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Minimality Conditions on Π_k^e-Connectedness in Graphs

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Abstract

Let k be a positive integer. A graph $G = (V, E)$ is said to be Π_k^e - connected if for any given edge subset F of $E(G)$ with $|F| = k$, the subgraph induced by F is connected. In this paper, we explore the minimality conditions on Π_k^e-connectedness of a graph and also its properties of prism and corona graphs are obtained.

Keywords: *Graph, subgraph, prism graph, corona graph, Π_k^e - connected graph.*

1 Introduction

In this article, we consider finite, undirected, simple and connected graphs $G = (V, E)$ with vertex set V and edge set E. As such $p =| V |$ and $q =| E |$ denote the number of vertices and edges of a graph G, respectively. An edge - induced subgraph is a subset of the edges of a graph G together with any vertices that are their endpoints. In general, we use $\langle X \rangle$ to denote the subgraph induced by the set of edges $X \subseteq E$. A graph G is connected if it has a $u - v$ path whenever $u, v \in V(G)$ (otherwise, G is disconnected). A graph with no cycle is acyclic. A tree T is a connected acyclic graph. An edge - independent set in a graph is a set of pairwise nonadjacent edges. A cut-edge or cut-vertex of a graph G is an edge or a vertex whose deletion increases the number of components. Difference between two sets A and B consists of the elements of A that are not in B and is denoted by $A \setminus B$. Unless mentioned otherwise, for terminology and notation the reader may refer Harary [8] and Bondy et.al. [3].

The concept of Π_k - connectedness was suggested by Sampathkumar [10] and [11], and studied by Chaluvaraju et al. [4] in the following manner. For any positive integer k. A graph G is said to be Π_k - connected if for any given subset S of $V(G)$ with $|S| = k$, the subgraph induced by S is connected.

Here, we shall introduce an edge analogue of this concept as follows: A graph G is said to be Π_k^e - connected if for any given edge subset F of $E(G)$ with $|F| = k$, the subgraph induced by F is connected. A Π_k^e - connected graph G is said to be edge minimal Π_k^e - connected if the graph G is not Π_{k-1}^e - connected. Let G be a nontrivial graph. Then a generalized vertex (edge)induced connected subsets of a graph is denoted as $\Pi_k(G)(\Pi_k^e(G))$. For more details on related concepts, we refer [1], [2], [5], [9] and [12].

2 Π_k^e - Connectedness

Theorem 2.1 *Let s and t be a positive integer. If the graph G is not Π_t^e - connected graph, then it is also not Π_s^e - connected graph, where $2 \leq s \leq t$.*

Proof. Let the graph G be not a Π_t^e - connected graph, then there exists t-edges whose edge induced subgraph, say F_1 is disconnected. Let F_2, $2 \leq |F_2| \leq t$ be set of edges formed by taking at least one edge from each component of F_1. Clearly the subgraph induced by F_2 is also disconnected. Hence G is not Π_s^e - connected graph, where $2 \leq s \leq t$.

Theorem 2.2 *For any connected graph G is an edge minimal Π_q^e - connected graph if and only if it has a cut edge.*

Proof. Let G be an edge minimal Π_q^e - connected graph then there exists an edge e such that the subgraph induced by $E(G)\setminus\{e\}$ is disconnected. Hence e is a cut edge of a graph G. Conversely, if the graph G has a cut edge e, then the subgraph induced by $E(G)\setminus\{e\}$ is disconnected and the graph on q- edges is connected. Hence the graph G is an edge minimal Π_q^e - connected graph.

Theorem 2.3 *For any tree T with $p \geq 3$ vertices is an edge minimal Π_{p-1}^e - connected graph.*

Proof. Let T be any tree with $p \geq 3$ vertices. Since the total number of edges in T is $p-1$, the graph induced by $p-1$ edges is isomorphic to T and hence connected. Let e be an edge whose end vertices have degree greater than one, then the subgraph induced by $E(T)\setminus\{e\}$ is disconnected. Hence the tree T with $p \geq 3$ vertices is an edge minimal Π^e_{p-1} - connected graph.

Theorem 2.4 *For any Complete graph K_p with $p \geq 4$ vertices is an edge minimal Π^e_k - connected graph, where $k = \frac{(p-2)(p-3)}{2} + 2$.*

Proof. Let K_p be a complete graph with $p \geq 4$ vertices. The maximum number of independent edges of a complete graph K_p is $\lfloor \frac{p}{2} \rfloor$. There fore only disconnected edge induced subgraph of K_p with maximum number of vertices is $K_2 \cup K_{p-2}$. Hence K_p is not Π^e_{k-1} - connected graph, where $k = \frac{(p-2)(p-3)}{2} + 2$. Addition of any edge makes the subgraph $K_2 \cup K_{p-2}$ is connected. Hence the complete graph K_p with $p \geq 4$ vertices is an edge minimal Π^e_k - connected graph, where $k = \frac{(p-2)(p-3)}{2} + 2$.

By the above two results, we have the following Theorem.

Theorem 2.5 *Let $\Pi^e_k(G)$ be an edge minimal Π^e_k- connected graph of a (p, q)-graph. Then*

$$q \leq \Pi^e_k(G) \leq \frac{(p-2)(p-3)}{2} + 2.$$

Theorem 2.6 *For any Cycle C_p with $p \geq 4$ vertices is an edge minimal Π^e_{p-1} - connected graph.*

Proof. Let C_p be any cycle on $p \geq 4$ vertices and e be any edge in C_p. The subgraph induced by $E(C_p)\setminus\{e\}$ is connected. Since an edge e is arbitrarily chosen, the cycle C_p with $p \geq 4$ vertices is Π^e_{p-1} - connected graph. Now we prove C_p is an edge minimal Π^e_{p-1} - connected graph, i.e., C_p is not Π^e_{p-2} - connected graph. Let e_1 and e_2 be two independent edges in C_p. The subgraph induced by $E(C_p)\setminus\{e_1, e_2\}$ is disconnected. Hence C_p with $p \geq 4$ vertices is an edge minimal Π^e_{p-1} - connected graph.

3 Π^e_k - Connectedness in Prism Graphs

The prism of a graph G is defined as the cartesian product $G \times K_2$. For more details, we refer [7].

Theorem 3.1 *For any Prism C^*_p of a cycle C_p with $p \geq 5$ vertices is Π^e_{3p-3} - connected graph.*

Proof. Let C_p be a cycle with $p \geq 5$ vertices and C_p^* be its prism. Let C_p^1 and C_p^2 be two copies of C_p in the prism. We have to prove the subgraph induced by any set of $(3p-3)$- edges is connected. The total number of edges in C_p^* is $3p$. Let $E(C_p^*)$ be the set of edges in C_p^* and e_1, e_2, e_3 be any three edges in $E(C_p^*)$. Instead of proving the subgraph induced by any set of $3p-3$ edges is connected, we prove the equivalent statement the subgraph induced by $E(C_p^*)\backslash\{e_1, e_2, e_3\}$ is connected for every set of three edges. Hence the following cases arise depending on the selection of edges in $E(C_p^*)$.

Case 1. Let $e_1 \in C_1$ and $e_2, e_3 \in C_2$. Then again we have the following two subcases,

Subcase 1.1. Let e_2 and e_3 are consecutive edges. Then both the subgraphs F_1 and F_2 induced by $E(C_1)\backslash\{e_1\}$ and $E(C_2)\backslash\{e_2, e_3\}$, respectively are connected. Now by adding all the edges between F_1 and F_2 from the C_p^*, we get a connected graph induced by $E(C_p^*)\backslash\{e_1, e_2, e_3\}$.

Subcase 1.2. Let e_2 and e_3 are nonconsecutive edges. Then the subgraph induced by $E(C_1)\backslash\{e_1\}$ is connected and the subgraph induced by $E(C_2)\backslash\{e_2, e_3\}$ is disconnected. But the edges between H_1 and H_2 from the C_p^* makes the subgraph induced by $E(C_p^*)\backslash\{e_1, e_2, e_3\}$ connected.

Case 2. Let $e_1, e_2 \in C_1$ and $e_3 \in C_2$. Proof follows on similar lines as in Case 1.

Case 3. Let all three edges e_1, e_2 and e_3 are in the first copy of C_p in the prism C_p^*. Then the subgraph induced by $E(C_1)\backslash\{e_1, e_2, e_3\}$ may be connected or disconnected. If the subgraph induced by $E(C_1)\backslash\{e_1, e_2, e_3\}$ is connected, then clearly the subgraph induced by $E(C_p^*)\backslash\{e_1, e_2, e_3\}$ is connected. If the subgraph induced by $E(C_1)\backslash\{e_1, e_2, e_3\}$ is disconnected having two or three components then the edges between each of these components and C_2 from the prism C_p^* makes the subgraph induced by $E(C_p^*)\backslash\{e_1, e_2, e_3\}$ connected.

Case 4. Let all three edges e_1, e_2 and e_3 are in the second copy of C_p in the prism C_p^*. Proof of this case follows on the similar lines as in Case 3.

Case 5. Let $e_1, e_2 \in C_1$ and e_3 belongs to the set of edges between the two copies of C_p in the prism. The subgraph H_1 induced by $E(C_1)\backslash\{e_1, e_2\}$ may be connected or disconnected depending on the edges e_1 and e_2 are consecutive or not. As seen before, the subgraph induced by $E(C_p^*)\backslash\{e_1, e_2, e_3\}$ connected, since $p \geq 5$, there exists edges between the two copies of C_p even after removal of e_3, connecting each component of H_1 with the second copy of C_p in the prism. Hence the subgraph induced by $E(C_p^*)\backslash\{e_1, e_2, e_3\}$ is connected.

Similarly we can prove the remaining cases as follows.

Case 6. $e_1 \in C_1$ and e_2, e_3 belongs to the set of edges between the two copies of C_p in the prism.

Case 7. All three edges e_1, e_2 and e_3 are in the set of edges between the two copies of C_p in the prism.

Case 8. $e_1, e_2 \in C_2$ and e_3 belongs to the set of edges between the two copies of C_p in the prism.

Case 9. $e_1 \in C_2$ and e_2, e_3 belongs to the set of edges between the two copies of C_p in the prism.

Hence the result follows.

Theorem 3.2 *Let T be a nontrivial tree. Then Prism T^* of T is an edge minimal Π^e_{3q} - connected graph.*

Proof. Let T be any nontrivial tree and T^* be its prism. The number of edges in T^* is equal to the number of edges in the first copy of a tree T + the number of edges in the second copy of a tree T + the number of edges between these two copies of a tree T, i.e., $|E(T^*)| = 2q + p = 3q + 1$. First, we prove T^* is not Π^e_{3q-1} - connected graph. Let e be an edge in the first copy of a tree T in its prism T^* and $f(e)$ in the second copy of T in T^* be the mirror image of e. The subgraph induced by $E(T^*)\backslash\{e, f(e)\}$ is disconnected, since e is a bridge in the first copy of a tree T and $f(e)$ is a bridge in the second copy of a tree T in the prism T^*. Hence there exist a set $E(T^*)\backslash\{e, f(e)\}$ of $3q-1$ edges whose induced subgraph is disconnected. Hence T^* is not Π^e_{3q-1} - connected graph. The subgraph induced by $E(T^*)\backslash\{e\}$ is connected for all e in the first copy or second copy of a tree T. Now suppose $e \in E(T_1 - T_2)$, where $E(T_1 - T_2)$ is the set of all edges between the two copies of a tree T in the prism T^*, clearly in this case also the subgraph induced by $E(T^*)\backslash\{e\}$ is connected for all e in the $E(T_1 - T_2)$. Hence the prism T^* of any nontrivial tree T is an edge minimal Π^e_{3q} - connected graph.

4 Π^e_k - Connectedness in Corona Graphs

The corona $G_1 \circ G_2$ was defined by Frucht and Harary [6] as the graph G obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2, and then joining the i^{th} node of G_1 to every node in the i^{th} copy of G_2.

Theorem 4.1 *Let C_p be a cycle with $p \geq 3$ vertices and $G(p_1, q_1)$ be a graph. Then the corona $C_p \circ G$ is an edge minimal $\Pi^e_{p[p_1+q_1+1]-1}$ - connected graph.*

Proof. Let $C_p : u_1, u_2, \ldots, u_p$, $p \geq 3$ be any cycle and $G(p_1, q_1)$ be any graph of order p_1 and size q_1. Let G_1, G_2, $\ldots$, G_p be p copies of G in the corona $C_p \circ G$. Let $E(C_p)$ be the set of edges in C_p, E_i be the set of edges from u_i to G_i in the corona and $E(G_i)$ be the set of edges in G_i. We first prove the corona $C_p \circ G$ is not $\Pi^e_{p[p_1+q_1+1]-2}$ - connected. Then there exists a set of $p[p_1 + q_1 + 1] - 2$ edges in $C_p \circ G$ whose edge induced subgraph is disconnected or equivalently we show the existence of a pair of edges e_1, e_2 in $C_p \circ G$ such that the subgraph induced by $E(C_p \circ G)\backslash\{e_1, e_2\}$ is disconnected, as the total number of edges in $C_p \circ G$ is $p[p_1 + q_1 + 1]$. Suppose e_1 and e_2 be any two edges on the cycle C_p in the corona, then clearly the subgraph induced by $E(C_p \circ G)\backslash\{e_1, e_2\}$ is disconnected. Hence the corona $C_p \circ G$ is not $\Pi^e_{p[p_1+q_1+1]-2}$ - connected. Now we prove $C_p \circ G$ is $\Pi^e_{p[p_1+q_1+1]-1}$ - connected or equivalently we prove for every edge e in $C_p \circ G$, the subgraph induced by $E(C_p \circ G)\backslash\{e\}$ is connected. Thus the following cases arise.

Case 1. If $e \in G_i$ then the subgraph induced by $E(C_p \circ G)\backslash\{e\}$ is connected, since every left out edge in G_i after removal of e is connected with u_i of C_p.

Case 2. If $e \in E_i$ then the subgraph induced by $E(C_p \circ G)\backslash\{e\}$ is connected, u_i is adjacent to all the vertices of G_i.

Case 3. If $e \in C_p$ then the removal e does not disconnect C_p. Hence the subgraph induced by $E(C_p \circ G)\backslash\{e\}$ is connected. Hence the proof.

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