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On Common Fixed Point Theorem In Complete Metric Space

Jay G. Mehta and M. L. Joshi

Harish Chandra Research Institute - Allahabad (India)
E-mail:jgm_hri@yahoo.in
M. and N. Virani Science College - Rajkot (india)
E-mail:mljoshi@vsc.edu.in

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Abstract

We prove a unique common fixed-point theorem for two pair of weakly compatible maps in a complete metric space, which generalizes the result of Brian Fisher by a weaker condition such as weakly compatibility instead of compatibility and contractive modulus instead of continuity of maps.

Keywords: *Common fixed point, Complete metric space, Contractive Modulus, Weakly compatible maps.*

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1 Introduction

The concept of the commutativity has generalized in several ways. For this Sessa S [6] has introduced the concept of weakly commuting and Gerald Jungck [2] initiated the concept of compatibility. It can be easily verified that when the two mappings are commuting then they are compatible but not conversely. In 1998, Jungck and Rhoades [4] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but not conversely. Brian Fisher [1] proved an important Common Fixed Point theorem.
The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research activity during the last three decades. In 1922, the Polish mathematician, Banach, proved a theorem which

ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banach fixed point theorem or the Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. Many authors have extended, generalized and improved Banach fixed point theorem in different ways. Jungck [2]introduced more generalized commuting mappings, called compatible mappings, which are more general than commuting and weakly commuting mappings.
The main purpose of this paper is to present fixed point results for two pair of maps satisfying a new contractive condition by using the concept of weakly compatible maps in a complete metric space.

2 Preliminaries

we recall the definitions of complete metric space, the notion of convergence and other results that will be needed in the sequel.

Definition 2.1 *Let f and g be two self-maps on a set X. Maps f and g are said to be commuting if $fgx = gfx$ for all $x \in X$.*

Definition 2.2 *Let f and g be two self-maps on a set X. If $fx = gx$, for some x in X then x is called coincidence point of f and g.*

Definition 2.3 *[4] Let f and g be two self-maps defined on a set, then f and g are said to be weakly compatible if they commute at coincidence points. that is, if $fu = gu$ for some $u \in X$, then $fgu = gfu$.*

Lemma 2.4 *[2] Let f and g be weakly compatible self mappings of a set X. If f and g have a unique point of coincidence, that is, $w = fx = gx$, then w is the unique common fixed point of f and g.*

Definition 2.5 *A sequence $\{x_n\}$ in a metric space (X, d) is said to be convergent to a point $x \in X$, denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} d(x_n, x) = 0$.*

Definition 2.6 *A sequence $\{x_n\}$ in a metric space (X, d) is said to be Cauchy sequence if $\lim_{t\to\infty} d(x_n, x_m) = 0$ for all $n, m > t$.*

Definition 2.7 *A metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.*

Definition 2.8 *A function $\phi : [0, \infty) \to [0, \infty)$ is said to be a contractive modulus if $\phi : [0, \infty) \to [0, \infty)$ and $\phi(t) < t$ fort > 0.*

Definition 2.9 *A real valued function ϕ defined on $X \subseteq R$ is said to be upper semi continuous if $\lim_{n\to\infty} \sup \phi(t_n) \leq \phi(t)$, for every sequence $\{t_n\} \in X$ with $t_n \to t$ as $n \to \infty$.*

It is clear that every continuous function is upper semi continuous but converse may not true.

Theorem 2.10 *[1] Suppose S, P, T and Q are four self maps of a metric space (X,d) satisfying the following conditions.*

1. *S(X) $\subseteq$ Q(X) and T(X) $\subseteq$ P(X) .*
2. *Pairs (S,P) and (T,Q) are commuting.*
3. *One of S,P,T and Q is continuous.*
4. $d(Sx, Ty) \leq c\lambda(x, y)$. *where* $\lambda(x, y) = max\{d(Px, Qy), d(Px, Sx), d(Qy, Ty)\}$ *for all* $x, y \in X$ *and* $0 \leq c < 1$.
   Further if
5. *X is complete.*

Then S,P,T and Q have a unique common fixed point $z \in X$. Also z is the unique common fixedpoint of (S,P) and of (T,Q).

3 Main Result

In this section we prove a common fixed point theorem for two pairs of weakly compatible mappings in complete metric spaces using a contractive modulus. This is the generalization of theorem 2.10 in the sense that instead of taking constant c, we take an upper semi continuous, contractive modulus.

Theorem 3.1 *Let (X,d) be a complete metric space. Suppose that the mappings P, Q, S and T are four self-maps of X satisfying the following conditions:*

1. *T(X) $\subseteq$ P(X) and S(X) $\subseteq$ Q(X);*
2. $d(Sx, Ty) \leq \phi(\lambda(x, y))$
   where ϕ is an upper semi continuous, contractive modulus and
   $\lambda(x, y) = max\{d(Px, Qy), d(Px, Sx), d(Qy, Ty), \frac{1}{2}(d(Px, Ty)+d(Qy, Sx))\}$.
3. *The pairs (S, P) and (T, Q) are weakly compatible*

Then P, Q, S and T have a unique common fixed point.

Proof : Suppose x_0 is an arbitrary point of X and define the sequence $\{y_n\}$ in X such that,

$$y_n = Sx_n = Qx_{n+1}$$

$$y_{n+1} = Tx_{n+1} = Px_{n+2}$$

By (ii), we have,

$$\begin{aligned} d(y_n, y_{n+1}) &= d(Sx_n, Tx_{n+1}) \\ &\leq \phi(\lambda(x_n, x_{n+1})) \end{aligned}$$

where
$\lambda(x_n, x_{n+1})$
$=\max\{d(Px_n, Qx_{n+1}), d(Px_n, Sx_n), d(Qx_{n+1}, Tx_{n+1}), \frac{1}{2}(d(Px_n, Tx_{n+1})+d(Qx_{n+1}, Sx_n))\}$
$=\max\{d(Tx_{n-1}, Sx_n), d(Tx_{n-1}, Sx_n), d(Sx_n, Tx_{n+1}), \frac{1}{2}(d(Tx_{n-1}, Tx_{n+1})+d(Sx_n, Sx_n))\}$
$=\max\{d(Tx_{n-1}, Sx_n), d(Sx_n, Tx_{n+1}), \frac{1}{2}(d(Tx_{n-1}, Tx_{n+1}) + 0)\}$
$=\max\{d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{1}{2}(d(y_{n-1}, y_{n+1}) + 0)\}$
$\leq \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}$
Since ϕ is a contractive modulus, $\lambda(x_n, x_{n+1}) = d(y_n, y_{n+1})$ is not possible.
Thus

$$d(y_n, y_{n+1}) \leq \phi(d(y_{n-1}, y_n)) \tag{1}$$

Since ϕ is an upper semi continuous, contractive modulus, equation (1) implies that the sequence $\{d(y_{n+1}, y_n)\}$ is monotonic decreasing and continuous.
Hence there exists a real number, say $r \geq 0$ such that $\lim_{n\to\infty} d(y_{n+1}, y_n) = r$
Therefore as $n \to \infty$, equation (1) implies that

$$r \leq \phi(r)$$

which is possible only if $r = 0$ because ϕ is a contractive modulus.
Thus $\lim_{n\to\infty} d(y_{n+1}, y_n) = 0$
Now we show that $\{y_n\}$ is a Chuchy sequence.
Let if possible we assume that $\{y_n\}$ is not a Chuchy sequence.
Then there exists an $\varepsilon > 0$ and subsequences $\{n_i\}$ and $\{m_i\}$ such that
$m_i < n_i < m_{i+1}$ and

$$d(y_{m_i}, y_{n_i}) \geq \varepsilon \quad \text{ and } \quad d(y_{m_i}, y_{n_i - 1}) < \varepsilon \tag{2}$$

So that $\varepsilon \leq d(y_{m_i}, y_{n_i}) \leq d(y_{m_i}, y_{n_i-1}) + d(y_{n_i-1}, y_{n_i}) < \varepsilon + d(y_{n_i-1}, y_{n_i})$
Therefore $\lim_{i\to\infty} d(y_{m_i}, y_{n_i}) = \varepsilon$
Now $d(y_{m_i-1}, y_{n_i-1}) \leq d(y_{m_i-1}, y_{m_i}) + d(y_{m_i}, y_{n_i}) + d(y_{n_i}, y_{n_i-1})$

By taking limit as $i \to \infty$, we get $\lim_{i\to\infty} d(y_{m_i-1}, y_{n_i-1}) = \varepsilon$
Now by (ii) and (2)

$$\varepsilon \leq d(y_{m_i}, y_{n_i}) = d(Sx_{m_i}, Tx_{n_i}) \leq \phi(\lambda(x_{m_i}, x_{n_i}))$$

$$i.e., \quad \varepsilon \leq \phi(\lambda(x_{m_i}, x_{n_i})) \tag{3}$$

where
$\lambda(x_{m_i}, x_{n_i})$
$=\max\{d(Px_{m_i}, Qx_{n_i}), d(Px_{m_i}, Sx_{m_i}), d(Qx_{n_i}, Tx_{n_i}), \frac{1}{2}(d(Px_{m_i}, Tx_{n_i})+d(Qx_{n_i}, Sx_{m_i}))\}$
$=\max\{d(Tx_{m_i-1}, Sx_{n_i-1}), d(Tx_{m_i-1}, Sx_{m_i}), d(Sx_{n_i-1}, Tx_{n_i}), \frac{1}{2}(d(Tx_{m_i-1}, Tx_{n_i})+$
$d(Sx_{n_i-1}, Sx_{m_i}))\}$
$=\max\{d(y_{m_i-1}, y_{n_i-1}), d(y_{m_i-1}, y_{m_i}), d(y_{n_i-1}, y_{n_i}), \frac{1}{2}(d(y_{m_i-1}, y_{n_i})+d(y_{n_i-1}, y_{m_i}))\}$
By taking limit as $i \to \infty$, we get $\lim_{i\to\infty} \lambda(x_{m_i}, x_{n_i}) = \max\{\varepsilon, 0, 0, \frac{1}{2}(\varepsilon + \varepsilon)\}$
Thus we have, $\lim_{i\to\infty} \lambda(x_{m_i}, x_{n_i}) = \varepsilon$
Therefore from (3) $\varepsilon \leq \phi(\varepsilon)$
This is a contradiction because $0 < \varepsilon$ and ϕ is contractive modulus.
Thus $\{y_n\}$ is a Chuchy sequence in X.
Since X is complete, there exists a point z in X such that $\lim_{n\to\infty} y_n = z$

$$Thus \quad \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Qx_{n+1} = z \quad and \quad \lim_{n\to\infty} Tx_{n+1} = \lim_{n\to\infty} Px_{n+2} = z$$
$$i.e., \quad \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Qx_{n+1} = \lim_{n\to\infty} Tx_{n+1} = \lim_{n\to\infty} Px_{n+2} = z$$

Since $T(X) \subseteq P(X)$, there exists a point $u \in X$ such that $z = Pu$.
Then by (ii), we have

$$\begin{aligned} d(Su, z) &\leq d(Su, Tx_{n+1}) + d(Tx_{n+1}, z) \\ &\leq \phi(\lambda(u, x_{n+1})) + d(Tx_{n+1}, z) \end{aligned}$$

where $\lambda(u, x_{n+1})$
$= \max\{d(Pu, Qx_{n+1}), d(Pu, Su), d(Qx_{n+1}, Tx_{n+1}), \frac{1}{2}(d(Pu, Tx_{n+1})+d(Qx_{n+1}, Su)\}$
$= \max\{d(z, Sx_n), d(z, Su), d(Sx_n, Tx_{n+1}), \frac{1}{2}(d(z, Tx_{n+1}) + d(Sx_n, Su))\}$
Taking the limit as $n \to \infty$ yields,
$\lambda(u, x_{n+1}) = \max\{d(z, z), d(z, Su), d(z, z), \frac{1}{2}(d(z, z) + d(z, Su))\} = d(Su, z)$
Thus as $n \to \infty$, $d(Su, z) \leq \phi(d(Su, z)) + d(z, z) = \phi(d(Su, z))$
If $Su \neq z$ then $d(Su, z) > 0$ and hence as ϕ is contractive modulus
$\phi(d(Su, z)) < d(Su, z)$.
Therefore $d(Su, z) < d(Su, z)$, which is a contradiction.
Thus $Su = z$. So $Pu = Su = z$.
So u is a coincidence point of P and S.
Since the pair of maps S and P are weakly compatible, $SPu = PSu$,
i.e.,$Sz = Pz$.

Again Since $S(X) \subseteq Q(X)$, there exists a point $v \in X$ such that $z = Qv$. Then by (ii), we have

$$\begin{aligned} d(z, Tv) &= d(Su, Tv) \\ &\leq \phi(\lambda(u, v)) \end{aligned}$$

where

$$\begin{aligned} \lambda(u, v) &= max\{d(Pu, Qv), d(Pu, Su), d(Qv, Tv), \frac{1}{2}(d(Pu, Tv) + d(Qv, Su))\} \\ &= max\{d(z, z), d(z, z), d(z, Tv), \frac{1}{2}(d(z, Tv) + d(z, z))\} \\ &= d(z, Tv) \end{aligned}$$

Thus $d(z, Tv) \leq \phi(d(z, Tv))$
If $Tv \neq z$ then $d(z, Tv) > 0$ and hence as ϕ is contractive modulus $\phi(d(z, Tv)) < d(z, Tv)$.
Therefore $d(z, Tv) < d(z, Tv)$, which is a contradiction.
Therefore $Tv = Qv = z$.
So v is a coincidence point of Q and T.
Since the pair of maps Q and T are weakly compatible, $QTv = TQv$, i.e., $Qz = Tz$.
Now we show that z is a fixed point of S.
By (ii), we have

$$\begin{aligned} d(Sz, z) &= d(Sz, Tv) \\ &\leq \phi(\lambda(z, v)) \end{aligned}$$

where

$$\begin{aligned} \lambda(z, v) &= max\{d(Pz, Qv), d(Pz, Sz), d(Qv, Tv), \frac{1}{2}(d(Pz, Tv) + d(Qv, Sz))\} \\ &= max\{d(Sz, z), d(Sz, Sz), d(z, z), \frac{1}{2}(d(Sz, z) + d(z, Sz))\} \\ &= d(Sz, z) \end{aligned}$$

Thus $d(Sz, z) \leq \phi(d(Sz, z))$
If $Sz \neq z$ then $d(Sz, z) > 0$ and hence as ϕ is contractive modulus $\phi(d(Sz, z)) < d(Sz, z)$.
Therefore $d(Sz, z) < d(Sz, z)$, which is a contradiction.
Therefore $Sz = z$.
Hence $Sz = Pz = z$.
Now, we show that z is a fixed point of T.
By (ii), we have

$$\begin{aligned} d(z,Tz) &= d(Sz,Tz) \\ &\leq \phi(\lambda(z,z)) \end{aligned}$$

where

$$\begin{aligned} \lambda(z,z) &= max\{d(Pz,Qz), d(Pz,Sz), d(Qz,Tz), \frac{1}{2}(d(Pz,Tz)+d(Qz,Sz))\} \\ &= max\{d(z,Tz), d(z,z), d(Tz,Tz), \frac{1}{2}(d(z,Tz)+d(Tz,z))\} \\ &= d(z,Tz) \end{aligned}$$

Thus $d(z,Tz) \leq \phi(d(z,Tz))$
If $z \neq Tz$ then $d(z,Tz) > 0$ and hence as ϕ is contractive modulus $\phi(d(z,Tz)) < d(z,Tz)$.
Therefore $d(z,Tz) < d(z,Tz)$, which is a contradiction.
Hence $z = Tz$.
Therefore $Tz = Qz = z$.
Therefore $Sz = Pz = Tz = Qz = z$. i.e. z is a common fixed point of P, Q, S and T.
Uniqueness : For uniqueness of z let if possible, we assume that z and w,($z \neq w$) are common fixed points of P, Q, S and T.
By (ii), we have

$$\begin{aligned} d(z,w) &= d(Sz,Tw) \\ &\leq \phi(\lambda(z,w)) \end{aligned}$$

where

$$\begin{aligned} (z,w) &= max\{d(Pz,Qw), d(Pz,Sz), d(Qw,Tw), \frac{1}{2}(d(Pz,Tw)+d(Qw,Sz))\} \\ &= max\{d(z,w), d(z,z), d(w,w), \frac{1}{2}(d(z,w)+d(w,z))\} \\ &= d(z,w) \end{aligned}$$

Thus $d(z,w) \leq \phi(d(z,w))$
Since $z \neq w$ then $d(z,w) > 0$ and hence as ϕ is contractive modulus $\phi(d(z,w)) < d(z,w)$.
Therefore $d(z,w) < d(z,w)$, which is a contradiction.
Therefore $z = w$.
Thus z is the unique common fixed point of P, Q, S and T.
Hence the theorem.

Corollary 3.2 *Let (X,d) be a complete metric space. Suppose that the mappings P, S and T are self-maps of X satisfying the following conditions:*

1. *T(X) $\subseteq$ P(X) and S(X) $\subseteq$ P(X);*
2. $d(Sx, Ty) \leq \phi(\lambda(x, y))$
   where ϕ is an upper semi continuous, contractive modulus and
   $\lambda(x, y) = max\{d(Px, Py), d(Px, Sx), d(Py, Ty), \frac{1}{2}(d(Px, Ty) + d(Py, Sx))\}$.
3. *The pairs (S, P) and (T, P) are weakly compatible*

Then P, S and T have a unique common fixed point.

proof: By taking $P = Q$ in theorem 3.1 we get the proof.

Corollary 3.3 *Let (X,d) be a complete metric space. Suppose that the mappings P and S are self-maps of X satisfying the following conditions:*

1. *S(X) $\subseteq$ P(X);*
2. $d(Sx, Sy) \leq \phi(\lambda(x, y))$
   where ϕ is an upper semi continuous, contractive modulus and
   $\lambda(x, y) = max\{d(Px, Py), d(Px, Sx), d(Py, Sy), \frac{1}{2}(d(Px, Sy) + d(Py, Sx))\}$.
3. *The pair (S, P) is weakly compatible*

Then P and S have a unique common fixed point.

proof: By taking $P = Q$ and $S = T$ in theorem 3.1 we get the proof.

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