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On Connected and Locally Connected of Smooth Topological Spaces

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Abstract

In this paper, concept of smooth topology on fuzzy set has been introduced. We define smooth connected and smooth locally connected and proved those properties are not hereditary properties. We study the relation between these concepts.

Keywords: *Smooth topology, Relative smooth and Smooth continuous function.*

1 Introduction:

The theory of fuzzy sets was introduced by Zadeh in 1965 [5]. Many authors have introduced the concept of smooth topology and gave new definitions to some of the properties of the fuzzy topology. Concluded from these definitions relationships, some those relationships are consistent with the original and others differs [1, 3 and 6]. In our paper we define smooth connected and smooth locally connected by generalization those concepts from the fuzzy topology. We are trying to check out some of these relationships through proved or disproved.

2 Preliminaries

Definition 2.1 [1, 3 and 6]: *A smooth topological space (sts) is a pair (X,τ) where X is a nonempty set and $\tau: I^X \to I$ $(I = [0,1])$ is mapping satisfy the following properties:*

1- $\tau(X) = \tau(\emptyset) = 1$.
2- $\forall A, B \in I^X, \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$.
3- For every subfamily $\{A_i : i \in J\} \subseteq I^X, \tau(\bigcup_{i\in J} A_i) \geq \bigwedge_{i\in J} \tau(A_i)$.

Definition 2.2: *Let (X,τ) be a sts and $A \subseteq X$ is fuzzy set, then we called A is neighborhood of fuzzy point $x_t \in X$ if A containing x_t and $\tau(A) = 1$.*

Definition 2.3 [7]: *Let (X,τ) be a sts and $A \subseteq X$ is fuzzy set. Let $\tau_A: P(A) \to I$ is define as following $\tau_A(B) = max\ \ \tau(G)$ when B=$G\cap A$ then τ_A is called smooth relative on A and (A,τ_A) is smooth subspace of smooth topology (X,τ).*

Definition 2.4 [6]: *Let $f: (X,\tau) \to (Y,\sigma)$ be a function, where τ and σ are smooth topology on X and Y, respectively, this function is called "$\tau - \sigma$ smooth continuous" iff for each $B \in I^Y$ with $\sigma(B) = 1$ then $\tau\left(f^{-1}(B)\right) = 1$*

3 Smooth Connected

Definition 3.1: *Let (X,τ) be a sts, X is connected if there exist no $A \in I^X$ such that $\tau(A) = \tau(A^c) = 1$.*

Theorem 3.1: *Let (X,τ) be a sts, X is connected then there exist no $A_1, A_2 \in I^X$ such that $\tau(A_1) = \tau(A_2) = 1, A_1 \cup A_2 = X$ and $A_1 \cap A_2 = \emptyset$.*

Proof: Let (X,τ) be a sts and X is connected. Let$A_1, A_2 \in I^X$ such that $\tau(A_1) = \tau(A_2) = 1$. Since

$A_1 \cup A_2 = X \Rightarrow \mu_{A_1}(x) + \mu_{A_2}(x) \geq 1, \forall x \in X$ and
$A_1 \cap A_2 = \emptyset \Rightarrow \mu_{A_1}(x) + \mu_{A_2}(x) \leq 1, \forall x \in X$
$\Rightarrow \mu_{A_1}(x) + \mu_{A_2}(x) = 1, \forall x \in X$

Therefore, A_2 is complement of $A_1 \Rightarrow$ there exist $A \in I^X$ such that $(A) = \tau(A^c) = 1$, but this is contradiction. Therefore our hypothesis is not true.

The converse of above theorem is not true in general and following example explains that.

Example 3.1: Let X be any infinite set and $\tau: I \rightarrow I^X$ is mapping define as following:

$$\tau(A) = \begin{cases} 1 \quad if\ A = \emptyset\ or\ X \\ 1 \quad if\ the\ set\ \{x: \mu_A(x) = 0\}\ is\ finite\ set \\ 0 \quad otherwise \end{cases}$$

τ is smooth on X[1]. For any $A_1, A_2 \in I^X$ with $\tau(A_1) = \tau(A_2) = 1$ then $A_1 \cap A_2 \neq \emptyset$ since X is infinite set. But X is not connected since if we take

$$A = \left\{\left(x, \mu_A(x)\right): \mu_A(x) = \frac{1}{2}, \qquad \forall x \in X\right\} \Rightarrow$$

$$A^c = \left\{\left(x, \mu_{A^c}(x)\right): \mu_{A^c}(x) = \frac{1}{2},\ \forall x \in X\right\} \Rightarrow \tau(A) = \tau(A^c) = 1.$$

Definition 3.2: *Let (X, τ) be a sts. A fuzzy set $A \subseteq X$ is said to be connected if there exist no $B \in P(A)$ such that $\tau(B) = \tau(B^c) = 1$, B^c is complement of B with respect to A ,i.e. $\mu_B(x) + \mu_{B^c}(x) = \mu_A(x), \forall x \in X$.*

Theorem 3.2: *Let (X, τ) be a sts and $A \subseteq X$, A is connected then there exist no $B_1, B_2 \in P(A)$ such that $\tau(B_1) = \tau(B_2) = 1, B_1 \cup B_2 = A$ and $B_1 \cap B_2 = \emptyset$.*

Proof: It's clear.

Theorem 3.3: *Let A be a fuzzy subset of sts (X, τ). If (A, τ_A) is connected then A is connected with respect to τ.*

Proof: Let (A, τ_A) is connected, <we want to prove A is connected with respect to τ>. Let A is not connected with respect to τ, then there exist $B \in P(A)$ such that $\tau(B) = \tau(B^c) = 1$, B^c is complement of B with respect to A, therefore $\tau_A(B) = \tau_A(B^c) = 1 \Rightarrow (A, \tau_A)$ is not connected, but this is contradiction, therefore A is connected with respect to τ.

The converse of above theorem is not true in general and following example explains that.

Example 3.2: In example (2.1). For any $A \subseteq X$ with $\tau(A) = 0$, then A is connected with respect to τ since the set $\{x: \mu_A(x) = 0\}$ $is\ infinite$ and this mean for any $B \subseteq A$, $\tau(B) \neq 1$. But the relative smooth is defined as following: $\tau_A(B) = 1, \forall B \in P(A) \Rightarrow (A, \tau_A)$ is not connected.

Remark 3.1: The converse of above theorem (theorem 2.3) is true if $\tau(A) = 1$.

Proof: Let A is connected with respect to τ and $\tau(A) = 1$, we want to prove (A, τ_A) is connected. Let (A, τ_A) is not connected then there exist $B \in P(A)$ such that$\tau_A(B) = \tau_A(B^c) = 1$.

$\tau_A(B) = 1 \Rightarrow \exists G, G \in I^X\ with\ \tau(G) = 1\ and\ B = G \cap A$.

$\tau_A(B^c) = 1 \Rightarrow \exists H, H \in I^X$ $with\ \tau(H) = 1\ and\ B^c = H \cap A.$
$\tau(B) = \tau(G \cap A) \geq \tau(G) \wedge \tau(A) = 1$ And so with respect to B^c, therefore $\tau(B) = \tau(B^c) = 1 \Rightarrow$ A is not connected with respect to τ, but this is contradiction, therefore (A, τ_A) is connected.

The connectedness in smooth topological space is weakly hereditary property and the following explain that.

Example 3.3: Let X={a, b} and $\tau: I^X \rightarrow I$ be defined as follows:

$$\tau(A) = \begin{cases} 1 & \text{if } A = \emptyset \text{ or } X \\ 1 & \text{if } \mu_A(a) > \mu_A(b) \\ \frac{1}{2} & \text{if } \mu_A(a) \leq \mu_A(b) \end{cases}$$

Then τ is smooth topology on X. for any $A \in I^X$ with $A \neq \emptyset$ or X, if $\tau(A) = 1$ then $\tau(A^c) = \frac{1}{2}$, therefore X is connected.

Let $A = \{(a, 0.5), (b, 0.2)\}$ and $B = \{(a, 0.4), (b, 0.2)\}$ then $\tau(B) = 1$, $B \in P(A) \Rightarrow \tau_A(B) = 1$ and B^c with respect to A is $\{(a, 0.1)\}$ $\Rightarrow \tau_A(B^c) = 1$ since $\tau(B^c) = 1 \Rightarrow$ A is not connected.

Theorem 3.4: *Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a surjective and $\tau - \sigma$ smooth continuous function where τ and σ are smooth topology on X and Y, respectively. If (X, τ) is smooth connected then so (Y, σ).*

Proof: Let (X, τ) is smooth connected and (Y, σ) is not smooth connected then there exist $G \in I^Y$ such that $\sigma(G) = \sigma(G^c) = 1 \Rightarrow \tau(f^{-1}(G)) = \tau(f^{-1}(G^c)) = 1 \Rightarrow$ X is not connected, but this is contradiction. Therefore our hypothesis is not true.

4 Locally Connected

Definition 4.1: *A sts (X, τ) is smooth locally connected iff every fuzzy point $x_t \in X$ has connected neighborhood.*

Remark 4.1: Every smooth connected is smooth locally connected, but the converse is not true, the following example explains that.

Example 4.1: Let X be any infinite set and $\tau: I^X \rightarrow I$ be defined as follows:

$$\tau(A) = \begin{cases} 1 & \text{if } \mu_A(x) = 1 \text{ or } 0\ \forall x \in X \\ 0 & \text{otherwise} \end{cases}$$

Then τ is smooth topology on X. Let $G = \{(x, 1)\}$ for fixed $x \in X$ then $G^c = \{(y, 1)\}$ for each $y \in X$ expect x such that $\mu_{G^c}(x) = 0$ then $\tau(G) = \tau(G^c) = 1 \Rightarrow X$ is not connected. Now, let $x \in X$ be a arbitrary such that $A = \{(x, 1)\}$ then $\tau(A) = 1$, A is connected since there exist no $B \subseteq A$ such that $\tau(B) = 1$ expect A, since $x \in X$ is arbitrary then above satisfy for each $x \in X$, therefore (X, τ) is smooth locally connected.

The locally connectedness in smooth topological space is not hereditary property and following example explains that.

Example 4.2: In example (2.3), X is connected therefore is smooth locally connected. Let $A = \{(a, 0.5)\}$ then A is not locally connected since for any $a_t \in A$ there exist no $B \subseteq A$ is connected.

Theorem 4.1: *Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a surjective and $\tau - \sigma$ smooth continuous function where τ and σ are smooth topology on X and Y, respectively. Let $f^{-1}: (Y, \sigma) \rightarrow (X, \tau)$ is smooth continuous then if (X, τ) is smooth locally connected then so (Y, σ).*

Proof: Let $y_r \in Y$, then there exist $x_t \in X$ such that $f(x_t) = y_r$. x_t Has connected neighborhood A and since f^{-1} is smooth continuous then f(A) is neighborhood of y_r.

Let f(A) is not connected then there exist $G \subseteq f(A)$ such that $\tau(G) = \tau(G^c) = 1$ and G^c is complement of G with respect to f(A).

f is smooth continuous $\Rightarrow \tau(f^{-1}(G)) = \tau(f^{-1}(G^c)) = 1$, but $f^{-1}(G^c) = (f^{-1}(G))^c$ [2], then A is not connected, but this is contradiction. Therefore our hypothesis is not true.

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