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Monotonicity of Functions Involving Generalization Gamma Function

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Abstract

The aim of this paper is to show some monotonicity properties of some function involving generalization gamma function. The results are analogue of results concerning of q-Gamma function who proved Chrysi G. Kokologianaki.

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1 Introduction and preliminaries

The gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \qquad x > 0$$

was first introduced by the Swiss mathematician Leonhard Euler (1707-1783) in his goal to generalize the factorial to non integer values. Later, because of its great importance, it was studied by other eminent mathematicians like Adrien-Marie Legendre (1752-1833), Carl Friedrich Gauss (1777-1855), Christoph Gudermann (1798-1852), Joseph Liouville (1809-1882), Karl Weierstrass (1815-1897), Charles Hermite (1822-1901), ... as well as many others.

The gamma function belongs to the category of the special transcendental functions and we will see that some famous mathematical constants are occurring in its study

Gamma function is one of the most important special functions with applications in various fields, like analysis, mathematical physics, probability theory and statistics. Many interesting historical information on this function can be found in Davis' survey paper [4].

The logarithmic derivative of the gamma function is called the digamma function. It is know as the psi function and is denoted by $\psi(x)$.

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

The following integral and series representations are valid (see [1]):

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}dt = -\gamma - \frac{1}{x} + \sum_{n\geq 1} \frac{x}{n(n+x)} \tag{1}$$

Euler, gave another equivalent definition for the $\Gamma(x)$ (see [6]),

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)\cdots(x+p)} = \frac{p^x}{x(1+\frac{x}{1})\cdots(1+\frac{x}{p})}, \quad x > 0, \tag{2}$$

where

$$\Gamma(x) = \lim_{p\to\infty}\Gamma_p(x). \tag{3}$$

The p-analogue of the psi function is defined as the logarithmic derivative of the Γ_p function (see [6]), that is

$$\psi_p(x) = \frac{d}{dx}\ln\Gamma_p(x) = \frac{\Gamma_p'(x)}{\Gamma_p(x)}. \tag{4}$$

Theorem 1.1 *The function ψ_p defined in (1.4) satisfies the following properties (see [6]). It has the following series representation $\psi_p(x) = \ln p - \sum_{k=0}^{p}\frac{1}{x+k}$ $= \ln p - \int_0^\infty \frac{e^{-xt}}{1-e^{-t}}(1-e^{-pt})dt$. It is increasing on $(0,\infty)$ and it is strictly completely monotonic on $(0,\infty)$. This means that the inequality*

$$(-1)^m\Big(\psi_p'(x)\Big) > 0 \tag{5}$$

holds for $m = 0, 1, 2, \ldots$

It's derivatives are given by (see [6]):

$$\psi_p^{(n)}(x) = \sum_{k=0}^{p}\frac{(-1)^{n-1}\cdot n!}{(x+k)^{n+1}} = (-1)^{n+1}\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}(1-e^{-pt})dt. \tag{6}$$

For $0 < x < y$

$$\psi_p(x) - \psi_p(y) < 0 \tag{7}$$

Recall that a function h is (strictly) completely monotonic on $(0, \infty)$ if

$$(-1)^n f^{(n)}(x) \geq 0,$$

for every $x \in (0, \infty)$.

2 Main Results

Theorem 2.1 *Let $A \leq 0$ and $b \geq 0$. Then the function*

$$f_p(x) = x^A \Big[\Gamma_p\Big(1 + \frac{b}{x}\Big)\Big]^x$$

decreases with respect to $x > 0$.

The function $f_p(x)$ is positive for $x > 0$. Taking the derivatives of $f_p(x)$ with respect to x we obtain:

$$f_p^{'}(x) = \Big[\frac{A}{x} - \frac{b}{x}\psi_p\Big(1 + \frac{b}{x}\Big)\Big] f_p(x)$$

The function

$$h_p(x) = \frac{A}{x} - \frac{b}{x}\psi_p\Big(1 + \frac{b}{x}\Big)$$

or by setting $y = \frac{1}{x}$

$$s_p(y) = Ay - by\psi_p(1 + by)$$

is negative for $y > 0$ if $A \leq 0$ because $s_p^{'}(y) < 0$ and $s_p(y) < s_p(0)$ for $y > 0$. This means that the function $h_p(x)$ is also negative for $x > 0$, so from eku1 we obtain the desired function.

Theorem 2.2 *(i) Let $x > 0, Mx + N > 0, 0 < a + bx \leq d + ex$ and A, c, f real numbers with $AM \leq 0$ and $0 < cb \leq fe$. Then the function*

$$G_p(x) = (Mx + N)^A \frac{\Big[\Gamma_p(a + bx)\Big]^c}{\Big[\Gamma_p(d + ex)\Big]^f}$$

decrease with x.

(ii) Let $A \leq 0, M \geq 0.0 < a < d$ and $c > 0$. Then the function

$$g_p(x) = (Mx + N)^A \Big[\frac{\Gamma_p(a + x)}{\Gamma_p(d + x)}\Big]^c$$

is logarithmically completely monotonic for $x > \max\{0, -N/M\}$.

The derivative of the function $G_p(x)$ with respect of x is

$$G_p^{'}(x) = \Big[\frac{AM}{Mx+N} + cb\psi_p(a+bx) - fe\psi_p(d+ex)\Big]G_p(x) \tag{8}$$

Using proposition (1.1) for $n = 0$ and if $AM \leq 0$ and $0 < cd \leq fe$ from deri1 we obtain the desired result.

(ii) The function $g_p(x)$ becomes from $G_p(x)$ for $b = e = 1$ and $f = c > 0$ so deri1 gives

$$g_p^{'}(x) = \frac{AM}{Mx+N} + c\Big[\psi_p(a+x) - \psi_p(d+x)\Big]g_p(x) < 0 \tag{9}$$

Let $\alpha(x) = \frac{AM}{Mx+N}$ and $\beta(x) = \psi_p(a+x) - \psi_p(d+x)$, then deri2 can be written as

$$\frac{g_p^{'}(x)}{g_p(x)} = \alpha(x) + c\beta(x)$$

or

$$(\ln g_p(x))^{'} = \alpha(x) + c\beta(x) < 0 \tag{10}$$

We can verify by induction that $\alpha^{(k)}(x) = \frac{(-1)^k k! AM^{k+1}}{(Mx+N)^{k+1}}$, for $k = 0, 1, 2, \dots$, so the function $\alpha^{'}(x)$ is completely monotonic, for $A \leq 0$ and $M \geq 0$. Also using proposition (1.1) the function $\beta^{'}(x)$ is completely monotonic for $x > 0$, if $0 < a < d$, so from palidhje we obtain the desired result.

Theorem 2.3 *The function*

$$\theta(x) = \psi_p'(x) + \log\left(e^{\frac{1}{(x+p+1)^2} - \frac{1}{x^2}} - 1\right) \tag{11}$$

is strictly increasing on $(0, \infty)$.

It is well known that for $x > 0$

$$\Gamma_p(x+1) = \frac{px}{x+p+1}\Gamma_p(x). \tag{12}$$

Taking the logarithm on both sides and differentiating yields

$$\psi_p(x+1) = \frac{1}{x} - \frac{1}{x+p+1} + \psi_p(x).$$

Therefore, the exponential function of θ satisfies $\mathrm{e}^{\theta(x)} = e^{\psi_p'(x)} \cdot e^{\log\left(e^{\frac{1}{(x+p+1)^2} - \frac{1}{x^2}} - 1\right)} = e^{\psi_p'(x)} \cdot \left(e^{\frac{1}{(x+p+1)^2} - \frac{1}{x^2}} - 1\right)$
$= e^{\psi_p'(x) + \frac{1}{(x+p+1)^2} - \frac{1}{x^2}} - e^{\psi_p'(x)} = e^{\psi_p'(x+1)} - e^{\psi_p'(x)}$. Let $s(x) = e^{\psi_p'(x+1)} - e^{\psi_p'(x)}$. Then

$$s'(x) = e^{\psi_p'(x+1)}\psi_p''(x+1) - e^{\psi_p'(x)}\psi_p''(x) = h(x+1) - h(x),$$

where $h(x) = e^{\psi_p'(x)}\psi_p''(x)$. Then $h'(x) = e^{\psi_p'(x)}((\psi_p''(x))^2 + \psi_p'''(x))$, from psi$_s eries2weconcludethat$h'(x)¿0$sothefunction$h$isstrictlyincreasing.Itmeans$s'(x)¿0$for$x$\in$ $(0, \infty)$ and this yields that s and θ are strictly increasing functions on $(0, \infty)$.

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