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Totally Transitive Maps - A Short Note

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Abstract

In this short note we discuss totally transitive maps. We show that all chaotic maps are not necessarily totally transitive, but there are chaotic maps with this property. Our discussion relates to symbol space Σ_2 *and real line.*

Keywords: *Chaos, Shift map, Topological transitivity, Total transitivity.*

1 Introduction

Any anomalous behavior in experiments and in computer models of behavior in any field of science was previously attributed to experimental error or noise. Now it is being re-evaluated for an explanation in terms of chaos theory or non-linear analysis. These new terms constitute a set of unifying principles, often called dynamical systems theory which describes phenomena that are common to physical and biological systems throughout science. Study of dynamical systems is aimed at understanding the long term behavior of states in a system for which there is a deterministic rule for how a state evolves.

Symbolic dynamics is also an example of chaotic dynamical systems. Of particular interest is the space Σ_2 which has been considered in a large number of works such as [1, 2, 3, 4, 6, 7]. Devaney [5] and Robinson [8] both have given brilliant description of the space Σ_2. So by symbolic dynamical system we mean here the sequence space $\Sigma_2 = \{\alpha : \alpha = (\alpha_0\alpha_1..........), \alpha_i = 0 \text{ or } 1\}$ and the shift map $\sigma : \Sigma_2 \to \Sigma_2$. The points in this space will be infinite sequences of 0's and 1's. We don't worry about convergence of these sequences; rather, the difficult notion here is to imagine such an infinite sequence as representing a single 'point' space. Also Σ_2 is a compact metric space by the metric

$d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^{i+1}}$, where $s = (s_0 s_1 ....)$ and $t = (t_0 t_1 ....)$ are any two points of Σ_2.

We now give some definitions and lemma which are required for this note.

Definition 2.1. (Shift map [5]) The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is defined by $\sigma(\alpha_0 \alpha_1 ....) = (\alpha_1 \alpha_2 ....)$, where $\alpha = (\alpha_0 \alpha_1 ....)$ is any point of Σ_2.

Definition 2.2. (Topologically transitive [5]) A continuous map $f : X \to X$ is called topologically transitive if for any pair of non-empty open sets $U, V \subset X$ there exists $k \geq 0$ such that $f^k(U) \cap V \neq \phi$, where (X, ρ) is a compact metric space.

Definition 2.3. (Totally transitive [9]) Let (X, ρ) be a compact metric space. A continuous map $f : X \to X$ is called totally transitive if f^n is topologically transitive for all $n \geq 1$.

We also require the following lemma.

Lemma 2.1. [5] *Let $s, t \in \Sigma_2$ and $s_i = t_i$, for $i = 0, 1, ..., m$. Then $d(s,t) < \frac{1}{2^m}$ and conversely if $d(s,t) < \frac{1}{2^m}$, $s_i = t_i$, for $i = 0, 1, ..., m$.*

In this short note we consider a particular property of the dynamical system, namely, the total transitivity. Here we show that the chaotic map may or may not satisfy this property. We get that the shift map is totally transitive but in intervals the situation is different.

2 The Main Result

Theorem 2.1 *The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is totally transitive on Σ_2.*

Proof. Let U and V be two non-empty open subsets of Σ_2 and $\varepsilon_1, \varepsilon_2 > 0$. Also let $s = (s_0 s_1 ....) \in U$ be a point such that $\min\{d(s, \beta)\} \geq \varepsilon_1$, for any β belongs to the boundary of U. Similarly, let $t = (t_0 t_1 ....) \in V$ be any point such that $\min\{d(t, \gamma)\} \geq \varepsilon_2$, for any γ belongs to the boundary of V. Next we choose k_1 and k_2 so large that $\frac{1}{2^{nk_1 - 1}} < \varepsilon_1$ and $\frac{1}{2^{nk_2}} < \varepsilon_2$, where n is any arbitrary positive integer. We now consider the point $\alpha = (s_0 s_1 .... s_{nk_1 - 1} t_0 t_1 .... t_{nk_2} ....)$. Then by Lemma 2.1, $d(s, \alpha) < \frac{1}{2^{nk_1 - 1}} < \varepsilon_1$.

Hence $\alpha \in U$, that is, $(\sigma^n)^{k_1}(\alpha) \in (\sigma^n)^{k_1}(U)$.

On the other hand, $(\sigma^n)^{k_1}(\alpha) = (t_0 t_1 .... t_{nk_2} ....)$.

Hence $d((\sigma^n)^{k_1}(\alpha), t) < \frac{1}{2^{nk_2}} < \varepsilon_2$, by applying Lemma 2.1 again. This gives $(\sigma^n)^{k_1}(\alpha) \in V$.

So we get that $(\sigma^n)^{k_1}(U) \cap V \neq \phi$.

Hence the shift map $\sigma : \Sigma_2 \to \Sigma_2$ is totally transitive on Σ_2.

A question now arises that, is all topologically transitive maps are totally transitive? The following example gives a suitable answer to this question.

Example 2.1. Let $f(x)$ be a continuous map from $[0, 1]$ onto itself defined by

$$f(x) = \begin{cases} 3x + \frac{2}{5},\ 0 \leq x \leq \frac{1}{5} \\ -3x + \frac{8}{5},\ \frac{1}{5} \leq x \leq \frac{2}{5} \\ \frac{2}{3}x - \frac{2}{3},\ \frac{2}{5} \leq x \leq 1 \end{cases}$$

It can be easily proved that the function f is topologically transitive on [0,1] and hence it is chaotic by the result of Vellekoop [10]. On the other hand it is not totally transitive, since the subintervals $[0, \frac{2}{5}]$ and $[\frac{2}{5}, 1]$ are invariant under f^2, so f^2 is not topologically transitive on [0,1].

3 Conclusions

In this short note we have given an example of an interval map which is topologically transitive but not totally transitive. But in the symbol space the shift map is totally transitive. So we can say that although total transitivity is a stronger property than topological transitivity, every chaotic map does not necessarily become totally transitive. We also know that shift map σ is topologically conjugate to the logistic map $F_4(x) = 4x(1-x)$ on intervals, but there is chaotic map on the same interval which is not totally transitive. Hence we conclude that in general not all transitive maps are totally transitive and also not all chaotic maps are totally transitive.

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