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SIMPLICIAL AND CROSSED LIE ALGEBRAS

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(*communicated by Ronald Brown*)

Abstract

In this paper we examine higher order Peiffer elements in simplicial Lie algebras and apply them to the Lie 2-crossed module and Lie crossed squares introduced by Ellis.

Introduction

Simplicial Lie algebras arise in simplicial homotopical algebra, [**6**], [**13**], and their homotopy theory is of considerable interest. Kassel and Loday [**10**] (see also [**11**] and [**12**]) introduced crossed modules of Lie algebras as computational algebraic objects equivalent to simplicial Lie algebras with associated Moore complex of length 1. Following work of Conduché in a group-theoretic setting, Ellis [**9**] captured the algebraic structure of a Moore complex of length 2 in his definition of a 2-crossed module of Lie algebras. Within the homotopy theory of simplicial Lie algebras, analogues of Samelson and Whitehead products are given by sums over shuffles $(a;b)$ of Lie products of the form $[s_b x, s_a y]$. In this paper we explain the relationship of these shuffles to crossed modules and crossed 2-modules. More precisely, let **L** be a simplicial Lie algebra with Moore complex **NL**. Let ∂ denote the boundary homomorphism of the Moore complex. For $n > 1$ let D_n be the ideal in L_n generated by the degenerate elements. We show in Proposition 2.3 that if $L_n = D_n$, then

$$NL_n = I_n$$

where I_n is an ideal in L_n generated by certain shuffles. We use this equality to prove the following theorem (in which the face homomorphisms of the simplicial Lie algebra L are denoted by d_i, and for $I \subseteq \{1, \dots, n\}$ the intersection $\cap_{i\in I}\mathrm{Ker}d_i$ is denoted by K_I).

Theorem 1. Let **L** be a simplicial Lie algebra.
(i) $L_2 = D_2$ then $\partial_2(NL_2) = [\mathrm{Ker}d_0, \mathrm{Ker}d_1]$.
(ii) $L_3 = D_3$ then

$$\partial_3(NL_3) = [K_{\{0,1\}}, K_{\{0,2\}}] + [K_{\{0,2\}}, K_{\{1,2\}}] + [K_{\{0,1\}}, K_{\{1,2\}}] + \sum_{I,J}[K_I, K_J].$$

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where $I \cup J = \{0,1,2\}$, $I \cap J = \emptyset$.
(iii) If $L_n = D_n$ then

$$\partial_n(NL_n) \supseteq \sum_{I,J}[K_I, K_J]$$

where I, J are nonempty subsets of $\{0,\ldots,n-1\}$ with $I \cup J = \{0,\ldots,n-1\}$.
Part (i) of this theorem is an analogue of a group-theoretic result of Brown and Loday [**2**]

The paper is organised as follows. In Section 1 we recall some basics on simplicial Lie algebras and crossed modules. In Section 2 we prove Proposition 2.3. In Section 3 we prove Theorem 1. In Section 4 we use Theorem 1 to construct a functor from simplicial Lie algebras to 2-crossed modules; we also explain how Theorem 1 yields a functor from simplicial Lie algebras to the crossed n-cubes of Lie algebras introduced in [**8**].

1. Review of simplicial Lie algebras

All Lie algebras will be over a fixed commutative ring k.

A simplicial Lie algebra [**6**] **L** is a sequence of Lie algebras,

$$\mathbf{L} = \{L_0, L_1, \ldots L_n, \ldots\},$$

together with face and degeneracy maps

$$\begin{array}{ll} d_i = d_i^n: & L_n \to L_{n-1}, \quad 0 \leqslant i \leqslant n \quad (n \neq 0) \\ s_i = s_i^n: & L_n \to L_{n+1}, \quad 0 \leqslant i \leqslant n. \end{array}$$

These maps are required to satisfied the simplicial identities

$$\begin{array}{rcll} d_i d_j &=& d_{j-1} d_i & \text{for} \quad i < j \\ d_i s_j &=& \begin{cases} s_{j-1} d_i & \text{for} \quad i < j \\ \text{identity} & \text{for} \quad i = j, j+1 \\ s_j d_{i-1} & \text{for} \quad i > j+1 \end{cases} \\ s_i s_j &=& s_{j+1} s_i & \text{for} \quad i \leqslant j. \end{array}$$

It can be completely described as a functor $\mathbf{L}$: $\Delta^{op} \to \mathbf{LieAlg}_k$ where Δ is the category of finite ordinals $[n] = \{0 < 1 < \cdots < n\}$ and increasing maps.

Elements $x \in L_n$ are called *n-dimensional simplices.* A simplex x is called *degenerate* if $x = s_i(y)$ for some y.

A simplicial map $f : \mathbf{L} \to \mathbf{L}'$ is a family of homomorphisms $f_n : L_n \to L'_n$ commuting with the d_i and s_i. We let **SLA** denote the category of simplicial Lie algebras.

An essential reference from our point of view is Carrasco's thesis, [**3**], where many of the basic techniques used here were developed systematically for the first time and the notion of hypercrossed complex was defined (although in a different context).

The following notation and terminology is derived from the analogous group theoretic case treated in [**3**], [**4**]. For the ordered set $[n] = \{0 < 1 < \ldots < n\}$, let $\alpha_i^n : [n+1] \to [n]$ be the increasing surjective map given by

$$\alpha_i^n(j) = \begin{cases} j & \text{if} \quad j \leqslant i \\ j-1 & \text{if} \quad j > i. \end{cases}$$

Let $S(n, n-r)$ be the set of all monotone increasing surjective maps from $[n]$ to $[n-r]$. This can be generated from the various α_i^n by composition. The composition of these generating maps is subject to the following rule $\alpha_j \alpha_i = \alpha_{i-1}\alpha_j,\ j < i$. This implies that every element $\alpha \in S(n, n-r)$ has a unique expression as $\alpha = \alpha_{i_1} \circ \alpha_{i_2} \circ \ldots \circ \alpha_{i_r}$ with $0 \leqslant i_1 < i_2 < \ldots < i_r \leqslant n-1$, where the indices i_k are the elements of $[n]$ such that $\{i_1, \ldots, i_r\} = \{i : \alpha(i) = \alpha(i+1)\}$. We thus can identify $S(n, n-r)$ with the set $\{(i_r, \ldots, i_1) \ : \ 0 \leqslant i_1 < i_2 < \ldots < i_r \leqslant n-1\}$. In particular, the single element of $S(n, n)$, defined by the identity map on $[n]$, corresponds to the empty 0-tuple ( ) denoted by $\emptyset_n$. Similarly the only element of $S(n, 0)$ is $(n-1, n-2, \ldots, 0)$. For all $n \geqslant 0$, let

$$S(n) = \bigcup_{0 \leqslant r \leqslant n} S(n, n-r).$$

We say that $\alpha = (i_r, \ldots, i_1) < \beta = (j_s, \ldots, j_1)$ in $S(n)$

$$\begin{array}{llll} \text{if} & i_1 = j_1, \ldots, i_k = j_k & \text{but} & i_{k+1} > j_{k+1} \quad (k \geqslant 0) \text{ or} \\ \text{if} & i_1 = j_1, \ldots, i_r = j_r & \text{and} & r < s. \end{array}$$

This makes $S(n)$ an ordered set. For instance, the orders in $S(2)$ and in $S(3)$ are respectively:

$$\begin{array}{lll} S(2) & = & \{\emptyset_2 < (1) < (0) < (1,0)\}; \\ S(3) & = & \{\emptyset_3 < (2) < (1) < (2,1) < (0) < (2,0) < (1,0) < (2,1,0)\}. \end{array}$$

We also define $\alpha \cap \beta$ as the set of indices which belong to both α and β.

The Moore complex

The Moore complex $\mathbf{NL}$ of a simplicial Lie algebra $\mathbf{L}$ is the complex

$$\mathbf{NL}: \qquad \cdots \to NL_n \xrightarrow{\partial_n} NL_{n-1} \to \cdots \to NL_1 \xrightarrow{\partial_1} NL_0 \xrightarrow{\partial_0} 0$$

where

$$NL_0 = L_0, \qquad NL_n = \bigcap_{i=0}^{n-1} \mathrm{Ker} d_i, \qquad \partial_n = d_n \text{ (restricted to } NL_n).$$

Truncated Simplicial Lie Algebras

By an *m-truncated Simplicial Lie Algebra*, we mean a collection of Lie algebras $\{L_0, \ldots, L_m\}$ and homomorphisms $d_i : L_n \to L_{n-1}$ for $0 \leqslant i \leqslant n,\ 0 \leqslant n \leqslant m$ and $s_i : L_n \to L_{n+1}$ for $0 \leqslant i \leqslant n,\ 0 \leqslant n \leqslant m-1$ which satisfy the simplicial identities.

The Semidirect Decomposition of a Simplicial Lie Algebra

The fundamental idea behind this can be found in Conduché [**5**]. A detailed investigation of it for the case of a simplicial group is given in Carrasco and Cegarra [**4**]. The algebra case of that structure is also given in Carrasco's thesis [**3**].

Proposition 1.1. *If* $\boldsymbol{L}$ *is a Simplicial Lie Algebra, then for any* $n \geqslant 0$

$$\begin{aligned} L_n \quad \cong \quad & (\dots(NL_n \rtimes s_{n-1}NL_{n-1}) \rtimes \dots \rtimes s_{n-2}\dots s_0 NL_1) \rtimes \\ & (\dots(s_{n-2}NL_{n-1} \rtimes s_{n-1}s_{n-2}NL_{n-2}) \rtimes \dots \rtimes s_{n-1}s_{n-2}\dots s_0 NL_0). \end{aligned}$$

Proof: This is by repeated use of the following lemma. □

Lemma 1.2. *Let* $\boldsymbol{L}$ *be a Simplicial Lie Algebra. Then* L_n *can be decomposed as a semidirect product:*

$$L_n \cong \mathrm{Ker} d_n^n \rtimes s_{n-1}^{n-1}(L_{n-1}).$$

Proof: The isomorphism is defined as follows:

$$\begin{aligned} \theta : \quad L_n \quad & \longrightarrow \quad \mathrm{Ker} d_n^n \rtimes s_{n-1}^{n-1}(L_{n-1}) \\ l \quad & \longmapsto \quad (l - s_{n-1}d_n l, s_{n-1}d_n l). \end{aligned}$$

□

The bracketting and the order of terms in this multiple semidirect product are generated by the sequence:

$$\begin{aligned} L_1 \quad & \cong \quad NL_1 \rtimes s_0 NL_0 \\ L_2 \quad & \cong \quad (NL_2 \rtimes s_1 NL_1) \rtimes (s_0 NL_1 \rtimes s_1 s_0 NL_0) \\ L_3 \quad & \cong \quad ((NL_3 \rtimes s_2 NL_2) \rtimes (s_1 NL_2 \rtimes s_2 s_1 NL_1)) \rtimes \\ & \qquad ((s_0 NL_2 \rtimes s_2 s_0 NL_1) \rtimes (s_1 s_0 NL_1 \rtimes s_2 s_1 s_0 NL_0)). \end{aligned}$$

and

$$\begin{aligned} L_4 \quad \cong \quad & (((NL_4 \rtimes s_3 NL_3) \rtimes (s_2 NL_3 \rtimes s_3 s_2 NL_2)) \rtimes \\ & ((s_1 NL_3 \rtimes s_3 s_1 NL_2) \rtimes (s_2 s_1 NL_2 \rtimes s_3 s_2 s_1 NL_1))) \rtimes \\ & s_0(\text{decomposition of } L_3). \end{aligned}$$

Note that the term corresponding to $\alpha = (i_r, \dots, i_1) \in S(n)$ is

$$s_\alpha(NL_{n-\#\alpha}) = s_{i_r \dots i_1}(NL_{n-\#\alpha}) = s_{i_r} \dots s_{i_1}(NL_{n-\#\alpha}),$$

where $\#\alpha = r$. Hence any element $x \in L_n$ can be written in the form

$$x = y + \sum_{\alpha \in S(n)} s_\alpha(x_\alpha) \qquad \text{with } y \in NL_n \text{ and } x_\alpha \in NL_{n-\#\alpha}.$$

Crossed Modules of Lie Algebras

The notion of crossed module of Lie algebras was defined by Kassel and Loday [**10**].

Let M and P be two Lie algebras. By an *action* of P on M we mean a **k**-bilinear map $P \times M \to M$, $(p, m) \mapsto p \cdot m$ satisfying

$$\begin{aligned}
[p, p'] \cdot m &= p \cdot (p' \cdot m) - p'(p \cdot m) \\
p \cdot [m, m'] &= [p \cdot m, m'] + [m, p \cdot m']
\end{aligned}$$

for all $m, m' \in M, p, p' \in P$. For instance, if P is a subalgebra of some Lie algebra Q (including possibly the case $P = Q$), and if M is an ideal in Q, then Lie multiplication in Q yields an action of P on M.

Suppose that M and N are Lie algebras with an action of M on N and action of N on M. For any Lie algebra Q we call a bilinear function $h : M \times N \to Q$ a *Lie pairing* [**7**] if

$$\begin{aligned}
h([m, m'], n) &= h(m, m' \cdot n) - h(m', m \cdot n), \\
h(m, [n, n']) &= h(n' \cdot m, n) - h(n \cdot m, n'), \\
h(n \cdot m, m' \cdot n') &= -[h(m, n), h(m', n')],
\end{aligned}$$

for all $m, m' \in M, n, n' \in N$. For example if M and N are both ideals of some Lie algebra then the function $M \times N \to M \cap N$, $(m, n) \mapsto [m, n]$ is a Lie pairing.

Recall from [**10**] the notion of a crossed module of Lie algebras. *A crossed module of Lie algebras* is a Lie homomorphism $\partial : M \to P$ together with an action of P on M such that ,

$$\text{CM1)} \quad \partial(p \cdot m) = [p, \partial m] \qquad \text{CM2)} \quad \partial m \cdot m' = [m, m']$$

for all $m, m' \in M, p \in P$.

The second condition (CM2) is called the *Peiffer identity*. A standard example of a crossed module is any ideal I in P giving an inclusion map $I \to P$, which is a crossed module. Conversely, given any crossed module $\partial : M \to P$, the image $I = \partial M$ of M is an ideal in P.

2. Hypercrossed Complex Pairings and Boundaries in the Moore Complex

Lemma 2.1. *Let* $\boldsymbol{L}$ *be a simplicial Lie algebra and let* $\overline{NL}_n^{(r)} = \bigcap_{\substack{i=0 \\ i \neq r}}^{n} \mathrm{Ker} d_i$ *for* $0 \leqslant r \leqslant n$. *Then the mapping*

$$\varphi : NL_n \longrightarrow \overline{NL}_n^{(r)}$$

in L_n, *given by*

$$\varphi(l) = l - \sum_{k=0}^{n-r-1} (-1)^{k+1} s_{r+k} d_n l,$$

is a **k***-linear isomorphism.* □

This easily implies:

Lemma 2.2.

$$d_n(NL_n) = d_r(\overline{NL}_n^{(r)}).$$

□

Proof of Theorem 1 (iii): For any $J \subset [n-1], J \neq \emptyset$, let r be the smallest element of J. If $r = 0$, then replace J by I and restart and if $0 \in I \cap J$, then re-define r to be the smallest nonzero element of J. Otherwise continue.

Letting $l_0 \in \bigcap\limits_{j\in J} \mathrm{Ker} d_j$ and $l_1 \in \bigcap\limits_{i\in I} \mathrm{Ker} d_i$, one obtains

$$d_i[s_{r-1}l_0, s_r l_1] = 0 \text{ for } i \neq r$$

and hence $[s_{r-1}l_0, s_r l_1] \in \overline{NL}_n^{(r)}$. It follows that

$$[l_0, l_1] = d_r[s_{r-1}l_0, s_r l_1] \in d_r(\overline{NL}_n^{(r)}) = d_n NL_n \quad \text{by the previous lemma,}$$

and this implies

$$\left[\bigcap_{i\in I} \mathrm{Ker} d_i, \bigcap_{j\in J} \mathrm{Ker} d_j\right] \subseteq \partial_n NL_n.$$

Hypercrossed complex pairings

We recall from Carrasco [**3**] the construction of a family of k-linear morphisms. This was done there for associative algebras but adapts well to the Lie context. We define a set $P(n)$ consisting of pairs of elements (α, β) from $S(n)$ with $\alpha \cap \beta = \emptyset$, (for the definition of $\alpha \cap \beta$, see section 1), where $\alpha = (i_r, \dots, i_1), \beta = (j_s, ..., j_1) \in S(n)$. The k-linear morphisms that we will need,

$$\{M_{\alpha,\beta} : NL_{n-\#\alpha} \times NL_{n-\#\beta} \longrightarrow NL_n : (\alpha, \beta) \in P(n), \quad n \geqslant 0\}$$

are given as composites by the diagrams

$$\begin{array}{ccc} NL_{n-\#\alpha} \times NL_{n-\#\beta} & \xrightarrow{M_{\alpha,\beta}} & NL_n \\ {\scriptstyle s_\alpha \times s_\beta}\big\downarrow & & \big\uparrow{\scriptstyle p} \\ L_n \times L_n & \xrightarrow[{[\ ,\ ]}]{} & L_n \end{array}$$

where

$$s_\alpha = s_{i_r} \dots s_{i_1} : NL_{n-\#\alpha} \longrightarrow L_n \ , \quad s_\beta = s_{j_s} \dots s_{j_1} : NL_{n-\#\beta} \longrightarrow L_n,$$

$p : L_n \to NL_n$ is defined by composite projections $p = p_{n-1} \dots p_0$, where

$$p_j = 1 - s_j d_j \qquad \text{with} \quad j = 0, 1, \dots n-1$$

and we denote the Lie bracket by $[\ ,\ ] : L_n \times L_n \to L_n$. Thus

$$\begin{aligned} M_{\alpha,\beta}(x_\alpha, y_\beta) &= p[\ ,\ ](s_\alpha \times s_\beta)(x_\alpha, y_\beta) \\ &= p([s_\alpha(x_\alpha), s_\beta(y_\beta)]) \\ &= (1 - s_{n-1}d_{n-1}) \dots (1 - s_0 d_0)([s_\alpha(x_\alpha), s_\beta(y_\beta)]). \end{aligned}$$

We now define the ideal I_n to be that generated by all elements of the form

$$M_{\alpha,\beta}(x_\alpha, y_\beta)$$

where $x_\alpha \in NL_{n-\#\alpha}$ and $y_\beta \in NL_{n-\#\beta}$ and for all $(\alpha,\beta) \in P(n)$.

Consider $M_{\alpha,\beta}(x_\alpha, y_\beta)$ and $M_{\beta,\alpha}(y_\beta, x_\alpha)$, here one uses $[s_\alpha(x_\alpha), s_\beta(y_\beta)]$, the other

$$[s_\beta(y_\beta), s_\alpha(x_\alpha)] = -[s_\alpha(x_\alpha), s_\beta(y_\beta)],$$

so the changing α and β gives the only minus sign.

Example For $n = 2$, suppose $\alpha = (1),\ \beta = (0)$ and $x, y \in NL_1 = \mathrm{Ker}d_0$. It follows that

$$\begin{aligned} M_{(1)(0)}(x,y) &= p_1p_0[s_1x, s_0y] \\ &= [s_1x, s_0y] - [s_1x, s_1y] \\ &= [s_1x, s_0y - s_1y] \end{aligned}$$

and these give the generator elements of the ideal I_2.

For $n = 3$, the linear morphisms are the following

$$\begin{array}{lll} M_{(1,0)(2)}, & M_{(2,0)(1)}, & M_{(2,1)(0)}, \\ M_{(2)(0)}, & M_{(2)(1)}, & M_{(1)(0)}. \end{array}$$

For all $x \in NL_1,\ y \in NL_2$, the corresponding generators of I_3 are:

$$\begin{aligned} M_{(1,0)(2)}(x,y) &= [s_1s_0x - s_2s_0x, s_2y], \\ M_{(2,0)(1)}(x,y) &= [s_2s_0x - s_2s_1x, s_1y - s_2y], \\ M_{(2,1)(0)}(x,y) &= [s_2s_1x, s_0y - s_1y + s_2y]; \end{aligned}$$

whilst for all $x,\ y \in NL_2$,

$$\begin{aligned} M_{(1)(0)}(x,y) &= [s_1x, s_0y - s_1y] + [s_2x, s_2y], \\ M_{(2)(0)}(x,y) &= [s_2x, s_0y], \\ M_{(2)(1)}(x,y) &= [s_2x, s_1y - s_2y]. \end{aligned}$$

Proposition 2.3. *Let $\boldsymbol{L}$ be a simplicial Lie algebra and $n > 0$, and D_n the ideal in L_n generated by degenerate elements. We suppose $L_n = D_n$, and let I_n be ideal generated by elements of the form*

$$M_{\alpha,\beta}(x_\alpha, y_\beta) \qquad \text{with } (\alpha,\beta) \in P(n)$$

where $x_\alpha \in NL_{n-\#\alpha},\ y_\beta \in NL_{n-\#\beta}$. Then

$$NL_n = I_n.$$

As corollary we, of course, have that the image of N_n is equal to the image of NL_n, i.e., $\partial_n(NL_n) = \partial_n(I_n)$.

We omit its proof which can be obtained by changing slightly the corresponding results in [**1**]

3. Proof of first two parts of Theorem 1

Proof of Theorem 1 (i): We know that any element l_2 of L_2 can be expressed in the form

$$l_2 = b + s_1 y + s_0 x + s_0 u$$

with $b \in NL_2, x, y \in NL_1$ and $u \in s_0 L_0$. We suppose $D_2 = L_2$. For $n = 1$, we take $\alpha = (1),\ \beta = (0)$ and $x, y \in NL_1 = \mathrm{Ker} d_0$. The ideal I_2 is generated by elements of the form

$$M_{(1)(0)}(x, y) = [s_1 x, s_0 y - s_1 y].$$

The image of I_2 by ∂_2 is known to be $[[\mathrm{Ker} d_0, \mathrm{Ker} d_1]]$ by direct calculation. Indeed,

$$\begin{array}{rcl} d_2[M_{(1)(0)}(x, y)] & = & d_2[s_1 x, s_0 y - s_1 y] \\ & = & [x, s_0 d_1 y - y] \end{array}$$

where $x \in \mathrm{Ker} d_0$ and $[x, s_0 d_1 y - y] \in \mathrm{Ker} d_1$ and all elements of $\mathrm{Ker} d_1$ have this form due to Lemma 2.1.

As $\partial = \partial_1$ restricted to NL_1, this is precisely $d_2(M_{(1)(0)}(x, y))$. In other words the ideal ∂I_2 is the 'Peiffer ideal' of the precrossed module $\partial : NL_1 \to NL_0$, whose vanishing is equivalent to this being a crossed module. The description of ∂I_2 as $[\mathrm{Ker} d_0, \mathrm{Ker} d_1]$ gives that its vanishing in this situation is module-like behaviour since a module, M, is a Lie algebra with $[M, M] = 0$. Thus if $(\mathbf{NL}, \partial)$ yields a crossed module this fact will be reflected in the internal structure of $\mathbf{L}$ by the vanishing of $[\mathrm{Ker} d_0, \mathrm{Ker} d_1]$. Because the image of this $M_{(1)(0)}(x, y)$ is the Peiffer element determined by x and y, we will call the $M_{\alpha,\beta}(x, y)$ in higher dimensions *higher order Peiffer elements* and will seek similar internal conditions for their vanishing.

We have seen that in all dimensions

$$\sum_{I,J} [K_I, K_J] \subseteq \partial_n(NL_n) = \partial I_n$$

and we will show shortly that this inclusion is an equality, not only in dimension 2 (as above), but also in dimension 3 and 4. The arguments are calculatory and do not generalise in an obvious way to higher dimensions although similar arguments can be used to get partial results there.

Proof of Theorem 1 (ii): By Proposition 2.3, we know the generator elements of the ideal I_3 and $\partial_3(I_3) = \partial_3(NL_3)$. For each pair $\alpha, \beta \in S(3)$ with $\emptyset_3 < \alpha < \beta$ and $\alpha \cap \beta = \emptyset$, we take $x \in NL_{3-\#\alpha}, y \in NL_{3-\#\beta}$ and set $M_{\alpha,\beta}(x, y) = p_3 p_2 p_1 [s_\alpha(x), s_\beta(y)]$ where $p_i(l) = l - s_i d_i(l)$. This element is thus in NL_3. The valid pairs together with their corresponding pairing functions is given in the following table:

	α	β	$M_{\alpha,\beta}(x,y)$
1	(1,0)	(2)	$[s_1s_0x - s_2s_0x, s_2y]$
2	(2,0)	(1)	$[s_2s_0x - s_2s_1x, s_1y - s_2y]$
3	(2,1)	(0)	$[s_2s_1x, s_0y - s_1y + s_2y]$
4	(2)	(1)	$[s_1x, s_0y - s_1y] + [s_2x, s_2y]$
5	(2)	(0)	$[s_2x, s_0y]$
6	(1)	(0)	$[s_2x, s_1y - s_2y]$

The explanation of this table is the following:

$\partial_3 M_{\alpha,\beta}(x,y)$ is in $[K_I, K_J]$ in the simple cases corresponding to the first 4 rows. In row 5, $\partial_3 M_{(2)(0)}(x,y) \in [K_{\{0,1\}}, K_{\{1,2\}}] + [K_{\{0,1\}}, K_{\{0,2\}}]$ and similarly in row 6, the higher Peiffer element is in the sum of the indicated $[K_I, K_J]$. To illustrate the sort of argument used we look at the case of $\alpha = (1,0)$ and $\beta = (2)$, i.e. row 1. For $x \in NL_1$ and $y \in NL_2$,

$$\begin{aligned} d_3[M_{(1,0)(2)}(x,y)] &= d_3[s_1s_0x - s_2s_0x, s_2y] \\ &= [s_1s_0d_1x - s_0x, y] \end{aligned}$$

and so

$$d_3[M_{(1,0)(2)}(x,y)] = [s_1s_0d_1x - s_0x, y] \in [\mathrm{Ker}d_2, \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1].$$

We have denoted $[\mathrm{Ker}d_2, \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1]$ by $[K_{\{2\}}, K_{\{0,1\}}]$ where $I = \{2\}$ and $J = \{0,1\}$. Rows 2, 3 and 4 are similar. For Row 5, $\alpha = (2)$, $\beta = (0)$ with $x, y \in NL_2 = \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1$,

$$\begin{aligned} d_3[M_{(2)(0)}(x,y)] &= d_3[s_2x, s_0y] \\ &= [x, s_0d_2y]. \end{aligned}$$

We can assume, for $x, y \in NL_2$,

$$x \in \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1 \quad \text{and} \quad y + s_0d_2y - s_1d_2y \in \mathrm{Ker}d_1 \cap \mathrm{Ker}d_2$$

and, multiplying them together,

$$\begin{aligned} [x, y + s_0d_2y - s_1d_2y] &= [x, y - s_1d_2y] + [x, s_0d_2y] \\ &= d_3[M_{(2)(1)}(x,y)] + d_3[M_{(2)(0)}(x,y)] \end{aligned}$$

and so

$$\begin{aligned} d_3[M_{(2)(0)}(x,y)] &\in [K_{\{0,1\}}, K_{\{1,2\}}] + d_3[M_{(2)(1)}(x,y)] \\ &\subseteq [K_{\{0,1\}}, K_{\{1,2\}}] + [K_{\{0,1\}}, K_{\{0,2\}}]. \end{aligned}$$

For Row 6, for $\alpha = (1)$, $\beta = (0)$ and $x, y \in NL_2 = \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1$,

$$\begin{aligned} d_3[M_{(1)(0)}(x,y)] &= d_3\big([s_1x, s_0y - s_1y] + [s_2x, s_2y]\big) \\ &= [s_1d_2x, s_0d_2y - s_1d_2x] + [x, y]. \end{aligned}$$

We can take the following elements

$$(s_0d_2y - s_1d_2y + y) \in \mathrm{Ker}d_1 \cap \mathrm{Ker}d_2 \quad \text{and} \quad (s_1d_2x - x) \in \mathrm{Ker}d_0 \cap \mathrm{Ker}d_2.$$

When we multiply them together, we get

$$\begin{aligned}[s_0d_2y - s_1d_2y + y, s_1d_2x - x] &= [s_0d_2y, s_1d_2x] - [s_1d_2y, s_1d_2x] + [y, x] \\ &\quad -[x, s_0d_2y] + [x, (s_1d_2y - y)] \\ &\quad +[y, (s_1d_2x - x)] \\ &= d_3[M_{(1)(0)}(x, y)] - d_3[M_{(2)(0)}(x, y)] + \\ &\quad d_3[M_{(2)(1)}(x, y) + M_{(2)(1)}(y, x)]\end{aligned}$$

and hence

$$d_3[M_{(1)(0)}(x, y)] \quad \in \quad [K_{\{0,2\}}, K_{\{1,2\}}] + [K_{\{0,1\}}, K_{\{1,2\}}] + [K_{\{0,1\}}, K_{\{0,2\}}].$$

So we have shown

$$\partial_3 I_3 \subseteq \sum_{I,J} [K_I, K_J] + [K_{\{0,1\}}, K_{\{0,2\}}] + [K_{\{0,2\}}, K_{\{1,2\}}] + [K_{\{0,1\}}, K_{\{1,2\}}].$$

The opposite inclusion can be verified by using proposition 2.3. Therefore

$$\begin{aligned}\partial_3(NL_3) = \;& [\mathrm{Ker}d_2, (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1)] + [\mathrm{Ker}d_1, (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2)] + \\ & [\mathrm{Ker}d_0, (\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2)] + [(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1), (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2)] + \\ & [(\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2), (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2)] + [(\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2), (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1)]\end{aligned}$$

This completes the proof of Theorem 1 (ii).

4. Application to 2-Crossed Modules and Crossed Squares of Lie Algebras

The following definition is due to Ellis [**9**].

Definition 4.1. *A 2-crossed module of Lie algebras consists of a complex of Lie algebras*

$$M_2 \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0$$

with ∂_2, ∂_1 morphisms of Lie algebras, where the algebra M_0 acts on itself by Lie bracket M_0 acts on M_1 and M_2 such that

$$M_2 \xrightarrow{\partial_2} M_1$$

is a crossed module in which M_1 acts on M_2 via M_0, further, there is a M_0-bilinear function giving

$$\{\quad,\quad\} : M_1 \times M_1 \longrightarrow M_2,$$

called a Peiffer lifting, which satisfies the following axioms:

$$
\begin{array}{llcl}
PL1: & \partial_2\{y_0, y_1\} & = & [y_0, y_1] - y_0 \cdot \partial_1(y_1), \\
PL2: & \{\partial_2(x_1), \partial_2(x_2)\} & = & [x_1, x_2], \\
PL3: & \{\partial_2(x), y\} & = & y \cdot x - \partial_1(y) \cdot x, \\
PL4: & \{y, \partial_2(x)\} & = & y \cdot x, \\
PL5: & \{y_0, y_1\} \cdot z & = & \{y_0 \cdot z, y_1\} + \{y_0, y_1 \cdot z\}, \\
PL6: & \{[y_0, y_1], y_2\} & = & \{y_0, y_2\} \cdot \partial_1 y_1 + \{y_1, [y_0, y_2]\} \\
 & & & -\{y_1, y_2\} \cdot \partial_1 y_0 - \{y_0, [y_1, y_2]\}, \\
PL7: & \{y_0, [y_1, y_2]\} & = & \{y_0, y_1\} \cdot \partial_1 y_2 - \{y_0, y_2\} \cdot \partial_1 y_1 \\
 & & & +\{y_1, [y_0, y_2] - y_0 \cdot \partial_1 y_2\} - \{y_2, [y_0, y_1] - y_0 \cdot \partial_1 y_1\},
\end{array}
$$

for all $x, x_1, x_2 \in M_2,\ y,\ y_0,\ y_1,\ y_2 \in M_1$ *and* $z \in M_0$.

We denote such a 2-crossed module of algebras by $\{M_2, M_1, M_0, \partial_2, \partial_1\}$.

The following result is an analogous result of the commutative algebra version, cf. [**1**].

Proposition 4.2. *Let* $\boldsymbol{L}$ *be a simplicial Lie algebra with the Moore complex* $\boldsymbol{NL}$*. Then the complex of Lie algebras*

$$NL_2/\partial_3(NL_3 \cap D_3) \xrightarrow{\overline{\partial}_2} NL_0$$

is a 2-crossed module of Lie algebras, where the Peiffer map is defined as follows:

$$
\begin{array}{rcl}
\{\ \ ,\ \ \}: \ NL_1 \times NL_1 & \longrightarrow & NL_2/\partial_3(NL_3 \cap D_3) \\
(y_0, y_1) & \longmapsto & \overline{[s_1 y_0, s_1 y_1 - s_0 y_1]}.
\end{array}
$$

Here the right hand side denotes a coset in $NL_2/\partial_3(NL_3 \cap D_3)$ *represented by the corresponding element in* NL_2.

Proof: We will show that all axioms of a 2-crossed module are verified. It is readily checked that the morphism $\overline{\partial}_2 : NL_2/\partial_3(NL_3 \cap D_3) \to NL_1$ is a crossed module. (In the following calculations we display the elements omitting the overlines.)

PL1:

$$
\begin{array}{rcl}
\overline{\partial}_2\{y_0, y_1\} & = & \partial_2[s_1 y_0, s_1 y_1 - s_0 y_1] \\
 & = & [y_0, y_1] - y_0 \cdot \partial_1 y_1.
\end{array}
$$

PL2: From $\partial_3(M_{(1)(0)}(x_1, x_2)) = [s_1 d_2(x_1), s_0 d_2(x_2) - s_1 d_2(x_2)] + [x_1, x_2]$, one obtains

$$
\begin{array}{rcl}
\{\overline{\partial}_2(x_1), \overline{\partial}_2(x_2)\} & = & [s_1 d_2 x_1, s_1 d_2 x_2 - s_0 d_2 x_2] \\
 & \equiv & [x_1, x_2] \mod \partial_3(NL_3 \cap D_3).
\end{array}
$$

PL3:

$$
\{\overline{\partial}_2(x), y\} = [s_1 \partial_2 x, s_1 y - s_0 y],
$$

but

$$\partial_3(M_{(2,0)(1)}(y,x)) = [s_0y - s_1y, s_1d_2x] - [s_0y - s_1y, x] \in \partial_3(NL_3 \cap D_3)$$

and

$$\partial_3(M_{(1,0)(2)}(y,x)) = [s_1s_0d_1y - s_0y, x] \in \partial_3(NE_3 \cap D_3),$$

so then

$$\begin{aligned} \{\overline{\partial}_2(x), y\} &\equiv [s_1(y), x] - [s_0(y), x] && \text{mod } \partial_3(NL_3 \cap D_3) \\ &= y \cdot x - \partial_1(y) \cdot x && \text{by the definition of the action,} \end{aligned}$$

PL4: since $\partial_3(M_{(2,1)(0)}(y,x)) = [s_1y, s_0d_2x - s_1d_2x] + [s_1(y), x]$,

$$\begin{aligned} \{y, \overline{\partial}_2(x)\} &= [s_1y, s_1\partial_2x - s_0\partial_2x] \\ &\equiv [s_1(y), x] \quad \text{mod } \partial_3(NL_3 \cap D_3) \\ &= y \cdot x \qquad \text{by the definition of the action.} \end{aligned}$$

PL5: By the definition of the action, we get

$$\{y_0, y_1\} \cdot z = \{y_0 \cdot z, y_1\} + \{y_0, y_1 \cdot z\}$$

with $x, x_1, x_2 \in NL_2/\partial_3(NL_3 \cap D_3)$, $y, y_0, y_1, y_2 \in NL_1$ and $z \in NL_0$. Verification of axioms PL6 and PL7 are omitted as they are routine. This completes the proof of the proposition. □

This only used the higher dimension Peiffer elements. A result in terms of $[K_I, K_J]$ vanishing can also be given:

Proposition 4.3. *If in a simplicial Lie algebra* $\mathbf{L}$*, one has* $[K_I, K_J] = 0$ *in dimension 2 for the following cases:* $I \cup J = [2], I \cap J = \emptyset; I = \{0,1\}, J = \{0,2\}$ *or* $I = \{1,2\}$*; and* $I = \{0,2\}, J = \{1,2\}$ *then*

$$NL_2 \longrightarrow NL_1 \longrightarrow NL_0$$

can be given the structure of a 2-crossed module.

Another application of higher order Peiffer elements is a Lie crossed square. First we recall from [**8**] the notion of crossed n-cubes of Lie algebras.

A crossed n-cube of Lie algebras is a family of Lie algebras, M_A for $A \subseteq <n> = \{1, ..., n\}$ together with homomorphisms $\mu_i : M_A \to M_{A-\{i\}}$ for $i \in <n>$ and for $A, B \subseteq <n>$, functions

$$h : M_A \times M_B \longrightarrow M_{A \cup B}$$

such that for all $\lambda \in \Lambda,\ a, a' \in M_A,\ b, b' \in M_B,\ c \in M_C,\ i, j \in <n>$ and $A \subseteq B$

$$
\begin{array}{lrcl}
1) & \mu_i a & = & a \quad \text{if } i \not\in A \\
2) & \mu_i \mu_j a & = & \mu_j \mu_i a \\
3) & \mu_i h(a,b) & = & h(\mu_i a, \mu_i b) \\
4) & h(a,b) & = & h(\mu_i a, b) = h(a, \mu_i b) \quad \text{if } i \in A \cap B \\
5) & h(a,a') & = & [a,a'] \\
6) & h(a,a) & = & 0 \\
7) & h(a+a',b) & = & h(a,b) + h(a',b) \\
8) & h(a,b+b') & = & h(a,b) + h(a,b') \\
9) & \lambda h(a,b) & = & h(\lambda a, b) = h(a, \lambda b) \\
10) & h(h(a,b),c) & + & h(h(b,c),a) + h(h(c,a),b) = 0.
\end{array}
$$

A morphism of crossed n-cubes is defined in the obvious way. We thus denote a category of crossed n-cubes by $\mathbf{Crs^n}$.

Theorem 1 (iii) can be used to verify that the following construction in a functor from simplicial Lie algebras to crossed n-cubes.

For a simplicial Lie algebra $\mathbf{L}$ and a given n, we write $\mathbf{M}(\mathbf{L},\ n)$ for the crossed n-cube, arising from the functor

$$\mathbf{M}(-,\mathbf{n}) : \mathbf{SLA} \longrightarrow \mathbf{Crs^n}.$$

Then the crossed n-cube $\mathbf{M}(\mathbf{L}, n)$ is determined by:

(i) for $A \subseteq <n>$,

$$\mathbf{M}(\mathbf{L}, n)_A = \frac{\bigcap_{j\in A} \mathrm{Ker} d^n_{j-1}}{d^{n+1}_{n+1}(\mathrm{Ker} d^{n+1}_0 \cap \{\bigcap_{j\in A} \mathrm{Ker} d^{n+1}_j\});}$$

(ii) the inclusion

$$\bigcap_{j\in A} \mathrm{Ker} d^n_{j-1} \longrightarrow \bigcap_{j\in A-\{i\}} \mathrm{Ker} d^n_{j-1}$$

induces the morphism

$$\mu_i : \mathbf{M}(\mathbf{L}, n)_A \longrightarrow \mathbf{M}(\mathbf{L}, n)_{A-\{i\}};$$

(iii) the functions, for $A, B \subseteq <n>$,

$$h : \mathbf{M}(\mathbf{L}, n)_A \times \mathbf{M}(\mathbf{L}, n)_B \longrightarrow \mathbf{M}(\mathbf{L}, n)_{A\cup B}$$

given by

$$h(\bar{x}, \bar{y}) = \overline{[x,y]},$$

where an element of $\mathbf{M}(\mathbf{L}, n)_A$ is denoted by $\bar{x}$ with $x \in \bigcap_{j\in A} \mathrm{Ker} d^n_{j-1}$.

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