseteq T$, $Y_7^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$;

7) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T' \cup T''}^\alpha \times (T' \cup T''))$, where, $\varnothing \neq T \subset T'$, $\varnothing \neq T \subset T''$, $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$ $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \supseteq \varnothing$ $Y_7^\alpha \cup Y_T^\alpha \supseteq T$, $Y_7^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_7^\alpha \cup Y_T^\alpha \cup Y_{T''}^\alpha \supseteq T''$ $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_{T''}^\alpha \cap T'' \neq \varnothing$.

8) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_T^\alpha \times T) \cup (Y_4^\alpha \times Z_4) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$, where $T \in \{Z_6, Z_5\}$, $Y_T^\alpha, Y_4^\alpha, Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \supseteq \varnothing$, $Y_7^\alpha \cup Y_T^\alpha \supseteq T$, $Y_7^\alpha \cup Y_T^\alpha \cup Y_4^\alpha \supseteq Z_4$, $Y_7^\alpha \cup Y_T^\alpha \cup Y_4^\alpha \cup Y_2^\alpha \supseteq Z_2$, $Y_7^\alpha \cup Y_T^\alpha \cup Y_4^\alpha \cup Y_1^\alpha \supseteq Z_1$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_4^\alpha \cap Z_4 \neq \varnothing$ $Y_2^\alpha \cap Z_2 \neq \varnothing$, $Y_1^\alpha \cap Z_1 \neq \varnothing$;

9) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$, where $\varnothing \neq Z_5 \subset Z_3$, $\varnothing \neq Z_5 \subset Z_4$, $Z_3 \setminus Z_4 \neq \varnothing$, $Z_4 \setminus Z_3 \neq \varnothing$ $Y_5^\alpha, Y_4^\alpha, Y_3^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \supseteq \varnothing$, $Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5$, $Y_7^\alpha \cup Y_5^\alpha \cup Y_3^\alpha \supseteq Z_3$ $Y_7^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \supseteq Z_4$, $Y_5^\alpha \cap Z_5 \neq \varnothing$, $Y_3^\alpha \cap Z_3 \neq \varnothing$, $Y_4^\alpha \cap Z_4 \neq \varnothing$, $Y_0^\alpha \cap \breve{D} \neq \varnothing$;

10) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T')) \cup (Y_{T''}^\alpha \times T'')$, where, $\varnothing \neq T, T''$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, $T \cup T' \subset T''$ $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \cup Y_T^\alpha \supseteq T$, $Y_7^\alpha \cup Y_{T'}^\alpha \supseteq T'$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$, $Y_{T''}^\alpha \cap T'' \neq \varnothing$;

11) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_T^\alpha \times T) \cup (Y_0^\alpha \times \breve{D})$, where $T \in \{Z_2, Z_1\}$, $Y_6^\alpha, Y_5^\alpha, Y_T^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and satisfies the conditions:, $Y_7^\alpha \cup Y_6^\alpha \supseteq Z_6$, $Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5$ $Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_T^\alpha \supseteq T$, $Y_6^\alpha \cap Z_6 \neq \varnothing$, $Y_5^\alpha \cap Z_5 \neq \varnothing$ $Y_T^\alpha \cap T \neq \varnothing$, $Y_0^\alpha \cap \breve{D} \neq \varnothing$;

12) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T')) \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T \cup T' \cup T''}^\alpha \times (T \cup T' \cup T''))$, where, $\varnothing \neq T, T'$, $T \setminus T' \neq \varnothing$, $T' \setminus T \neq \varnothing$, $T \subset T''$, $(T \cup T') \setminus T'' \neq \varnothing$, $T'' \setminus (T \cup T') \neq \varnothing$, , $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_{T \cup T' \cup T''}^\alpha \notin \{\varnothing\}$ and satisfies the conditions:, $Y_7^\alpha \cup Y_T^\alpha \supseteq T$, $Y_7^\alpha \cup Y_{T'}^\alpha \supseteq T'$ $Y_7^\alpha \cup Y_{T'}^\alpha \cup Y_{T''}^\alpha \supseteq T''$, $Y_T^\alpha \cap T \neq \varnothing$, $Y_{T'}^\alpha \cap T' \neq \varnothing$ $Y_{T''}^\alpha \cap T'' \neq \varnothing$;

13) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$, where $Z_5 \subset Z_3$, $Z_5 \subset Z_4$, $Z_3 \setminus Z_4 \neq \varnothing$, $Z_4 \setminus Z_3 \neq \varnothing$, $Z_4 \subset Z_2$, $Z_1 \setminus Z_2 \neq \varnothing$, $Z_2 \setminus Z_1 \neq \varnothing$, , $Y_5^\alpha, Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \supseteq \varnothing$, $Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5$ $Y_7^\alpha \cup Y_5^\alpha \cup Y_3^\alpha \supseteq Z_3$, $Y_7^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \supseteq Z_4$, $Y_7^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_1^\alpha \supseteq Z_1$, $Y_5^\alpha \cap Z_5 \neq \varnothing$, $Y_3^\alpha \cap Z_3 \neq \varnothing$, $Y_4^\alpha \cap Z_4 \neq \varnothing$ $Y_1^\alpha \cap Z_1 \neq \varnothing$;

14) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$, where,

$Z_6 \subset Z_4$, $Y_6^\alpha, Y_5^\alpha, Y_4^\alpha, Y_3^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5$, $Y_7^\alpha \cup Y_6^\alpha \supseteq Z_6$, $Y_7^\alpha \cup Y_5^\alpha \cup Y_3^\alpha \supseteq Z_3$, $Y_5^\alpha \cap Z_5 \neq \varnothing$, $Y_6^\alpha \cap Z_6 \neq \varnothing$, $Y_3^\alpha \cap Z_3 \neq \varnothing$ $Y_0^\alpha \cap \breve{D} \neq \varnothing$;

15) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$, where $Y_6^\alpha, Y_5^\alpha, Y_4^\alpha, Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \cup Y_6^\alpha \supseteq Z_6$, $Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5$, $Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_2^\alpha \supseteq Z_2$, $Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_1^\alpha \supseteq Z_1$ $Y_6^\alpha \cap Z_6 \neq \varnothing$, $Y_5^\alpha \cap Z_5 \neq \varnothing$, $Y_2^\alpha \cap Z_2 \neq \varnothing$ $Y_1^\alpha \cap Z_1 \neq \varnothing$;

16) $\alpha = (Y_7^\alpha \times \varnothing) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$, where, $Z_5 \subset Z_4$, $Y_6^\alpha, Y_5^\alpha, Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and satisfies the conditions: $Y_7^\alpha \supseteq Z_7$ $Y_7^\alpha \cup Y_5^\alpha \supseteq Z_5$, $Y_7^\alpha \cup Y_6^\alpha \supseteq Z_6$, $Y_7^\alpha \cup Y_5^\alpha \cup Y_3^\alpha \supseteq Z_3$, $Y_7^\alpha \cup Y_5^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \cup Y_2^\alpha \supseteq Z_2$, $Y_5^\alpha \cap Z_5 \neq \varnothing$, $Y_6^\alpha \cap Z_6 \neq \varnothing$, $Y_3^\alpha \cap Z_3 \neq \varnothing$, $Y_2^\alpha \cap Z_2 \neq \varnothing$;

Proof: This corollary immediately follows from the theorem 2.1.

The corollary is proved.

Lemma 3.1: *The number of automorphisms of those semilattices, which are defined by the diagrams 1), 2), 3), 4), 5), 12), 13), 14) and 16) in fig. 2 is equal to 1, those semilattices which are defined diagrams 6), 7), 8), 9), 10) and 11) in fig. 2 is equal to 2 and that semilattice which is defined by the diagram 15) in fig. 2 is equal to 4.*

Proof: Let us prove the given lemma in case of the semilattice which is defined by the diagram 16) in fig. 2. The proofs of the other cases are almost identical of the ongoing one.

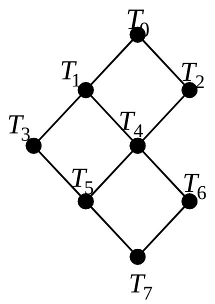


Fig. 5

Suppose $Q = \{T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0\}$ (see fig. 5). Our purpose is to prove that the number of automorphisms of the given semilattice Q is equal to 1. Indeed, if $T_i(n_i, m_i)$ denote the element T_i of the semilattice Q such that $n_i = |Q_{T_i}|$, $m_i = |\ddot{Q}_{T_i}|$ (see

(1.1)) and φ is arbitrary automorphism of the semilattice Q, then $\varphi(T_i)=T_j$ only if $n_i = n_j$ and $m_i = m_j$, i.e. $(n_i, m_i) = (n_j, m_j)$.

For the semilattice Q we have:

$T_0 = T_0(1,8)$, $T_1 = T_1(2,6)$, $T_2 = T_2(2,5)$, $T_3 = T_3(3,3)$,
$T_4 = T_4(4,4)$, $T_5 = T_5(6,2)$, $T_6 = T_6(5,2)$, $T_7 = T_6(8,1)$,

So, we have $\varphi(T_i)=T_i$ for every $i = 0,1,2,...,6$, because $(n_i, m_i) = (n_j, m_j)$ equality is satisfied if and only if when $i = j$.

Therefore the number of automorphisms of the given semilattice Q is equal to 1.

Lemma is proved.

Denote by the symbol $G_X(D,\varepsilon)$ a maximal subgroup of the semigroup $B_X(D)$ whose unit is an idempotent binary relation ε of the semigroup $B_X(D)$.

Theorem 3.1: *For any idempotent binary relation ε of the semigroup $B_X(D)$, the order a subgroup $G_X(D,\varepsilon)$ of the semigroup $B_X(D)$ is one, or two, or four.*

Proof: Let ε be an arbitrary binary relation of the semigroup $B_X(D)$. Now, if we denote by Φ the group of all complete automorphisms of the semilattice $V(D,\varepsilon)$, then by virtue of Theorem 1.3. We have that the groups $G_X(D,\varepsilon)$ and Φ are anti-isomorphic.

To prove the theorem, we will consider the following cases with regard to the idempotent binary relation ε:

1) The idempotent binary relation ε satisfies the conditions 1), 2), 3), 4), 5), 12), 13), 14) and 16) of the Theorem 2.1 and theorem 2.2., then the diagram of the semilattice $V(D,\varepsilon)$ has form 1, 2, 3, 4, 5, 12, 13, 14 and 16 in Fig. 2. Therefore in this case the number of automorphisms of the semilattice $V(D,\varepsilon)$ is equal to one (see Lemma 3.1). Now, taking into account Theorem 1.3, we obtain $|G_X(D,\varepsilon)| = 1$.
2) The idempotent binary relation ε satisfies the conditions 6), 7), 8), 9), 10) and 11) of the Theorem 2.1 and theorem 2.2. So that the diagram of the emilattice $V(D,\varepsilon)$ has form 6-11 in Fig. 2. Therefore in this case the number of automorphisms of the semilattice $V(D,\varepsilon)$ is equal to two (see

Lemma 3.1). Now, taking into account Theorem 1.3, we obtain $\left|G_X(D,\varepsilon)\right| = 2$.

3) The idempotent binary relation is of type 15) so that the number of automorphisms of the semilattice $V(D,\varepsilon)$ has form 15 in Fig. 2. Clearly, in this case the number of automorphisns of the semilattice $V(D,\varepsilon)$ is four (see Lemma 3.1). Now, taking into account Theorem 1.3, we obtain $\left|G_X(D,\varepsilon)\right| = 4$.

Since the diagrams shown in Fig. 2 exhaust all the diagrams of the $XI-$subsemilattices of the semilattice D, the idempotent binary relations of the semigroup $B_X(D)$ are exhausted by types 1-16 from Theorem 2.1 and Theorem 2.2. Hence it follows that for any idempotent ε of the semigroup $B_X(D)$, the order a subgroup $G_X(D,\varepsilon)$ of the semigroup $B_X(D)$ is one, or two, or four.

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