GENERATION THEORY FOR SEMIGROUPS OF HOLOMORPHIC MAPPINGS IN BANACH SPACES

SIMEON REICH AND DAVID SHOIKHET

Abstract. We study nonlinear semigroups of holomorphic mappings in Banach spaces and their infinitesimal generators. Using resolvents, we characterize, in particular, bounded holomorphic generators on bounded convex domains and obtain an analog of the Hille exponential formula. We then apply our results to the null point theory of semi-plus complete vector fields. We study the structure of null point sets and the spectral characteristics of null points, as well as their existence and uniqueness. A global version of the implicit function theorem and a discussion of some open problems are also included.

Introduction

Nonlinear semigroup theory is not only of intrinsic interest, but is also important in the study of evolution problems. In recent years many developments have occurred, in particular, in the area of nonexpansive semigroups in Banach spaces.

As a rule, such semigroups are generated by accretive operators and can be viewed as nonlinear analogs of the classical linear contraction semigroups. See, for example, [10, 9] and [55]. Another class of nonlinear semigroups consists of those semigroups generated by holomorphic mappings. Such semigroups appear in several diverse fields, including, for example, the theory of Markov stochastic branching processes [28, 64], Krein spaces [72, 73], the geometry of complex Banach spaces [7, 67], control theory and optimization [32]. These semigroups can be considered natural nonlinear analogs of semigroups generated by (bounded) linear operators.

These two distinct classes of nonlinear semigroups are also related by the fact that holomorphic self-mappings are nonexpansive with respect to Schwarz-Pick pseudometrics.

1991 *Mathematics Subject Classification.* 32H15, 34G20, 46G20, 47H10, 47H15, 47H20.

Key words and phrases. Banach space, Cauchy problem, exponential formula, holomorphic generator, hyperbolic metric, Lie generator, nonlinear semigroup, null point, resolvent, spectrum.

Received: September 20, 1995.

©1996 Mancorp Publishing, Inc.

In this connection, see [61] and [59] for the case of hyperbolic spaces and, in particular, the Hilbert ball. For real analytic semiflows see [6]. In the finite dimensional case, a characterization of holomorhpic generators in terms of Finsler metrics is given in [2].

The present work is devoted to semigroups of holomorphic mappings in Banach spaces. When the generators are Fredholm operators, several results were obtained in [43]. We use a different approach in the spirit of the Hille-Yosida theory. Variants of this approach may be found, for example, in [57, 58, 53, 54, 44] and in the references mentioned there.

It the first section we recall some basic properties of holomorphic mappings in Banach spaces. We also include several known results in the fixed point theory of such mappings which will be used in the sequel.

In §2 we consider nonlinear semigroups of holomorphic mappings and their infinitesimal generators. We also introduce semi-plus complete vector fields (Definition 2.4) and compare them with infinitesimal generators. We show, in particular, that for bounded holomorphic mappings these two notions coincide (Proposition 2.2). Moreover, it follows that any strongly continuous semigroup with a bounded holomorphic generator is, in fact, continuous with respect to the topology of local uniform convergence over D. A crucial point in this section is Lemma 2.1, which shows that any set of uniformly bounded generators is sequentially closed with respect to this topology.

Since a bounded holomorphic mapping is locally Lipschitzian, the theory of bounded holomorphic generators turns out to be closely connected to globally Lipschitzian generators. Therefore in §3 we give several geometric and analytic criteria for a Lipschitzian holomorphic mapping to be a generator. These will be needed later.

The principal results of our paper are established in Section 4. Theorem 4.1 provides the following characterization of bounded holomorphic generators on a bounded convex domain D in a Banach space X: A bounded mapping $f \in \text{Hol}\,(D, X)$ generates a one-parameter semigroup of holomorphic self-mappings of D if and only if for each positive r its resolvent $(I + rf)^{-1}$ exists and is a holomorphic self-mapping of D.

The question whether the sum of two generators is also a generator is of interest in many areas. This is certainly true in the case of generators of groups of holomorphic automorphisms because the set of all such generators is known [35] to be a real Banach Lie algebra. The latter fact is no longer true for semicomplete vector fields. Nevertheless, it is a consequence of Theorem 4.1 that the family of bounded semigroup generators is a real convex cone (Corollary 4.4).

The above-mentioned question is related to the method of product formulas which generalizes the exponential representation of semigroups. Combining a Lie algebraic approach with our results, we also obtain a complete analog of the Hille exponential formula for semigroups generated by holomorphic mappings (Theorem 4.2).

Another important consequence of Theorem 4.1 is that if F is a holomorphic self-mapping of D, then $f = I - F$ is a generator of a one-parameter semigroup (Proposition 4.3). Thus the well-developed fixed point theory for holomorphic self-mappings can be viewed as a special case of the null point theory of semi-plus complete vector fields. We study this subject in Sections 5, 6 and 7.

More precisely, §5 is devoted to the structure of the null point sets of generators and their difference approximations. In §6 we study the spectral characteristics of null points. We show that such local properties can influence the global structure of the whole null point set and the asymptotic behavior of the semigroup. Some new sufficient conditions for the existence and uniqueness of null points are presented in §7.

Section 8 is devoted to a global version of the implicit function theorem. In particular, Theorem 8.1 is a complete generalization of the uniform fixed point principle in [42]. In the last section we discuss several open problems.

1. Holomorphic mappings in Banach spaces

1.1. **Some basic properties.** Let X and Y be complex Banach spaces, and let $D \subset X$ and $\tilde{D} \subset Y$ be domains, i.e. nonempty connected open subsets of X and Y, respectively.

Definition 1.1. *A mapping $f\colon D \mapsto \tilde{D}$, defined on D with values in $\tilde{D}$, is said to be holomorphic on D if it is Fréchet differentiable at each point in D.*

The Fréchet derivative $f'(x)$ at $x \in D$ is a bounded (complex) linear operator of X into Y.

The set of holomorphic mappings of D into $\tilde{D}$ will be denoted by
Hol $(D, \tilde{D})$.

Definition 1.2. *A subset $K \subset D$ is said to be strictly inside D, in symbols $K \subset\subset D$, if*

$$\inf\{\|x-y\| : x \in K \text{ and } y \in X \setminus D\} > 0.$$

Sometimes such a subset K is said to be completely interior to D (see, for example, [22] *and* [35]*).*

The following concepts and propositions can be found, for example, in [34], as well as in [22] and [35].

Proposition 1.1. (Power series representation) *Let $f \in \mathrm{Hol}\,(D, \tilde{D})$ and let $\tilde{D}$ be bounded. Then for each $x_0 \in D$ and for each ball $B \subset\subset D$ centered at x_0, the following representation holds:*

$$f(x) = \sum_{k=0}^{\infty} P_f^{(k)}(x_0) \cdot (x - x_0), \qquad x \in B, \tag{1.1}$$

where $P_f^{(k)}(x_0)$, $k = 0, 1, 2, \dots$, are homogeneous forms (polynomials) of order k and $P_f^{(0)}(x_0) = f(x_0)$.

Furthermore, the Fréchet derivatives $f^{(k)}(x_0)$ of all orders $k = 0, 1, 2, \ldots$ exist and

$$P_f^{(k)}(x_0)v = \frac{1}{k!} f^{(k)}(x_0)(v, v, \ldots, v), \quad v \in X.$$

In addition, the convergence in (1.1) is uniform in B.

Proposition 1.2. (The Cauchy inequalities) *In the setting of Proposition 1.1, let r be the radius of the ball B, and let $\|f(x)\|_Y \leq M$ for all $x \in B$. Then for each $k = 0, 1, 2, \ldots$ we have*

$$\|P_f^{(k)}(x_0)\|_{L(X^k,Y)} \leq M r^{-k} \tag{1.2}$$

where by $L(X^k, Y)$ we denote the space of all multilinear bounded operators from X^k into Y.

Definition 1.3. *Let $f \in \mathrm{Hol}\,(D, Y)$ and let $f(x_0) = 0 \in Y$ for some point $x_0 \in D$ (we will write in this case $x_0 \in \mathrm{Null}_D f$). We say that x_0 is a null point of f in D of order m if in the formula (1.1), $P_f^{(k)}(x_0) \equiv 0$ for all $k = 0, 1, \ldots, m-1$ and $P_m(x_0) \neq 0$.*

Proposition 1.3. (Schwarz lemma, see [41]) *Let $f \in \mathrm{Hol}\,(D, Y)$ have a null point $x_0 \in D$ of order m, and suppose that $\|f(x)\|_Y \leq M$ for all $x \in D$. Then for each ball $B_r \subset\subset D$ (of radius r) centered at x_0 and each $x \in B_r$ we have*

$$\|f(x)\|_Y \leq M(\|x - x_0\|/r)^m. \tag{1.3}$$

1.2. **Topology of local uniform convergence and T-attractivity.** In this section we follow in principle the notations and definitions given in [22] and [35]. As above, let D and $\tilde{D}$ be domains in X and Y, respectively.

For $f \in \mathrm{Hol}\,(D, Y)$ and $K \subset\subset D$, we set $\|f(x)\|_K = \sup\limits_{x \in K} \|f(x)\|$.

Definition 1.4. *A net $\{f_j\}_{j \in \mathcal{A}} \subset \mathrm{Hol}\,(D, \tilde{D})$ is said to converge to a mapping $f \in \mathrm{Hol}\,(D, Y)$ in the topology of local uniform convergence over D (or briefly T-converge) if for every ball $B \subset\subset D$*

$$\lim_{j \in \mathcal{A}} \|f_j - f\|_B = 0.$$

We write in this case $f = T\text{-}\lim_{j \in \mathcal{A}} f_j$.

Proposition 1.4. (Vitali's property [22] and [35]) Let D and $\tilde{D}$ be bounded domains in X and Y. Let $\{f_j\}_{j \in \mathcal{A}}$ be a net of holomorphic mappings of D into $\tilde{D}$. Then the following assertions are equivalent:

1) $\{f_j\}_{j \in \mathcal{A}}$ T-converges to $f \in \mathrm{Hol}\,(D, Y)$, i.e. $f = \mathrm{T}\text{-}\lim_{j \in \mathcal{A}} f_j$;

2) There exists a ball $B \subset\subset D$ such that the net $\{f_j\}_{j \in \mathcal{A}}$ is fundamental in the norm determined by B, i.e. $\lim_{j, j' \in \mathcal{A}} \|f_j - f_{j'}\|_B = 0$.

Proposition 1.5. (Continuity of composition in the T-topology [35]) *Let $\{f_j\}_{j \in \mathcal{A}}$ and $\{g_j\}_{j \in \mathcal{A}}$ be nets in $\mathrm{Hol}\,(D, D_1)$ and $\mathrm{Hol}\,(D_1, D_2)$, respectively, such that*

$$T\text{-}\lim_{j \in \mathcal{A}} f_j = f \quad \text{and} \quad T\text{-}\lim_{j \in \mathcal{A}} g_j = g.$$

Then T-$\lim_{j\in\mathcal{A}} g_j f_j = gf \in \mathrm{Hol}\ (D, D_2)$.

We don't mention here other important classical properties such as uniqueness theorems, maximum principles and Weierstrass theorems, but the reader may find them in many books, e.g. [34, 22, 35, 33] and [41].

Now we turn to the special case when $X = Y$ and $\tilde{D} = D$. In this case Hol (D, D) is the set of all holomorphic self-mappings F of D, and the family $\{F^n\}$, $n = 0, 1, \dots,$ of the iterates of F ($F^n = F \circ F^{n-1}$, $n = 1, 2 \dots,$ $F^0 = I|_D$, where I denotes the identity on X), is contained in $\mathrm{Hol}(D, D)$.

Definition 1.5. *Let $F \in \mathrm{Hol}\ (D, D)$ have a fixed point $x_0 \in D$, i.e. $F(x_0) = x_0$. This point will be called a T-attractive fixed point of F if the sequence $\{F^n\}$ T-converges to x_0 in D.*

The local nature of such a point is brought out by the following assertion.

Theorem A. ([70, 71, 40]) *Let D be a bounded domain in X and let F $\in\mathrm{Hol}(D, D)$ have a fixed point $x_0 \in D$. Set $A = f'(x_0)$. Then* 1) *The spectral radius $r(A)$ of the linear operator $A\colon X \mapsto X$ is less than or equal to 1,*
2) $r(A) < 1$ *if and only if x_0 is a T-attractive fixed point of F.*

The existence of a T-attractive fixed point may be guaranteed by the well-known Earle-Hamilton theorem.

Theorem B. ([21]) *Let $F \in\mathrm{Hol}(D, \tilde{D})$, where $\tilde{D} \subset\subset D$ is strictly inside D. Then F has a unique fixed point in D and it is T-attractive.*

Finally, concerning the description of the fixed point set of holomorphic self-mappings we note that this problem has been considered by many mathematicians (see, for example, [63, 31, 69, 74, 75, 76, 8, 33, 23, 3, 65] and [51]).

We mention here one of the most important results due to P. Mazet and J. P. Vigué:

Theorem C. ([52]) *Let D be a bounded convex domain in X, and let F $\in \mathrm{Hol}\ (D, D)$ have a fixed point $x_0 \in D$. Suppose that one of the following hypotheses holds:*

(i) *X is reflexive;*
(ii) $\mathrm{Ker}\ (I - F'(x_0)) \oplus \mathrm{Im}\ (I - F'(x_0)) = X$.

Then
1) *The fixed point set $\mathcal{F} = \mathrm{Fix}_D F$ of the mapping F is a connected complex analytic submanifold of D which is tangent to* $\mathrm{Ker}\ (I - F'(x_0))$*;*
2) *There is a holomorphic self-mapping $\Phi\colon D \mapsto D$ which is a retraction onto $\mathcal{F}$, i.e. $\Phi(D) = \mathcal{F}$ and $\Phi^2 = \Phi$.*

This theorem has recently been extended to unbounded domains (see [19]).

Now let B denote the open unit ball of a complex Hilbert space and let B^n be the product of n Hilbert balls. We also mention two results about holomorphic self-mappings of the Hilbert ball and its powers. See also [29, 30].

Theorem D. ([63]) *The fixed point set of a holomorphic self-mapping of B is affine.*

Theorem E. ([48]) *A holomorphic self-mapping of B^n with a continuous extension to $\overline{B}^n$ has a fixed point in $\overline{B}^n$.*

1.3. **The infinitesimal Carathéodory-Reiffen-Finsler pseudometric.** Let D be a domain in a complex Banach space X, and let Δ be the open unit disc in $\mathbb{C}$.

Definition 1.6. *The real-valued nonnegative function $\alpha_D(\cdot,\cdot)$ defined on $D \times X$ by the formula*

$$\alpha_D(x,v) = \sup\{|f'(x)v| \colon f \in \mathrm{Hol}\,(D,\Delta)\}$$

is called the infinitesimal Carathéodory-Reiffen-Finsler pseudometric on D (or the CRF pseudometric for short).

Proposition 1.6. ([22, 23, 17]) *The infinitesimal CRF pseudometric satisfies the following properties:*

a) $\alpha_D(x,tv) = |t|\alpha_D(x,v)$;

b) α_D *is continuous;*

c) *If $f \in \mathrm{Hol}\,(D_1,D_2)$, where D_1 and D_2 are domains in X_1 and X_2, respectively, then*

$$\alpha_{D_2}(f(x), f'(x)\cdot v) \le \alpha_{D_1}(x,v)$$

for all $x \in D_1$ and $v \in X$ (contraction property).

For any two points x and y in D consider a curve $\gamma\colon [0,1] \mapsto D$ which joins x and y and has a piecewise continuous derivative. Such a curve is said to be admissible. Define its length by

$$L_\alpha(\gamma) = \int_0^1 \alpha_D(\gamma(t),\gamma'(t))\,dt.$$

Definition 1.7. The function $\rho(\cdot,\cdot)\colon D\times D \mapsto R$ defined by the formula

$$\rho_D(x,y) = \inf\{L_\alpha(\gamma)\colon \gamma \text{ is an admissible curve joining } x \text{ and } y\}$$

is called the integrated form of the infinitesimal CRF pseudometric.

Proposition 1.7. *The integrated form $\rho_D(x,y)$ satisfies the following properties:*

a) *$\rho_D(\cdot,\cdot)$ is a pseudometric on D, i.e. $\rho_D(x,y) \ge 0$ and $\rho_D(x,y) \le \rho_D(x,z)+\rho_D(z,y)$ for all x,y and z in D;*

b) *If $f \in \mathrm{Hol}\,(D_1,D_2)$ where D_1 and D_2 are domains in X_1 and X_2, respectively, then $\rho_{D_2}(f(x),f(y)) \le \rho_{D_1}(x,y)$ (Schwarz-Pick contraction inequality);*

c) $\rho_\Delta(x,y) = \tanh^{-1}|\frac{x-y}{1-x\bar{y}}|$;

d) $\rho_\Delta(0,x) = \rho_\Delta(0,|x|) = \tanh^{-1}|x|$;

e) $\lim_{s\to 0^+} \frac{\rho_\Delta(0,s)}{s} = 1$;

f) *If $x \in D$ and $B_r(x) \subset D$ is a ball centered at x with radius r, then*

$$\rho_D(x,y) \le \tanh^{-1}\Big(\frac{\|x-y\|}{r}\Big)$$

whenever $y \in B_r(x)$;

g) *If D is a bounded domain in X, which is contained in some open ball $B_R(x)$ centered at $x \in D$ with radius R, then*

$$\rho_D(x,y) \ge \tanh^{-1}\Big(\frac{\|x-y\|}{R}\Big)$$

for all $y \in D$.

Thus when D is bounded, $\rho_D(\cdot,\cdot)$ is a metric defined on D. It is called the CRF metric.

A system which assigns a pseudometric to each domain in each normed linear space such that Δ is assigned the Poincaré metric and property b) of Proposition 1.7 is satisfied, is called a Schwarz-Pick system. There are other Schwarz-Pick systems in addition to the CRF system. Of particular interest are the so-called Carathéodory and Kobayashi pseudometrics as they form the smallest and largest Schwarz-Pick systems. All of them also satisfy the properties b) and g) of Proposition 1.7.

As a matter of fact, in our investigations we do not need a concrete representation of Schwarz-Pick systems. We will only use the properties of Proposition 1.7. Moreover, since we will mainly deal with convex domains in a Banach space, we note that all the Schwarz-Pick systems in this case coincide (see [49] and [18]). We call this common pseudometric the hyperbolic pseudometric of D. If D is bounded, then as noted above, it is, in fact, a metric.

2. Nonlinear semigroups with holomorphic generators

2.1. **Continuous and discrete one-parameter semigroups.** Let X be a Banach space and let D be a subset of X

Definition 2.1. *A family $S = \{F_t\}$, where either $t \in R^+$ $(= [0,\infty))$ or $t \in N$ $(= \{0,1,2,\dots\})$, of self-mappings F_t of D is called a (one-parameter) semigroup if*

$$F_{s+t} = F_s \circ F_t, \qquad s,t \in R^+ \quad (s,t \in N), \tag{2.1}$$

and

$$F_0 = I_D, \tag{2.2}$$

where I_D is the identity operator on D.

A semigroup $S = \{F_t\}$, $t \in R^+$, is said to be (strongly) continuous if the vector-valued function $F_t(x)\colon R^+ \mapsto X$ is continuous in t for each $x \in D$.

If $t \in N$ we say that the semigroup S is discrete. In other words, a discrete semigroup $S = \{F_t\}$, $t \in N$, is the family of iterates of a self-mapping $F = F_1\colon D \mapsto D$.

Definition 2.2. *Let $S = \{F_t\},\ t \in R^+$, be a continuous semigroup defined on D. If the strong limit*

$$f(x) = \lim_{t \to 0^+} \frac{x - F_t(x)}{t} \tag{2.3}$$

exists for each $x \in D$, then f will be called the generator of the (continuous) semigroup S.

For a fixed $t > 0$, the mapping

$$f_t = t^{-1}(I - F_t) \colon D \mapsto X, \qquad t > 0, \tag{2.4}$$

will be called a difference approximation of the generator f in (2.3).

For a discrete semigroup $\{F_n\},\ n \in N$, the generator f is usually defined as the complement of F_1, i.e. $f = I - F_1$. But as it is mentioned in [36], in approximation theory it is necessary to connect the order n with the "time" t. Therefore we recall the following definition.

Definition 2.3. *Let $S = \{F_n\},\ n \in N$, be a discrete semigroup of self-mappings $F_n \colon D \mapsto D$. For a given $\tau > 0$ we define $F(n\tau) = F_n$ and we say that a mapping f is a τ-generator of $\{F(n\tau)\}_1^\infty$ with respect to the unit time τ, if*

$$f = \frac{I - F(\tau)}{\tau}. \tag{2.5}$$

Thus if F is a self-mapping of D, its complement $f = I - F$ is a 1-generator of the semigroup $\{F^n\},\ n \in N$.

Note also that for a continuous semigroup $S = \{F_t\}$ with generator f, its difference approximation f_t, defined by (2.4), is a t-generator of the discrete semigroup $\{F_{tn}\},\ n \in N$.

It is an important problem in the general theory of evolutions to determine when a generator of a discrete semigroup is also a generator of a continuous semigroup.

Finally, when we need to emphasize that $S = \{F_t\},\ t \in R^+$, is a semigroup generated by a given f, we will write $S = S_f$.

Now let D be a domain in X and let $S_f = \{F_t\},\ t \in R^+$, be a continuous semigroup generated by a holomorphic mapping f in D, i.e.

$$f = \lim_{t \to 0^+} t^{-1}(I - F_t) \in \mathrm{Hol}\,(D, X).$$

The first question which arises at this point is whether each $F_t : D \mapsto D$ is also holomorphic.

The second one is whether S_f is the unique semigroup satisfying (2.3).

In order to trace the analogy with the linear case we note that a holomorphic linear mapping is bounded by definition. Therefore it is well known that both these questions have affirmative answers in this case. Moreover, it is known that the semigroup generated by a linear bounded operator is uniformly continuous and the difference approximations (2.4) converge to the generator in the uniform operator topology when t tends to 0^+.

In this section we will establish a similar fact for the nonlinear holomorphic case. This, in turn, will yield affirmative answers to both questions mentioned above.

2.2. **Semicomplete vector fields.** To begin with, we note that it follows from the semigroup properties (2.1), (2.2) and Definition 2.2 that F_t is the solution of the right-hand Cauchy problem

$$\frac{\partial^+ F_t(x)}{\partial t} + f(F_t(x)) = 0, \qquad F_0(x) = x. \tag{2.6}$$

Definition 2.4. ([35, 7]) *A holomorphic mapping $f\colon D \mapsto D$ is said to be a complete (semi-plus complete) vector-field if the Cauchy problem*

$$\begin{cases} \dfrac{\partial F_t(x)}{\partial t} + f(F_t(x)) = 0, & t \in R, \quad (t \in R^+,) \\ \\ F_0(x) = x, & x \in D, \end{cases} \tag{2.7}$$

has a solution $\{F_t(x)\} \subset D$, $t \in R$ $(t \in R^+)$, for each $x \in D$.

The semigroup properties (2.1) and (2.2) imply the following fact:

Proposition 2.1. *Let $f\colon D \mapsto X$ be the generator of a continuous semigroup, and assume that the convergence in (2.3) of the difference approximations (2.4) is uniform on each compact subset of D. Then f is a semi-plus complete vector field.*

Proposition 2.2. *Let $f \in \mathrm{Hol}(D, X)$ be bounded. Then*
(1) *f is the generator of a one-parameter semigroup (group) iff it is a semi-plus complete (complete) vector field;*
(2) *Moreover, the difference approximations $\{f_t\}$ converge to f uniformly on each closed subset strictly inside D.*

Proof. The first assertion is simple enough and it follows from some classical facts. Indeed, if f is a semi-plus complete (complete) vector field, then the uniqueness of the solution of the Cauchy problem (2.7) implies the semigroup (group) property of this solution with respect to $t \in R^+$ $(t \in R)$ (see, for example, [13]). Condition (2.3) is obvious. Conversely, if f generates a semigroup (group) $\{F_t\}$, $t \in R^+$ $(t \in R)$, then $F_t(x)$ is a solution of the right-hand Cauchy problem (2.6). In addition, if f is bounded, then the right-hand derivative $\partial^+ F_t(x)/\partial t$ of F_t is a continuous bounded function of $t \in R^+$ $(t \in R)$. It is more or less known (see, for example, [77]), that in this case the left-hand derivative $\partial^- F_t(x)/\partial t$ also exists and coincides with $\partial^+ F_t(x)/\partial t$. Thus F_t is the unique solution of the Cauchy problem (2.7) (because a holomorphic mapping is locally Lipschitzian) and it is holomorphic (see, for example, [16]).

Now we turn to assertion (2). Let U be an arbitrary subset strictly inside D. Since f is bounded on D it follows from the Cauchy inequalities that f is Lipschitzian on U. Hence on some disk $\Omega \subset \mathbb{C}$ centered at $0 \in \mathbb{C}$ there is a unique solution $\Phi(t,x)$ of the Cauchy problem

$$(2.8)\qquad \begin{cases} \dfrac{\partial \Phi(t,x)}{\partial t} + f(\Phi(t,x)) = 0, & (t,x) \in \Omega \times U, \\ \\ \Phi(0,x) = x, \qquad x \in D, \end{cases}$$

which is holomorphic and bounded on $\Omega \times U$. Thus we have

$$\Phi(t,x) = x - tf(x) + \omega(t,x)$$

for $(t,x) \in \Omega \times U$, where $\omega(t,x)$ is holomorphic in $t \in \Omega$ and bounded for each $x \in U$. By the Schwarz lemma (Proposition 1.3) we have

$$\|\omega(t,x)\| \leq |t|^2 \sup_{\substack{t \in \Omega \\ x \in U}} \|\omega(t,x)\| \varepsilon^{-2},$$

where ε is the radius of Ω. Thus for $t \in \Omega \cap R^+$ we have the inequality

$$\|f_t(x) - f(x)\| \leq t \sup_{\substack{t \in \Omega \\ x \in U}} \|\omega(t,x)\| \varepsilon^{-2},$$

which proves the assertion. □

In the sequel we denote by $\mathrm{HG}(D)$ the family of all mappings in $\mathrm{Hol}\,(D,X)$ which are generators of continuous semigroups on D (see Definition 2.2).

We state now our main auxiliary lemma.

Lemma 2.1. *If* $\{f_n\} \subset \mathrm{HG}\ (D)$ *is a sequence which* T*-converges to a bounded* $f \in \mathrm{Hol}\ (D,X)$*, then* f *also belongs to* $\mathrm{HG}(D)$

Proof. Since $\{f_n\} \subset \mathrm{HG}\ (D)$ is a sequence which converges in the topology of local uniform convergence over D, it is clear that $\{f_n\}$ is uniformly bounded on each ball $B \subset\subset D$.

We need to show that for each $x \in D$ the Cauchy problem

$$(2.9)\qquad \begin{cases} \dfrac{\partial F(t,x)}{\partial t} + f(F(t,x)) = 0, \\ \\ F(0,x) = x, \end{cases}$$

has a solution $F_t(x)$ for $t \geq 0$.

Let $\{F_n(t,\cdot)\},\ t \geq 0$, be the semigroup generated by f_n, for each $n \geq 1$.

Fix an arbitrary $x_0 \in D$ and choose $r > 0$ such that $\overline{B_{2r}(x_0)} \subset D$. Then the family $\{f_n\}$ is uniformly Lipschitzian on $B_{2r}(x_0)$. Hence for each $x \in \overline{B_r(x_0)}$ we can find $\delta > 0$ such that $\{F_n(t,x)\} \subset B_{2r}(x_0)$ for all $0 \leq t \leq 2\delta$ and $n \geq 1$. Therefore for each $\varepsilon > 0$ there is $n_0 > 0$ such that for all $n > n_0$, $t \in [0, 2\delta]$ and $x \in \overline{B_r(x_0)}$, the following inequality holds:

$$(2.10)\qquad \begin{aligned} &\left\| \frac{\partial F_n(t,x)}{\partial t} + f(F_n(t,x)) \right\| \leq \\ &\left\| \tfrac{\partial F_n(t,x)}{\partial t} + f_n(F_n(t,x)) \right\| + \|f_n(F_n(t,x)) - f(F_n(t,x))\| < \varepsilon. \end{aligned}$$

It is known that (2.10) means that the Cauchy problem (2.9) has a solution $F(\cdot,\cdot)$ on $[0,2\delta]\times\overline{B_r(x_0)}$, and that for all $t\in[0,2\delta]$ and $x\in\overline{B_r(x_0)}$,

$$\|F_n(t,x)-F(t,x)\|\le\varepsilon\frac{e^{tL}-1}{L}, \tag{2.11}$$

where L is the Lipschitz constant for f on $\overline{B_{2r}(x_0)}$ (see, for example, [13] and [16]).

It also follows from the uniqueness of the solution of the Cauchy problem that for all $t,\tau\ge 0$ such that $t+\tau\le 2\delta$ the following equality holds for $x\in\overline{B_r(x_0)}$:

$$F(t+\tau,x)=F(t,F(\tau,x)). \tag{2.12}$$

But (2.11) implies that for each fixed $t\in[0,2\delta]$ the sequence of holomorphic mappings $F_n(t,\cdot)$ converges to $F\in\mathrm{Hol}\,(B_r(x_0)),D)$ uniformly on $B_r(x_0)$. Hence by Vitali's property (see Proposition 1.4) it converges to a holomorphic extension of F on all of D in the topology of local uniform convergence over D. By the uniqueness property of holomorphic mappings, (2.12) holds for all $x\in D$.

Now we want to show that $F(\cdot,x),\ x\in D$, can be extended as a semigroup to all of R^+.

Indeed, take an arbitrary $t\in R^+$ and write it (uniquely) in the form $t=n\delta+r$, where $0\le r<\delta,\ n=0,1,\dots$. For such t, setting $F(t,x)=[F(\delta,F(r,x))]^n$, we have that $F(t,x)$ is defined on D by composing and iterating holomorphic mappings. Hence $F(t,\cdot)$ is holomorphic on D too. To show that it is a semigroup, take $s,t\ge 0$ and set $t=n\delta+r,\ s=m\delta+p$, and $s+t=k\delta+q$, where $0\le r,p,q<\delta$ and $n,m,k=0,1,2,\dots$.

Then the equality

$$F(t+s,x)=F(t,F(s,x)) \tag{2.12$'$}$$

for all $t,s\ge 0$ is equivalent to

$$[F(\delta,F(q,x))]^k=[F(\delta,F(r,[F(\delta,F(p,x))]^m))]^n \tag{2.13}$$

or

$$[F(\delta,\cdot)]^k\circ F(q,\cdot)=[F(\delta,\cdot)]^n\circ F(r,\cdot)\circ[F(\delta,\cdot)]^m\circ F(p,\cdot). \tag{2.13$'$}$$

There are two possibilities

a) $m+n=k$ and $r+p=q$.

b) $m+n=k-1$ and $r+p=q+\delta$.

Since $0\le p<\delta$, $0\le q<\delta$ and $0\le r<\delta$, it follows from (2.12) that the two pairs of mappings $F(\delta,\cdot)$ and $F(r,\cdot)$, as well as $F(q,\cdot)$ and $F(p,\cdot)$, are commutative. Therefore (2.13′) (hence (2.13)) holds in both cases a) and b). Thus (2.12′) holds for all $t,s\ge 0$, and we have obtained a semigroup $F(\cdot,\cdot)$ $(=F(t,x))$, $t\ge 0,\ x\in D$, which solves the Cauchy problem (2.9) for $0\le t\le\delta$. But it follows from the semigroup property that (2.9) holds for all $t\ge 0$. The proof is complete. □

Remark 2.1. Our goal in this paper is to study the class HG(D) of semi-plus continuous vector fields. As we saw above, this class contains all bounded holomorphic generators of continuous semigroups. It is a very important problem for different applications (see, for example, [12, 64] and [36]) to find out if it contains the class of 1-generators of discrete semigroups. In other words, the question is: If $f = I - F \in \mathrm{Hol}\,(D, X)$, where F is a self-mapping of D, can the Cauchy problem (2.7) be solved on R^+?

We show in the sequel that if D is a bounded convex domain, then HG(D) contains all τ-generators of discrete semigroups with unit of "time" $\tau > 0$. This will provide an affirmative answer to this question.

3. Lipschitzian mappings and the flow invariance condition

Here we consider the class of holomorphic mappings on D which are also defined on $\overline{D}$, the closure of D, and are Lipschitzian on $\overline{D}$.

This class will be denoted by HL(D, X).

Definition 3.1. ([50] **and** [56]) *Let $f \in HL(D, X)$. We say that f satisfies the flow invariance condition if the following holds:*

$$\lim_{h \to 0^+} \frac{dist(x - hf(x), \overline{D})}{h} = 0, \qquad x \in \overline{D}. \tag{3.1}$$

Proposition 3.1. *Let D be a bounded convex subset of X and let $f \in \mathrm{HL}\ (D, X)$. Then the following are equivalent:*

1) *f satisfies (3.1);*
2) *f is the generator of a continuous semigroup $S = \{F_t\}$, $t \in R^+$, $F_t\colon \overline{D} \mapsto \overline{D}$;*
3) *There exists $\varepsilon > 0$ such that for all $r \in (0, \varepsilon)$, $(I + rf)(\overline{D}) \supset \overline{D}$;*
4) *There exists $\varepsilon > 0$ such that for each $r \in (0, \varepsilon)$ the mapping $(I + rf)^{-1}\colon \overline{D} \mapsto \overline{D}$ is well defined and belongs to* HL (D).

Proof. The equivalence of conditions 1), 2) and 3) follows from Theorem 6 in [50].

The implication 1) ⇒ 4) was proved in [38].

The implication 4) ⇒ 3) is evident. □

Remark 3.1. The mapping $J_r = (I + rf)^{-1}$ is called a (nonlinear) resolvent of the mapping $(-f)$. Its existence and Proposition 3.1 may be used to obtain some very interesting consequences and conclusions (see, for example, [50, 56, 58, 62, 37] and [38]).

However, two circumstances are unpleasant in this situation and restrict our possibilities.

The first one is that we must impose the additional restriction that f be defined on $\overline{D}$ and, moreover, that it be Lipschitzian there. This already does not allow us to generalize the well-developed theory of holomorphic self-mappings on open domains. Besides it leaves open the questions mentioned above.

The second one is that the number ε in conditions 3) and 4) of Proposition 3.1 depends on the Lipschitz constant of the mapping f. Thus we cannot consider the behavior of the resolvent $J_r = (I + rf)^{-1}$ as r tends to infinity, or at least for r large enough, as it is done in the linear Hille-Yosida theory.

Nevertheless, if $f \in \mathrm{HG}\,(D)$ we are able to establish the existence of the resolvent $J_r = (I + rf)^{-1}$ for all $r \geq 0$, and conversely, we will show that the existence of the resolvent on D implies that $f \in \mathrm{HG}\,(D)$.

4. The resolvent method

4.1. **A Hille-Yosida type theorem.** In this section we establish our main results. We denote by $\mathrm{HR}\,(D)$ the family of all mappings $f \in \mathrm{Hol}\,(D, X)$ for which the resolvent $(I + rf)^{-1}$ is well-defined and belongs to $\mathrm{Hol}\,(D, D)$ for all $r > 0$. The following result includes the Hille-Yosida theorem for linear contraction semigroups with bounded generators.

Theorem 4.1. *Let D be a bounded convex domain in X and let $f \in \mathrm{Hol}\,(D, X)$ be bounded. Then $f \in \mathrm{HG}\,(D)$ if and only if $f \in \mathrm{HR}\,(D)$.*

In other words, f generates a one-parameter semigroup of holomorphic self-mappings of D if and only if for each $r > 0$ its resolvent $(I + rf)^{-1}$ exists and is a holomorphic self-mapping of D.

To prove our theorem we need some auxiliary assertions. First we give some simple geometric estimates for bounded convex domains in a Banach space.

Lemma 4.1. *Let D be a bounded convex domain in X, x a point in D and $0 \leq s < 1$. Then the subset $K = \{(1-s)x + s\omega : \omega \in \overline{D}\}$ is strictly inside D.*

Proof. Assume without loss of generality that $x = 0$, and suppose that K is not strictly inside D. This means that there exist sequences $\{y_n\} \subset \partial D$ and $\{\omega_n\} \subset \overline{D}$ such that

$$z_n = y_n - s\omega_n \to 0. \tag{4.1}$$

Since $x = 0 \in D$, there is a ball $B_r(0)$ with radius r, centered at the origin, which is contained in D. It follows from (4.1) that there is $n > 0$ such that

$$\|z_n\| < (1-s)r.$$

Hence $x_n = (1-s)^{-1} z_n \in B_r$. But now we have $y_n = s\omega_n + (1-s)x_n$, where $\omega_n \in \overline{D}$, $x_n \in D$, and this implies that $y_n \in D$, which is a contradiction. $\square$

For any two subsets K_1 and K_2 of X we denote $\inf\{\|x - y\| \colon x \in K_1$ and $y \in K_2\}$ by $\mathrm{dist}(K_1, K_2)$. Thus $K \subset\subset D$ if $\mathrm{dist}(K, \partial D) > 0$. Recall also that the ball $\{x \in X \colon \|x - z\| < R\}$ centered at the point z with radius R is denoted by $B_R(z)$.

Lemma 4.2. *Let D be a bounded convex domain in X with $0 \in D$. Let $B_\varepsilon(0)$ and $B_R(0)$ be two balls such that $B_\varepsilon(0) \subset\subset D \subset\subset B_R(0)$. If ρ is the*

hyperbolic metric on D, then
1) For all $(x,y)\in D\times D$ and any $0\le\kappa\le 1$ the following inequality holds:

$$\rho(\kappa x,\kappa y)\le \frac{2R}{\varepsilon(1-\kappa)+2R}\,\rho(x,y);$$

2) For $0\le M<\infty$ and $\mathcal{M}=\{x\in D\colon \rho(0,x)<M\}$,

$$\operatorname{dist}(\mathcal{M},\partial D)\ge \frac{\varepsilon(1-(\tanh M)^2)}{4}\,.$$

Proof. 1) Denote $L=2R/\varepsilon$. Then the points $L^{-1}(x-y)$ belong to D for all $x,y\in D$. Fix $y\in D$, $0\le\kappa\le 1$, and consider the affine mapping g defined by

$$g(x)=\kappa x+(1-\kappa)\kappa L^{-1}(x-y).$$

It is clear that $g\in$Hol$\,(D,D)$ because $0\in D$ and D is convex.

Let $\alpha(\cdot,\cdot)$ denote the infinitesimal CRF pseudometric on D. Since $g'(x)=(1-(1-\kappa)L^{-1})\kappa I$ and $g(y)=\kappa y$, it follows by Proposition 1.6 that

$$\begin{aligned}\alpha(y,v)\ge\alpha(g(y),g'(y)\cdot v)&=\alpha\Big(\kappa y,(1+(1-\kappa)L^{-1})\kappa v\Big)\\&=(1+(1-\kappa)L^{-1})\alpha(\kappa y,\kappa v).\end{aligned}$$

Since y is arbitrary we can substitute x for y and obtain

$$\alpha(\kappa x,\kappa v)\le L(1-K+L)^{-1}\alpha(x,v)$$

for all $x\in D$ and $v\in X$. Using the integrated form for the hyperbolic metric we now get the required inequality.
2) Denote $\ell=\tanh M<1$ and $s=\frac{\varepsilon(1-\ell^2)}{4}$.

If $y\notin\overline{D}$, then by Mazur's theorem (see, for example, [77]) there is a real linear functional $\tilde{x}$ such that

$$\langle y,\tilde{x}\rangle>1, \tag{4.2}$$

and

$$\langle x,\tilde{x}\rangle\le 1 \text{ for all } x\in D. \tag{4.3}$$

Consider the complex linear functional x^* defined by

$$\langle x,x^*\rangle=\langle x,\tilde{x}\rangle-i\langle ix,\tilde{x}\rangle.$$

This functional is bounded by (4.3) because $B_\varepsilon(0)\subset D$. Now we define the function g by $g(x)=\langle x,x^*\rangle(2-\langle x,x^*\rangle)^{-1}$. It is clear that $g(0)=0$ and $g\in$Hol$\,(D,\Delta)$. Hence for all $x\in\mathcal{M}$ we have

$$\tanh^{-1}|g(x)|=\rho_\Delta(0,g(x))=\rho_\Delta(g(0),g(x))\le\rho(0,x)<M$$

(see Proposition 1.6). Thus for $x\in\mathcal{M}$,

$$|g(x)|<\tanh M=\ell<1.$$

Hence

$$\begin{aligned}\operatorname{Re}\langle x, x^*\rangle &= \operatorname{Re}\Big(1-\frac{1-g(x)}{1+g(x)}\Big) = \\ &= 1-\frac{1}{|1+g(x)|^2}\operatorname{Re}\left(1-g(x)+\overline{g(x)}-|g(x)|^2\right) = \\ &= 1-\frac{1-|g(x)|^2}{|1+g(x)|^2} \le 1-\frac{1-\ell^2}{4} = 1-\frac{s}{\varepsilon}\end{aligned}$$

whenever $x \in \mathcal{M}$. Therefore if $x \in \mathcal{M}$ and $\|y-x\| < s$, we obtain

$$\operatorname{Re}\langle y, x^*\rangle = \operatorname{Re}\langle x, x^*\rangle - \operatorname{Re}\langle x-y, x^*\rangle \le 1-\frac{s}{\varepsilon}+\frac{s}{\varepsilon} = 1,$$

which contradicts (4.2). Hence $|y-x| \ge s$ for all $x \in \mathcal{M}$ and $y \in \partial D$, as claimed. □

Now we continue with several results on holomorphic mappings. Theorem 4.1 will follow by combining these results.

Proposition 4.1. *Let D be a bounded convex domain in X. Suppose that a net $\{g_t\}_{t\in\mathcal{A}} \subset \mathrm{Hol}\,(D,X)$ satisfies the following conditions:*

- (i) *$g_t(D) \supseteq D$ for all $t \in \mathcal{A}$;*
- (ii) *For each $t \in \mathcal{A}$ there exists a single valued mapping $g_t^{-1} \in \mathrm{Hol}\,(D,D)$;*
- (iii) *There exists at least one point $z \in D$ such that the net of points $\{g_t^{-1}(z)\}$ is strictly inside D;*
- (iv) *$\{g_t\}_{t\in\mathcal{A}}$ converges to $g \in \mathrm{Hol}\,(D,X)$ uniformly on each closed subset strictly inside D.*

Then

1) *There exists a single valued mapping $g^{-1}\colon D \mapsto D$ which belongs to $\mathrm{Hol}\,(D,D)$.*
2) *g_t^{-1} converges to g^{-1}.*

Proof. **Step 1.** First we show that there exists a point $x_0 \in D$ such that $g(x_0) = y_0 \in D$.

Indeed, (iii) implies that there is $D_1 \subset\subset D$ such that $\{x_t = g_t^{-1}(z)\} \subset D_1$. By (iv),

$$\sup_{x\in D_1} \|g_t(x)-g(x)\| \to 0.$$

Thus we have $\|g(x_t)-z\| = \|g(x_t)-g_t(x_t)\| \to 0$ (recall that for all t, $z = g_t(x_t)$).

It follows that $g(x_t) \in D$ for all $t \ge t_0$.

Let $x_0 = x_{t_0}$ and $y_0 = g(x_0)$.

Step 2. Now we show that the mapping g is invertible on some neighborhood of the point x_0.

We know that $\{g_t\}$ converges uniformly on some neighborhood of the point x_0 to g_0. Using the Cauchy inequalities we see that the net of the linear operators $\{A_t = g_t'(x_0)\}$ converges to A in the operator topology. In addition, for all t such that the element $y_{t_0} = g_t(x_0)$ is close enough to y_0 there is a number $r > 0$ for which the ball $B_r(y_t)$ with its center at the point y_t and radius r is contained in D. Once again, using the Cauchy inequalities we see that for such t, $\|(g_t^{-1})'(y_t)\|$ is uniformly bounded (recall that $g_t^{-1}\colon B_r(y_t) \mapsto D$ and that D is bounded). But it follows from the chain rule that $(g_t^{-1})'(y_t) = A_t^{-1}$. It is known that the last conclusion implies that A is invertible and that A_t^{-1} converges to A^{-1} in the operator topology. Thus we have that g is invertible in some neighborhood of the point x_0, by the Inverse Function Theorem. In addition, there are neighborhoods U of the points x_0 and $V \subset\subset D$ of the the point y_0, such that $V \subset \cap_{t \geq t_1} g_t(U)$ and g^{-1} exists in V (see, for example, [4]).

Step 3. Finally, note that it is enough to prove our assertion for V (see Proposition 1.4).

Take an arbitrary $y \in V$ and set $x = g^{-1}(y)$, $y_t = g_t(x)$. Then $\{y_t\}$ converges to y. Note also that because D is bounded and $V \subset\subset D$, the net $\{g_t^{-1}(y)\}$ is uniformly Lipschitz on V, i.e. there is $0 \leq K < \infty$ such that $\sup_{y\in V} \|(g_t^{-1})'(y)\| \leq K$. Then we obtain

$$\|g^{-1}(y) - g_t^{-1}(y)\| \leq \|g^{-1}(y) - g_t^{-1}(y_t)\| + \|g_t^{-1}(y_t) - g_t^{-1}(y)\| \leq K\|y_t - y\|$$

because $g^{-1}(y) = g_t^{-1}(y_t) = x$. This concludes the proof of our proposition. □

The next proposition proves the necessity part of our theorem. As a matter of fact, we are able to prove a stronger result.

Proposition 4.2. *Let D be a bounded convex domain and let $\{G_t\colon D \mapsto D\}$, $0 \leq t \leq \delta$, be a net of holomorphic self-mappings of D. Then*

1) *If h_t, $t \in (0, \varepsilon)$, is a t-generator of the discrete semigroup $\{G_t^n\}$, $n = 1, 2, \dots$, i.e.*
$$h_t = \frac{1}{t}(I - G_t),$$
then $h_t \in$ HR (D);
2) *If $G_0 = I$ and $\{G_t\}$ is right differentiable at $t = 0$ in the T-topology over D, i.e. there exists*
$$h = T\text{-}\lim_{t\to 0^+} \frac{I - G_t}{t},$$
and $h \in$ Hol (D, X) is bounded, then $h \in$ HR (D). Moreover, for each $r > 0$,

$$J_r[h] := (I + rh)^{-1} = T\text{-}\lim_{t\to 0^+} \Big(I + \frac{r}{t}(I - G_t)\Big)^{-1}. \tag{4.4}$$

Proof. 1) To see this, we first note that the equation
$$(I + rh_t)x = y, \qquad y \in D, \quad r > 0,$$
which determines the resolvent $J_r[h_t]$ is equivalent to the equation
$$x = \frac{r}{r+t} G_t(x) + \frac{t}{r+t} y. \tag{4.5}$$

By lemma 4.1, the mapping G defined by $x \mapsto \frac{r}{r+t} G_t(x) + \frac{t}{r+t} y$ maps D strictly inside D. Hence Theorem B (see §1) implies that for each $y \in D$

the equation (4.5) has a unique solution $x = x(y)$, which is T-attractive, i.e. $G^n(y) \mapsto x(y)$, where $G^0 = I,\ G^{n+1} = G^n \circ G$, as $n \to \infty$. But the iterates G^n holomorphically depend on $y \in D$. Thus setting $J_r[h_t](y) := x(y)$ we see that $J_r[h_t] \in \text{Hol}\ (D, D)$.

2) Setting $g_t = (I+rh_t)^{-1}$ for a fixed $r > 0$, we see that g_t satisfies conditions (i) and (ii) of proposition 4.1 by assertion 1. Condition (iv) of Proposition 4.1 also holds by assumption. Thus to prove our assertion it is enough to show that $\{g_t\}$ satisfies condition (iii) of Proposition 4.1.

Without loss of generality assume that $0 \in D$. Then we claim that $\{g_t^{-1}(0)\}$ is strictly inside D.

Indeed, the net $x_t := g_t^{-1}(0) \in D$ may be defined by the equation

$$x_t = \frac{r}{r+t} G_t(x_t)$$

(see (4.5)). Let $\rho(\cdot,\cdot)$ be the hyperbolic metric on D. It follows by assertion 1 of Lemma 4.2 that for each $t \in (0, \delta)$,

$$\rho(0, x_t) = \rho(0, \frac{r}{r+t} G_t(x_t)) \le s(t)\rho(0, G_t(x_t)),$$

where

$$s(t) = \frac{2R(r+t)}{\varepsilon t + 2R(r+t)}\,.$$

Since G_t is nonexpansive with respect to the ρ metric, the triangle inequality implies that

$$\rho(0, x_t) \le s(t)[\rho(0, G_t(0)) + \rho(G_t(0), G_t(x_t))] \le$$

$$\le s(t)[\rho(0, G_t(0)) + \rho(0, x_t)].$$

Since $0 \le s(t) < 1$ for all $t \in (0, \delta)$, we get

$$\rho(0, x_t) \le \frac{s(t)}{1 - s(t)} \rho(0, G_t(0)). \tag{4.6}$$

Note that $\lim_{t\to 0^+} s(t) = 1$ and $\lim_{t\to 0^+} \frac{1-s(t)}{t} = \frac{\varepsilon}{2Rr}$. In addition, T-$\lim_{t\to 0^+} \frac{G_t(0)}{t} = -h(0)$ by assumption. Thus we obtain

$$\begin{aligned}
&\limsup_{t\to 0^+} \frac{s(t)}{1-s(t)} \rho(0, G_t(0)) \\
&\qquad = \limsup_{t\to 0^+} s(t) \Big(\frac{t}{1-s(t)} \Big) \Big(\frac{\|G_t(0)\|}{t} \Big) \Big(\frac{\rho(0, G_t(0))}{\|G_t(0)\|} \Big) \\
&\qquad = \frac{2Rr\|h(0)\|}{\varepsilon} \limsup_{t\to 0^+} \frac{\rho(0, G_t(0))}{\|G_t(0)\|} \\
&\qquad \le \frac{2Rr\|h(0)\|}{\varepsilon} \lim_{t\to 0^+} \frac{\tanh^{-1} \frac{\|G_t(0)\|}{\varepsilon}}{\|G_t(0)\|} = \frac{2Rr\|h(0)\|}{\varepsilon^2}
\end{aligned}$$

(see Proposition 1.7). Together with (4.6) this implies that for sufficiently small t,

$$\rho(0, x_t) < M < \infty.$$

Now an appeal to assertion 2 of Lemma 4.2 concludes our proof.

The necessity part of Theorem 4.1 is now clear: if $f \in \mathrm{HG}\ (D, D)$ is bounded and $S_f = \{F_t\}$ is the semigroup generated by f, then setting $G_t = F_t$ and $h = f$ in Proposition 4.2, we obtain that $f \in \mathrm{HR}\ (D)$. □

To prove the converse we need the following proposition which provides a positive answer to the question mentioned in Remark 2.1.

Proposition 4.3. *If $F \in \mathrm{Hol}\ (D, D)$, then $f = I - F$ is a semi-plus complete vector field, i.e. $f \in \mathrm{HG}\ (D)$.*

Proof. Consider the sequence of mappings $\{f_n\}$ defined by

$$f_n(y) = y - \Big(\frac{1}{n}z + \Big(1 - \frac{1}{n}\Big)F(y)\Big), \qquad y \in D,$$

where $z \in D$ is fixed. The sets $D_n = \{\frac{1}{n}z + \Big(1 - \frac{1}{n}\Big)F(y) \colon y \in D\}$ are all strictly inside D by lemma 4.2.1.

Now fix a positive integer n and an arbitrary x in D. There is a convex domain $U \subset\subset D$ such that

$$D_n \cup \{x\} \subset \overline{U} \subset\subset D. \tag{4.7}$$

Since f_n belongs to Hol (D, X) and is bounded on D, it is Lipschitzian on $\overline{U}$. We also have for each $y \in \overline{U}$ and $0 \le h \le 1$,

$$\mathrm{dist}\ (y - hf_n(y), \overline{U}) = \mathrm{dist}\ \Big(\frac{1}{n}z + \Big(1 - \frac{1}{n}\Big)F(y)\Big), \overline{U}) = 0.$$

Proposition 3.1 now implies that the Cauchy problem

$$\begin{cases} \dfrac{\partial \Phi(t,x)}{\partial t} + f_n(\Phi(t,x)) = 0, \\[2ex] \Phi(0,x) = x, \end{cases}$$

has a global solution on R^+. Since x was an arbitrary point in D, this means that each f_n is a semi-plus complete vector field. Since $\{f_n\}$ converges uniformly on D to f, the result follows by Lemma 2.1. □

Returning now to the sufficiency part of Theorem 4.1, we suppose that $f \in \mathrm{HR}\ (D)$ is bounded. Then for each $r > 0$, $I - J_r$ is a semi-plus complete vector field by Proposition 4.3. It is easy to see that so is $(I - J_r)/r$. Moreover, it follows from the implicit function theorem that $J_r \colon R^+ \mapsto \mathrm{Hol}\ (D, D)$ is T-continuous at 0 and T-$\lim_{r \to 0^+} J_r = I$.

In addition, by the definition of J_r we have the equality

$$I - J_r = rf(J_r)$$

and hence $\{(I - J_r)/r\}_{r>0}$ T-converges to f by the boundedness of f.

Lemma 2.1 now yields the sufficiency part of Theorem 4.1 which is thus completely established.

Remark 4.1. In our proof of Theorem 4.1 we have obtained some properties of generators and their resolvents which we would like to list here:

Let $f \in \mathrm{HG}\,(D)$ be bounded and let $J_r = (I + rf)^{-1}$ denote the resolvent of f. Then

1) $I - J_r = rf(J_r),\ r \geq 0$;

2) The so-called Yosida approximations $\{Y_r = (I - J_r)/r\}$ T-converge to f as $r \to 0^+$, i.e.

$$\text{T-}\lim_{r\to 0^+} \frac{I - J_r}{r} = f;$$

3) $$J_r = \text{T-}\lim_{t\to 0^+} \Big(I + \frac{r}{t}(I - J_t)\Big)^{-1}, \qquad r > 0;$$

4) $$J_r = \text{T-}\lim_{t\to 0^+} \Big(I + \frac{r}{t}(I - F_t)\Big)^{-1}, \qquad r > 0,$$

where $\{F_t\} = S_f$.

The last two properties are obtained immediately from Proposition 4.2 by using property 2) and the definition of the generator. Moreover, combining this proposition with Theorem 4.1 we deduce the following result.

Corollary 4.1. *Let D be a bounded convex domain in X and let $f \in \mathrm{Hol}\,(D, X)$ be bounded. Then f belongs to $HG(D)$ if and only if there exist a positive δ and a T-continuous (on $[0, \delta)$) curve $G_t\colon [0, \delta) \mapsto \mathrm{Hol}\,(D, D)$ such that*

$$T\text{-}\lim_{t\to 0^+} G_t = I$$

and

$$T\text{-}\lim_{t\to 0^+} \frac{1}{t}(I - G_t) = f.$$

Now we touch upon the case of a not necessarily convex domain. We mention two results which follow from Theorem 4.1 and a theorem of Mazet [51].

Corollary 4.2. *Let D be a bounded domain in X and let $f \in \mathrm{Hol}(D, X)$ be bounded. Suppose that f has a null point $a \in D$, i.e $f(a) = 0$. If $f \in HR\,(D)$, then there exists a neighborhood $U \subset D$ of the point a such that $f \in HG\,(U)$, i.e. the Cauchy problem*

$$\begin{cases} \dfrac{\partial F_t(x)}{\partial t} + f(F_t(x)) = 0, \\[2ex] F_0(x) = x, \end{cases}$$

has a global solution $\{F_t(x)\} \subset U$, where $t \geq 0$ and $x \in U$.

Corollary 4.3. *Let D be a bounded domain and let $F\colon D \mapsto D$ have a fixed point $a \in D$. Then there exists a neighborhood U of the point a such that* *the Cauchy problem*

$$\left\{ \begin{array}{l} \dfrac{\partial F_t}{\partial t} = F(F_t(x)) - F_t(x), \\ \\ F_0(x) = x, \end{array} \right. \tag{4.8}$$

has a global solution $\{F_t(x)\} \subset U$ *for all* $t \geq 0$ *and* $x \in U$*.*

Remark 4.2. The natural question which arises here is whether the situation in Corollary 4.3 is indeed only local. In other words, can the solution be holomorphically extended to all of D? We do not know a complete answer to this question, but generally speaking the answer is negative. More precisely, suppose that $F \in \mathrm{Hol}\,(D, D)$ has two fixed points $a \in \mathrm{Fix}_D F$ and $b \in \mathrm{Fix}_D F$ such that $a \notin \mathcal{M}_b$ where $\mathcal{M}_b$ is a connected component of $\mathrm{Fix}_D F$ which contains b (hence $b \notin \mathcal{M}_a$, where $\mathcal{M}_a$ is a connected component of $\mathrm{Fix}_D F$ which contains a).

It can be shown that there are at least two different solutions F_t and $\tilde{F}_t$ of the Cauchy problem (4.8) defined in neighborhoods U_a and U_b of the two points a and b, for all $t \geq 0$ ($U_a \cap U_b$ is, of course, empty). Moreover, F_t and $\tilde{F}_t$ T-converge in U_a and U_b, respectively, to mappings Q_a and Q_b, which are retractions onto $\mathcal{M}_a \cap U_a$ and $\mathcal{M}_b \cap U_b$, respectively (see example 4.1).

Example 4.1. A well-known example of a holomorphic self-mapping which has more than one fixed point in the one dimensional case is given in [52]. Let D be the annulus $\{z \in \mathbb{C}\colon 2^{-1} < |z| < 2\}$, and consider $F\colon D \mapsto D$ defined by the formula $F(z) = z^{-1}$. Then it is easy to see that the Cauchy problem (4.8)

$$\left\{ \begin{array}{l} \dfrac{\partial z}{\partial t} = \dfrac{1}{z} - z, \\ \\ z(0) = x, \end{array} \right.$$

has two holomorphic solutions $z_1(t,x)$ and $z_2(t,x)$ on the neighborhoods U_1 and U_{-1} of the points 1 and -1, and as $t \to \infty$,

$$z_1(t,x) \to 1 \text{ for } x \in U_1, \text{ and}$$

$$z_2(t,x) \to -1 \text{ for } x \in U_{-1}.$$

Corollary 4.4. *Let D be a bounded convex domain in X, and let f and g belong to $HG(D)$. If f and g are bounded, then for all $\alpha, \beta \geq 0$, $\alpha f + \beta g \in HG(D)$, i.e. the subset of $HG(D)$ consisting of bounded mappings is a real convex cone.*

Proof. Let $S_f = \{F_t\}_{t \geq 0},\ S_g = \{G_t\}_{t \geq 0}$, and let α, β be positive. Since

$$\alpha f + \beta g = \text{T-}\lim_{t \to 0^+} \alpha f_t + \beta g_t,$$

where $f_t = \frac{1}{t}(I - F_t)$ and $g_t = \frac{1}{t}(I - G_t)$, it is sufficient to prove the inclusion $\alpha f_t + \beta g_t \in \mathrm{HG}\,(D)$ for each $t > 0$. Fix $t > 0,\ r > 0$, and consider the equation

$$x + r(\alpha f_t(x) + \beta g_t(x)) = y,$$

where $y \in D$.

A simple chain of calculations shows that this equation is equivalent to the following equation:

$$x = \frac{r\alpha}{t + r(\alpha + \beta)} F_t(x) + \frac{r\beta}{t + r(\alpha + \beta)} G_t(x) + \frac{t}{t + r(\alpha + \beta)}\, y,$$

the right-hand side of which is a convex combination of self-mappings of D.

By Lemma 4.2 and the Earle-Hamilton theorem this equation has a unique solution $x\,(= x(y))$ which determines the resolvent $J_r(\alpha f_t + \beta g_t)\colon D \mapsto D$ of the mapping $\alpha f_t + \beta g_t$.

Applying theorem 4.2 we arrive at our assertion. □

For the case when D is a ball in X and a bounded $f \in \mathrm{Hol}\ (D, X)$ has a uniformly continuous extension to $\overline{D}$, we are able to formulate a simple boundary condition which implies that f is a generator of a flow in D. In formulating this condition we use the duality mapping J of X:

$$Jx = \{x^* \in X^* \colon (x, x^*) = |x|^2 = |x^*|^2\}.$$

Corollary 4.5. *Let D be a ball in X centered at the origin and let a bounded $f \in \mathrm{Hol}\ (D, X)$ admit a uniformly continuous extension to $\overline{D}$. If f satisfies the following boundary condition:*

$$\inf\{\mathit{Re}\langle f(x), x^*\rangle \colon x^* \in Jx\} \geq 0$$

for all $x \in \partial D$, then it is a semi-plus complete vector field.

We refer the reader to [5] for a full discussion, including the proof of Corollary 4.5 and other related results.

Remark 4.3. As we saw in §2 (see (2.11)), if $f_1 \in \mathrm{HG}\,(D)$ and $f_2 \in \mathrm{HG}\,(D)$ are close in the T-topology, then the semigroups S_{f_1} and S_{f_2} generated by them are also close as solutions of the Cauchy problems.

Thus, using property 2) of Remark 4.1, we obtain the formula

$$S_f(t) = \text{T-}\lim_{r \to 0^+} S_{Y_r}(t), \tag{4.9}$$

uniformly on compact t intervals, where $\{Y_r\}$ are the Yosida approximations, $r > 0$.

This formula is an analogue of the Yosida formula on representations of linear semigroups [77].

Combining the Lie algebraic methods developed, for example, in [35, 67, 7] and [20] with our previous results we obtain a complete analog of the Hille exponential formula for semigroups generated by holomorphic mappings. This will be done in the next section.

4.2. **Lie generators and the exponential formula.** Let $f \in \mathrm{HG}\,(D)$ be bounded on D. Then f generates a continuous semigroup $S_f = \{F_t\}$, $t \in R^+$, on D. It induces a linear vector field $\tilde{f} : E = \{g \in \mathrm{Hol}\ (D, X) :\ g$ is bounded on $D\} \mapsto E$, written symbolically as $\tilde{f} = f\frac{\partial}{\partial x}$, which is the differential operator on E defined by the formula

$$(\tilde{f}g)(x) = \Big(f(x)\frac{\partial}{\partial x}\Big)\Big(g(x)\Big) = g'(x) \circ f(x). \tag{4.10}$$

On the other hand, the semigroup $S_f = \{F_t\}$ generated by f induces the linear semigroup $\{\hat{F}_t\}_{t\geq 0}\colon E \mapsto E$, $t \geq 0$, defined by

$$(\hat{F}_t g)(x) = g(F_t(x)), \qquad x \in D, \quad t \geq 0. \tag{4.11}$$

It is clear by formal differentiation of formula (4.11) that the linear operator $\tilde{f}$ must be the generator of the semigroup $\{\hat{F}_t\}$. Indeed,

$$\begin{aligned} \frac{\partial(\hat{F}_t g)(x)}{\partial t} &= g'_x(F_t(x)) \circ \frac{\partial F_t(x)}{\partial t} = \\ &= -g'_x(F_t(x)) \circ f(F_t(x)) = -(\tilde{f}g)(F_t(x)) = -\tilde{f} \circ (\hat{F}_t g)(x). \end{aligned} \tag{4.12}$$

This generator $\tilde{f}$ is called the Lie generator induced by f. However, we want formula (4.12) to make sense in the T-topology of the space E.

First we note that the space E with the T-topology is a sequentially complete locally convex space with the seminorms

$$p_B(g) = \sup_{x\in B} \|g(x)\|,$$

where B is a ball strictly inside D [22]. Thus T-lim in E coincides with the strong limit in this space.

Lemma 4.3. *Let $G(\cdot,\cdot)\,(= G(t,x))$ be a function of two variables t and x continuous in $t \in [0,a)$ and holomorphic in $x \in D$.*

Suppose that $G(\cdot,\cdot)$ satisfies the following conditions:

(i) $G([0,a) \times D) \subset D$;
(ii) $\lim_{t\to 0^+} G(t,x) = x$ *for each* $x \in D$;
(iii) $\lim_{t\to 0^+} \frac{1}{t}(x - G(t,x)) = f(x)$ *for each* $x \in D$;
(iv) *For each ball $K \subset\subset D$ there exists a disk $\Omega_K \subset \mathbb{C}$ centered at $0 \in \mathbb{C}$ such that $G(\cdot,\cdot)$ admits a holomorphic extension to $\Omega_K \times K$ and $G(\Omega_K \times K) \subseteq D$.*

Then the induced collection of linear operators on E, $\{\hat{G}(t)(g) = G(t,\cdot) \circ g,\ g \in E\}$, satisfies the following formula:

$$\lim_{t\to 0^+} \frac{\hat{I} - \hat{G}(t)}{t} = \tilde{f}(x), \tag{4.13}$$

where $\tilde{f}(x)$ is the Lie generator induced by f and $\hat{I}$ is the identity on E.

The limit in (4.13) is the strong limit in E, as a locally convex space with the topology T.

Proof. For each $K \subset\subset D$ and $g \in E$ consider the mapping $h(\lambda,x) = g(x) - g(G(\lambda,x)) - \lambda g'(x) \circ f(\lambda,x)$, where

$$f(\lambda,x) = \frac{x - G(\lambda,x)}{\lambda}, \qquad \lambda \in \Omega_K = \Omega, \quad x \in K.$$

It follows from the conditions (ii) and (iv) that $f(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are holomorphic on $\Omega\times K$ and bounded, i.e.

$$\sup_{\substack{\lambda\in\Omega\\ x\in K}}\|h(\lambda,x)\| = M_{K,\Omega}(h) < \infty.$$

(Note that $g'(x)$ is bounded on K by the Cauchy inequalities.)

In addition, $h(0,x)=0$ and $h'_\lambda(0,x)=0$ for each $x\in K$.

Hence it follows by the generalized Schwarz lemma (Proposition 1.3) that

$$\|h(\lambda,x)\| \le \frac{M_{K,\Omega}(h)}{\delta^2}|\lambda|^2, \qquad x\in K,$$

where δ is the radius of Ω. Thus

$$\tfrac{1}{t}\|h(t,x)\| \to 0 \text{ as } t\to 0^+,$$

i.e.

$$p_K(\hat f_t(g) - g'(\cdot)\circ f_t) \to 0 \text{ as } t\to 0^+, \tag{4.14}$$

where $\hat f_t = \frac{1}{t}(\hat I - \hat G(t))$.

Once again by the Cauchy inequalities we have

$$p_K(\tilde f(g) - g'(\cdot)\circ f_t) = \sup_{x\in K}\|g'(x)f(x) - g'(x)f_t(x)\| \le$$

$$\le \sup_{x\in K}\{\|g'(x)\|\cdot\|f(x)-f_t(x)\|\} \to 0 \text{ as } t\to 0^+.$$

Together with (4.14) this implies the required formula (4.13). □

The first conclusion from this lemma is the following one. If $\{F_t\}$ is the semigroup defined as above by a semi-plus complete continuous vector field f, then as we saw in Proposition 2.2 it satisfies the conditions (i)–(iv) of the lemma. Thus substituting F_t for $G(t,\cdot)$ we have the formula

$$\tilde f = \lim_{t\to 0^+}\frac{\hat I - \hat F_t}{t}, \tag{4.15}$$

i.e. $\tilde f(g) =$T-$\lim_{t\to 0^+}\frac{1}{t}(g-\hat F_t(g))$, $g\in E$.

The second conclusion is an analog with respect to the collection of resolvents $\{J_r\}$, $r\in[0,\infty)$, where $J_r=(I+rf)^{-1}$ is defined on R^+ by Theorem 4.1 if D is a convex bounded domain in X. We want to show that this collection also satisfies the conditions (i)–(iv) of the above lemma. Indeed conditions (i)–(iii) were proved in §4.1. To prove that condition (iv) is also satisfied, we may assume without loss of generality that $0\in D$. Let K_r denote a ball with radius $r>0$. Fix two concentric balls K_s and $K_{s+\varepsilon}$ such that $K_{s+\varepsilon}\subset\subset D$, and consider the equation $x+\lambda f(x)=y$ written in the form

$$x = y-\lambda f(x) = h_1(\lambda,x), \tag{4.16}$$

where $y\in K_s$, $x\in K_{s+\varepsilon}$. Then for each $y\in K_s$ and for all $\lambda\in\Omega_\delta\subset\mathbb{C}$, where $\delta=\varepsilon/\sup\{\|f(x)\|\colon x\in D\}$ is the radius of the disk Ω_δ, the mapping $h_1(\cdot,\cdot)$ $(=h_1(\lambda,x))$ maps $\Omega_\delta\times K_{s+\varepsilon}$ into $K_{s+\varepsilon}$. In addition, $h_1(0,y)=y$.

Hence the equation (4.16) has a unique solution (see Corollary 8.1) $J_\lambda(y)$ which is holomorphic in $\Omega_\delta \times K_\rho$ and

$$J_\lambda(K_s) \subseteq K_{s+\varepsilon} \subset\subset D. \tag{4.17}$$

Thus substituting now J_t instead of $G(t,\cdot)$ in Lemma 4.3 we obtain the formula

$$\tilde{f} = \lim_{t\to\infty} \frac{\hat{I} - \hat{J}_t}{t}, \tag{4.18}$$

where $\hat{J}_t(g) = g(J_t)$, $g \in E$.

Now we want to show that there exists $\rho > 0$ such that for all $t \in [0,\rho)$,

$$\lim_{n\to\infty} \hat{J}^n_{\frac{t}{n}} = \hat{F}_t. \tag{4.19}$$

Once again using the Schwarz lemma for the mapping $h_2(\cdot,\cdot)$ $\Big(= g(x) - g(J_\lambda(x))\Big)$ and denoting $\sup\{\|f(x)\|\colon x \in D\}$ by $M(f)$, we obtain the inequality

$$\sup_{x\in K_s} \|\frac{1}{\lambda}[g(x) - g(J_\lambda(x))]\| \le \frac{2}{\varepsilon} \sup_{K_{s+\varepsilon}} \|g(x)\| \cdot M(f), \tag{4.20}$$

whenever $|\lambda| < \delta = \frac{\varepsilon}{M(f)}$.

Setting $\hat{Y}_\lambda = \frac{1}{\lambda}(\hat{I} - \hat{J}_\lambda)$, a linear operator in E, and $L = 2M(f)$, we may rewrite (4.20) as

$$p_{K_s}(\hat{Y}_\lambda(g)) \le \frac{L}{\varepsilon} p_{K_{s+\varepsilon}}(g), \tag{4.21}$$

whenever $|\lambda| < \delta = \frac{\varepsilon}{M(f)}$.

Now we take two positive numbers μ and η and integers $m,n > 0$, $m \le n$, such that $K_{\mu+\eta} \subset\subset D$. Let $\varepsilon = \frac{\eta}{m}$ and $\rho = \frac{\eta}{M(f)}$. Then for $t \in [0,\rho)$ we have $\lambda = t/h < \delta = \varepsilon/M(f)$ and the following chain of inequalities (a consequence of (4.21)):

$$\begin{aligned} p_{K_\mu}(\hat{Y}^m_{\frac{t}{n}}(g)) &= p_{K_\mu}(\hat{Y}^{m-1}_{\frac{t}{n}}(g)) \le \frac{L\cdot m}{\eta} p_{K_{\mu+\frac{\eta}{m}}}(\hat{Y}^{m-1}_{\frac{t}{n}}(g)) \le \\ &\le \Big(\frac{L\cdot m}{\eta}\Big)^2 p_{K_{\mu+\frac{2\eta}{m}}}(\hat{Y}^{m-2}_{\frac{t}{n}}(g)) \le \Big(\frac{L\cdot m}{\eta}\Big)^m p_{K_{\mu+\eta}}(g). \end{aligned} \tag{4.22}$$

Then using the binomial formula we have

$$\hat{J}^n_{\frac{t}{n}}(g) = [\hat{I} - \frac{t}{n}\hat{Y}_{\frac{t}{n}}]^n(g) = \sum_{m=0}^{n} \binom{n}{m} (-1)^m \Big(\frac{t}{n}\Big)^m \hat{Y}^m_{\frac{t}{n}}(g). \tag{4.23}$$

But for a fixed m we have

$$\binom{n}{m}\Big(\frac{t}{n}\Big)^m = \frac{1}{m!} t^m \frac{n(n-1)(n-2)\cdots(n-m+1)}{n^m} \to \frac{1}{m!} t^m. \tag{4.24}$$

It follows from (4.19) that the series (4.23) converges in the seminorm p_{K_μ} for t small enough. In addition, observe that for each fixed K and $t \in [0, \rho)$ (see (4.18))

$$p_{K_\mu}([\hat{Y}_{\frac{t}{n}}]^k(g) - \tilde{f}^k(g)) \to 0, \text{ as } n \to \infty. \tag{4.25}$$

Hence we have from (4.24) and (4.25) that for $t \in [0, \rho)$,

$$[\hat{J}_{\frac{t}{n}}]^n(g) \to \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \hat{f}^k(g) = \exp[-t\hat{f}(g)] = \hat{F}_t(g).$$

Now it is clear by induction that for such $t \in [0, \rho)$,

$$\hat{F}_t(I) = I(F_t) = F_t \text{ and } [\hat{J}_{\frac{t}{n}}]^n(I) = [J_{\frac{t}{n}}]^n = \Big(I + \frac{t}{n} f\Big)^{-n}.$$

Thus we have, for $t \in [0, \rho)$,

$$\lim_{n\to\infty} [\hat{J}_{\frac{t}{n}}]^n = \lim_{n\to\infty} \Big(I + \frac{t}{n} f\Big)^{-n}(x) = F_t(x)$$

uniformly on K_μ and hence uniformly on each ball strictly inside D, because the family $\Big(I + \frac{t}{n} f\Big)^{-n}$ is uniformly bounded on D.

Finally, let t be an arbitrary positive fixed number. Choose $m > 0$ such that $t/m \in [0, \rho)$. Then $n_1 = \frac{n}{m}$ tends to infinity as n tends to infinity. Since the composition of self-mappings of D is a continuous operation (see Proposition 1.5), the semigroup property of $\{F_t\}$, $t \geq 0$, yields

$$\text{T-}\lim_{n_1\to\infty} [J_{\frac{t}{n}}]^n = \text{T-}\lim_{n_1\to\infty} \Big[[J_{\frac{t}{mn_1}}]^{n_1}\Big]^m = [F_{t/m}]^m = F_t.$$

Thus we have proved the following theorem:

Theorem 4.2. *Let D be a bounded convex domain in X, and let $f \in \mathrm{HR}\,(D)$ be bounded. Then f generates a semigroup $S_f = \{F_t\}$, $F_t \in \mathrm{Hol}\,(D, D)$, and*

$$F_t = T\text{-}\lim_{n\to\infty} \Big(I + \frac{t}{n} f\Big)^{-n}.$$

5. Null point sets of holomorphic generators

5.1. **Structure of the null point sets of semi-complete vector fields.** By $\mathrm{Null}_D f$ we denote the analytic set defined as the null point set of $f \in \mathrm{Hol}\,(D, X)$. Even in the finite dimensional case it is a complicated problem to recognize when an analytic set $\mathcal{N}$ consists only of irreducible components (see, for example [14]). It is known that this is the case when $\mathcal{N}$ is locally a complex analytic manifold.

The results of §4 and Theorem C lead to the following global description of the null point sets of semi-complete vector fields.

Theorem 5.1. *Let D be a convex bounded domain in X, and let $f \in \mathrm{HG}\,(D)$ be bounded. Suppose that $a \in \mathrm{Null}_D f$ and that one of the following hypotheses holds:*

1) *X is reflexive,*
2) $\mathrm{Ker}\, A \oplus \mathrm{Im}\, A = X$, *where $A = f'(a)$.*

Then $\mathrm{Null}_D f$ *is a connected complex analytic submanifold in* D*, which is tangent to* $\mathrm{Ker}\ A$.

Proof. It is sufficient to note that $\mathrm{Null}_D f = \mathrm{Fix}_D J_r(f)$ for all $r > 0$, where $\mathrm{Fix}_D J_r(f)$ is the fixed point set of the resolvent $J_r(f)$ of f in D. In addition, it follows by the chain rule and property 1 of Remark 4.1 that $I - J_r(A) = rJ_r(A)$, where $J_r(A) = [J_r(f)]'(a)$ is the resolvent of the linear operator A for $r > 0$. Thus $\mathrm{Ker}\ (I - [J_r(f)]'(a)) = \mathrm{Ker}\ A$ and the theorem is proved. □

Corollary 5.1. *Let* D, X *and* f *be as above. If* $a \in D$ *is an isolated point of* $\mathit{Null}_D f$*, then it is unique. In particular, if* $a \in \mathit{Null}_D f$ *is regular, i.e.* $f'(a)$ *is invertible, then* a *is unique.*

5.2. **Stationary points of nonlinear semigroups.** Now we consider the stationary point set $\mathcal{F}$ of a semigroup $S = \{F_t\}$ with a holomorphic generator. This set is defined as the common fixed point set of $\{F_t\}$ for all t, i.e.

$$\mathcal{F}_D = \cap\{\mathrm{Fix}_D F_t : t \in R^+\}.$$

If the generator $f \in \mathrm{Hol}\ (D, X)$ is semi-plus complete, then it follows from the uniqueness of the solution to the Cauchy problem that the stationary point set of S coincides with the null point set of f, i.e. (see, for example, [13] and [2])

$$\mathcal{F}_D = \mathrm{Null}_D f. \tag{5.1}$$

Note that actually this also holds for the more general case, when f is a generator in the sense of Definition 2.2.

Proposition 5.1. *Let* D *be a domain in* X *and let* $S = \{F_t\}$, $t \geq 0$, $F_t \in \mathrm{Hol}\ (D, D)$, *be a semigroup with a generator* $f \in \mathrm{Hol}\ (D, X)$. *Then the stationary point set* $\mathcal{F}_D$ *of* S *in* D *coincides with the null point set of the generator* f*, i.e. formula (5.1) holds.*

Proof. As we mentioned above, the mapping F_t satisfies the equation (2.6).

As a matter of fact, it can be shown that it also satisfies another differential equation:

$$\frac{\partial^+ F_t(x)}{\partial t} = -\frac{\partial F_t}{\partial x} \circ f(x). \tag{5.2}$$

Therefore, if $z \in \mathrm{Null}_D f$, then by (5.2)

$$\frac{\partial^+ F_t(z)}{\partial t} = 0.$$

This in turn implies that $F_t(z)$ is a constant and hence $F_t(z) = F_0(z) = z$ for all $t \geq 0$. The converse statement is evident. □

Remark 5.1. The following example shows that formula (5.1) is no longer true for the closure of D even in the case when f is continuous in $\overline{D}$.

Example 5.1. Let D be the unit disk in the complex plane $\mathbb{C}$, i.e. $D = \{x \in \mathbb{C} : |x| < 1\}$. Consider $f(x) = x - 1 + \sqrt{1 - x}$. It is clear that $f \in \mathrm{Hol}\ (D, C)$ and that it is continuous on $\overline{D}$.

In addition, $\mathrm{Null}_{\overline{D}} f = \{0, 1\}$. However, the Cauchy problem (2.7) has the solution

$$F_t \colon \overline{D} \mapsto \overline{D}, \qquad t \geq 0,$$

defined by the formula

$$F_t(x) = 1 - [1 - e^{-\frac{1}{2}t} + e^{-\frac{1}{2}t}\sqrt{1-x}]^2,$$

and for all $t > 0$ we have

$$F_t(1) = 1 - [1 - e^{-\frac{1}{2}t}]^2 < 1$$

Thus $\mathcal{F}_{\overline{D}} \neq \mathrm{Null}_{\overline{D}} f$.

Thus from Theorem 5.1 we can obtain the global description of the (interior) stationary point set of a semigroup $\{F_t\}$, $t \geq 0$, generated by a holomorphic bounded mapping.

Here we establish another interesting feature of this set.

Let us consider a semigroup $S = \{F_t\}$, $t \in R^+$, generated by $f \in \mathrm{Hol}\,(D, X)$. Let f_t be, as above, the difference approximations of f, i.e. $f_t = \frac{1}{t}(I - F_t)$, $t > 0$. If $\mathrm{Null}_D f$ is not empty, then

$$\mathrm{Null}_D f \subseteq \mathrm{Null}_D f_t, \quad t > 0. \tag{5.3}$$

Moreover, it is natural to expect that for sufficiently small t, $\mathrm{Null}_D f_t$ approximates $\mathrm{Null}_D f$ in some sense.

As a matter of fact, in the linear case, as well as in the holomorphic case, there is a stabilization phenomenon of $\mathrm{Null}_D f_t$ for sufficiently small t.

Theorem 5.2. *Let $f \in \mathrm{HG}\,(D)$ be bounded and let $f_t = t^{-1}(I - F_t)$, where $\{F_t\}$ is the continuous semigroup generated by f.*

Suppose that $\mathrm{Null}_D f \neq \emptyset$ and that one of the following conditions holds:

1) *X is reflexive,*
2) Ker $f'(a) \oplus \mathrm{Im}\, f'(a) = X$ *for some $a \in Null_D f$.*

Then there exists $\delta > 0$ such that for all $t \in (0, \delta)$,

$$Null_D f_t = Null_D f. \tag{5.4}$$

Proof. Since both $\mathrm{Null}_D f_t$ and $\mathrm{Null}_D f$ are connected complex submanifolds of D and

$$\mathrm{Null}_D f \subseteq \mathrm{Null}_D f_t,$$

it suffices to show that their tangent spaces coincide. A simple calculation shows that for $a \in \mathrm{Null}_D f$, $(f_t)'(a) = \frac{1}{t}(I - e^{-tA})$, where $A = f'(a)$. Thus our claim is that there exists a positive δ such that for all $t \in (0, \delta)$,

$$\mathrm{Fix}\,(e^{-tA}) = \mathrm{Ker}\ A. \tag{5.5}$$

In order to prove (5.5) when X is reflexive, we first note that the semigroup $e^{-tA} = (F_t)'(a)$ is uniformly bounded by the Cauchy inequalities. We then let P denote the projection of X onto Ker A obtained from the mean ergodic theorem.

Now let $g_t = \frac{1}{t}\int_0^t e^{-sA}\,ds$. There is a positive δ such that g_t is invertible for all $0 < t < \delta$. For such t, let P_t be the mean ergodic projection onto $\operatorname{Fix}(e^{-tA})$. A computation shows that for all natural numbers m,

$$g_{mt} = \Big(\frac{1}{m}\sum_{j=1}^{m} e^{-(j-1)tA}\Big) g_t.$$

Letting $m \to \infty$, we see that $P = P_t g_t = g_t P_t$. Hence $P_t = g_t^{-1} P$ and $\operatorname{Ker}\, P \subset \operatorname{Ker}\, P_t$. Since

$$X = \operatorname{Fix} P \oplus \operatorname{Ker}\, P = \operatorname{Fix} P_t \oplus \operatorname{Ker}\, P_t$$

and $\operatorname{Fix} P \subset \operatorname{Fix} P_t$, it follows that $\operatorname{Ker}\, A = \operatorname{Fix} P = \operatorname{Fix} P_t = \operatorname{Fix} e^{-tA}$.

When hypothesis 2) holds, the following simple direct argument is due to V. Khatskevich.

In this case there is a positive ϵ such that

$$\|Az\| \geq \varepsilon \|z\| \tag{5.6}$$

for all $z \in \operatorname{Im} A$.

Let $x = y + z$, where $y \in \operatorname{Ker}\, A$ and $z \in \operatorname{Im} A$, belong to $\operatorname{Fix}(e^{-tA})$. Then $e^{-tA}z = z$ and

$$Az = t\Big(\frac{A^2}{2!} - \frac{tA^3}{3!} + \cdots\Big)z.$$

If $0 < t < \min\{1, \epsilon/(e^{\|A\|} - 1 - \|A\|)\}$ and $z \neq 0$, it follows that $\|Az\| < \epsilon\|z\|$, which contradicts (5.6). Hence $z = 0$ and $x = y$ belongs to $\operatorname{Ker}\, A$. $\square$

6. Local and spectral characteristics of stationary points

6.1. **Cartan's uniqueness theorem.** The following simple consequences of the above results indicate that some local characteristics of a null point of a generator can influence the global structure of the whole null point set and the global behavior of the semigroup.

Theorem 6.1. *Let D be a convex bounded domain in X, and let $f \in \operatorname{HG}(D)$ be bounded and have a null point $a \in D$. If $f'(a) = 0$, then $f \equiv 0$.*

Proof. Indeed, it is clear that $a \in \operatorname{Null}_D f$ is a fixed point of the resolvent $J_r = (I + rf)^{-1} \in \operatorname{Hol}(D, D)$, $r > 0$. In addition $(J_r)'(a) = I|_D$. Thus by Cartan's theorem (see, for example, [22], [41]), $J_r \equiv I|_D$. This implies that $f \equiv 0$ in D. $\square$

Moreover, we can establish a continuous form of this assertion. It is a generalization of the Harris-Schwarz lemma [26].

Theorem 6.2. *Let D be a convex bounded domain in X, and let $\{f_n\} \subset \operatorname{HG}(D)$ be a uniformly bounded sequence of holomorphic mappings such that for some $a \in D$ the following conditions hold:*

a) $\{f_n(a)\}$ *strongly converges to zero;*

b) $\{f_n'(a)\}$ *converges to* 0 *in the operator topology.*

Then $\{f_n\}$ T-converges to 0 in D, i.e. $T\text{-}\lim_{n\to\infty} f_n \equiv 0$.

6.2. **Harris' spectrum of a semi-complete vector field.** Following L. Harris [27] we give the following definition:

Definition 6.1. *Let D be an open subset of X, $a \in D$, and let $h \in \mathrm{Hol}\,(D, X)$. The spectrum of h with respect to a, denoted by $\sigma_a(h)$, is the set of all $\lambda \in \mathbb{C}$ such that it is not possible to find open sets $U \subset D$, with $a \in U$, and $V \subseteq X$ with the property that $\lambda I - h$ is a biholomorphism of U onto V.*

Proposition 6.1. ([27]) *$\sigma_a(h) = \sigma(h'(a))$ is the spectrum of the linear operator $h'(a)$.*

Theorem 6.3. *Let $f \in \mathrm{HG}\,(D)$ be bounded and let $a \in \mathrm{Null}_D f$. Then*

1) *$\sigma_a(f)$ lies in the right half-plane;*
2) *If $0 \notin \sigma_a(f)$, then a is the unique null point of f in D.*
3) *$\sigma_a(f)$ lies strictly inside the right half-plane iff a is a globally asymptotically stable (in the Lyapunov sense) stationary point of the semigroup $S_f = \{F_t\}$, $t \geq 0$. More precisely, $\{F_t\}$ T-converges to a in D.*

Proof. Set $A = f'(a)$. It is easy to see that A is the infinitesimal generator of a uniformly continuous semigroup $U_t = e^{-tA}$ and that $U_t = (F_t(x))'_{x=a}$. Thus it follows by the Cauchy inequalities that U_t is a uniformly bounded semigroup of linear operators. It is well known that the resolvent $R(\lambda, A) = (\lambda I - A)^{-1}$ is defined on the open left half-plane, i.e. $\mathrm{Re}\,\lambda \geq 0$ for all $\lambda \in \sigma(A)$. Thus assertion 1) follows from Proposition 6.1.
2) If $0 \notin \sigma_a(f)$, then the operator $A = f'(a)$ is invertible. Hence a is an isolated null point of f in D, and our assertion follows from corollary 5.1.
3) Suppose now that $\sigma_a(f) = \sigma(A)$ lies strictly inside the right half-plane of $\mathbb{C}$. As it is well known this fact implies the estimate

$$\|e^{-tA}\| \leq Ne^{-\nu t} \tag{6.1}$$

for some $N > 0$ and $\nu > 0$ (see, for example, [77], [15]).

Rewrite now the Cauchy problem in the form of a perturbed equation:

$$\begin{aligned} x'(t) &= -Ax(t) + g(x(t)), \\ x(0) &= x \in D, \end{aligned} \tag{6.2}$$

where $g = A - f$.

Since $f(a) = 0$, there is some ball $B_r(a) \subset\subset D$, centered at a with radius r, such that g admits the representation

$$g(x) = \sum_{k=2}^{\infty} P_f^{(k)}(a) \circ (x - a)$$

where $P_f^{(k)}(x)$, $k \geq 2$, are homogeneous forms of order k (see §1). Setting $M = \sup_{x \in D} \|g(x)\|$, we have, by Proposition 1.3,

$$\|g(x)\| \leq Mr^{-2}\|x - a\|^2$$

for all $x \in B_r(a)$. Choosing now $\rho < r\sqrt{\nu(MN)^{-1}}$, where ν and N are as in (6.1), we obtain the inequality

$$\|g(x)\| < \frac{\nu}{N}\|x-a\|$$

for all $x \in \overline{B_\rho(a)} = \{x \in D\colon \|x-a\| \le \rho\}$. Thus Theorem VII.2.1. from [15], p. 403, implies that problem (6.2) has a uniformly asymptotically stable solution on $\overline{B_\rho(a)} \times R^+$. In other words, the net $F_t|_{B_\rho(a)} = x(t)$ converges uniformly to the point a uniformly on $B_\rho(a)$. An appeal to Proposition 1.4 (§1) concludes now the proof of our assertion in one direction.

Conversely, let $\{F_t\}$ T-converge to $a \in \mathrm{Null}_D f$. Then it follows from Theorem A that for all $t > 0$ the spectral radius $r_\sigma(U_t) < 1$, where

$$U_t = e^{-tA} = (F_t)'_{x=a}.$$

By Dunford's theorem on the spectrum it follows that $\sigma(A) = \sigma_a(f)$ lies strictly inside the right half-plane and we are done. $\square$

Definition 6.2. ([46]) *Let D be a domain in X and let $f \in \mathrm{Hol}\,(D, X)$. A point $a \in Null_D f$ is said to be regular if $0 \notin \sigma(f'(a))$, i.e. $f'(a)$ is an invertible linear operator. It is said to be strictly regular if $\sigma(A)$ does not intersect the imaginary axis of the complex plane $\mathbb{C}$.*

According to this definition we obtain the following direct consequence of Theorems 6.3 and 5.1:

Corollary 6.1. *Let D be a bounded convex domain in X and let f be a bounded semi-complete vector field in D. Suppose that f is a Fredholm mapping and that $a \in Null_D f$.*

Then $Null_D f = \{a\}$ if and only if the point a is regular.

Remark 6.1. If f is not Fredholm, but X is reflexive, we have, in general, two singular situations. Namely, if $a \in \mathrm{Null}_D f$ and $0 \in \sigma_a(f)$, then either

1) a is the unique null point in D, or
2) there are infinitely many null points of f in D and they form a connected complex submanifold of D.

The following example shows that situation 1) actually may exist in the case of an infinite dimensional space (even if it is reflexive). Despite its uniqueness, such a point has no "good" property such as regularity.

Example 6.1. Let X be the complex Hilbert space ℓ^2 with basis $\{e_i\}_{i=1}^\infty$, and let $0 < \alpha_i < 1$ satisfy $\alpha_i \to 1$ as $i \to \infty$. Let D be the unit ball in X and define the linear mapping $A\colon \overline{D} \mapsto X$ by $Ae_i = (1-\alpha_i)e_i$. This mapping has a unique null point $x = 0$, but it is not regular (0 is a point of the continuous spectrum of A).

It is clear that A is the generator of a semigroup of self-mappings of D.

Now we turn to the same questions concerning the approximation of fixed points.

Let D be as above, and let $F\colon D \mapsto D$ be a holomorphic self-mapping of D. Its iterates $F^n\colon D \mapsto D$, $F^n = F^{n-1} \circ F$, $n = 1, 2, \ldots$, $F^0 = I$, are well defined and holomorphic. However, even when X is finite dimensional and F has a unique fixed point, there are many situations when the sequence of iterates $\{F^n(x)\}_{n=0}^{\infty}$ does not converge to the fixed point a for $x \neq a$.

For example, let D be a unit ball and $F = e^{i\varphi}I,\ 0 < \varphi < 2\pi$.

More generally, such a situation arises when the spectrum $\sigma(B)$ of the linear operator $B = f'(a)$ contains points of the unit circle other than 1 (see, for example, [70, 71] and [1]).

There are many other approximative methods (explicit and implicit) for finding the fixed point. They can be found, for example, in [23, 52, 41, 39] and [60].

We include here only one observation in this direction.

Let $F \in \mathrm{Hol}\ (D, D)$ have a fixed point $a \in D$ such that $1 \notin \sigma(B)$ where $B = F'(a)$. Then this point is a regular null point for the mapping $f = I - F \in \mathrm{Hol}\ (D, X)$, which is a bounded semi-plus complete vector field. As a matter of fact, it is also strictly regular. Indeed, if $\lambda \in \sigma(A)$ where $A = f'(a) = I - B$, then $1 - \lambda \in \sigma(B)$ and $\mathrm{Re}(1 - \lambda) \leq 1$ by Theorem 6.3. But $|1 - \lambda| \leq 1$ by Theorem A and $1 - \lambda \neq 1$ according to our assumption. Hence $\mathrm{Re}\,\lambda > 0$ and we are done.

Thus we have proved the following theorem.

Theorem 6.4. *Let D be a bounded convex domain in X, and let $F \in$ Hol (D, D) have a fixed point $a \in D$ which is a regular null point of $f = I - F$.*

Then the semigroup family $\{F_t\}$ defined by the Cauchy problem

$$\begin{aligned} &\frac{\partial F_t}{\partial t} = F(F_t) - F_t, \\ &F_0 = I|_D, \end{aligned} \tag{6.4}$$

T-converges to a as t tends to infinity.

As a simple example, consider again the mapping $F = iI$, mentioned above, whose iterates do not converge to zero for each $x \neq 0$. At the same time the Cauchy problem (6.4) has the solution $F_t(x) = e^{it} \cdot e^{-t}x$ which evidently uniformly converges to zero as t tends to infinity.

More complicated (nonlinear) examples will be considered below in §7, when we don't know a priori the location of the fixed point.

6.3. **Trotter-Kato type theorems.**

Theorem 6.5. *Let $\{f_n\}_1^{\infty} \subseteq \mathrm{HG}\,(D)$ be a uniformly bounded sequence of semi-plus complete vector fields which have a common null point $x_0 \in D$, i.e. $f_n(x_0) = 0$ for all $n = 1, 2, \ldots$. Set $R(\lambda, f_n) = (\lambda I - f_n)^{-1}$, which is defined on some neighborhood $V_{n,\lambda} \subset X$, $V_{n,\lambda} \ni 0$, where* Re $\lambda < 0$ *(see Theorem* 6.3*). Suppose that for some λ_0,* Re $\lambda_0 < 0$*, there is a number n_0 and a neighborhood $V \ni 0$ such that $V \subset \bigcap_{n \geq n_0} V_{n,\lambda_0}$ and the sequence $\{R(\lambda_0, f_n)\}$ converges to $\{R_{\lambda_0}\}$ uniformly on V. Then*
1) *there exists $f \in \mathrm{HG}\,(D)$ such that $\mathrm{R}_{\lambda_0} = (\lambda_0 I - f)^{-1}$;*
2) *$\{f_n\}$ converges to f in the topology of local uniform convergence over D.*

Proof. Set $A_n = f_n'(x_0)$. Then as we mentioned above $\mathrm{R}\,(\lambda, A_n) = (\lambda I - A_n)^{-1} = [R(\lambda, f_n)]'(0)$. Hence by the Cauchy inequalities we have that $\{R(\lambda_0, A_n)\}$ converges to the linear operator $B = [R_{\lambda_0}]'(0)$ in the operator norm. By the linear Trotter-Kato theorem, B is the resolvent of some linear operator A, i.e. $B = (\lambda_0 I - A)^{-1}$. Thus R_{λ_0} is invertible in a neighborhood of zero and hence there is a neighborhood $U \ni x_0$, $\overline{U} \subset D$ such that $\overline{U} \subset \cap_{n \geq n_0}$ $\mathrm{Im}[\mathrm{R}\,(\lambda_0, f_n)]$ and $g_n = \lambda_0 I - f_n$ converges uniformly on $\overline{U}$ to $g = [R_{\lambda_0}]^{-1} \in$ Hol $(\underline{U}, X)$ (see, for example, [4]). This means that $\{f_n\}$ converges uniformly on $\overline{U}$, and hence locally uniformly on D to the mapping $f = \lambda_0 I - g \in$ Hol (D, X), with $f(x_0) = 0$. By Lemma 2.1, $f \in \mathrm{HG}\,(D)$ and it is evident that $(\lambda_0 I - f)^{-1} = R_{\lambda_0}$. □

Corollary 6.2. *Let $\{f_n\}_1^\infty \subseteq \mathrm{HG}\,(D)$ be a uniformly bounded sequence such that $f_n(x_0) = 0$, $x_0 \in D$, for $n = 1, 2, \ldots$. Suppose that for some $r_0 > 0$ the sequence $\{J_{n,r_0} = (I + r_0 f_n)^{-1}\} \subset$ Hol (D, D) converges to $J_{r_0} \in$ Hol (D, D) in the topology of local uniform convergence over D. Then for all $r > 0$, the sequence $\{J_{n,r} = (I + r f_n)^{-1}\}$ converges to J_r uniformly on each compact subset of R^+ and $J_r = (I + rf)^{-1}$, where $f \in HG(D)$.*

7. Existence and uniqueness of a null point

7.1. **Boundary conditions.** Concerning the existence of a fixed point of holomorphic self-mappings we mentioned above in §1 two results: Theorem B and Theorem E.

Using the resolvent method we are able to generalize them and treat the existence of a null point of semi-complete bounded vector fields.

Moreover, for existence and uniqueness we can point out more general conditions which allow us to consider a wider class of mappings (even in the case of self-mappings).

Recall that a point $a \in \mathrm{Null}_D f$ is said to be regular if $f'(a)$ is an invertible linear operator.

Theorem 7.1. *Let D be a bounded domain in X and let $f \in \mathrm{HR}\,(D)$. Suppose that there exist $K \subset\subset D$ and $\varepsilon > 0$ such that*

$$\|f(x)\| \geq \varepsilon \text{ for all } x \in D \setminus K. \tag{7.1}$$

Then f has a unique null point in D and it is regular.

Proof. Let $J_r \colon D \mapsto D$ be the resolvent of f, $r > 0$. Then for r large enough

$$J_r(D) \subset K. \tag{7.2}$$

Indeed, for all $r > 0$ and $x \in D$, $f(J_r(x)) = r^{-1}(x - J_r(x))$ and therefore there exists $r_0 > 0$ such that for all $x \in D$, $\|f(J_r(x))\| < \varepsilon$ whenever $r > r_0$. Hence for such r, (7.1) implies (7.2). Using the Earle-Hamilton theorem (see §1, Theorem B) and the observation that $\mathrm{Fix}_D J_r = \mathrm{Null}_D f$ we obtain the existence and the uniqueness of a null point $a \in D$ of the mapping f.

Now it follows by Theorems A and B that the linear operator $I - B$, where $B = (J_r(x))'_{x=a}$ is invertible. Further, using the chain rule we see that $B = (I + rA)^{-1}$ where $A = f'(a)$.

Hence $A = r^{-1}(I - B)^{-1}B^{-1}$ is invertible. This means that the point $a \in \mathrm{Null}_D f$ is regular, and the theorem is proved. □

Recall that a mapping $f\colon \overline{D} \mapsto X$ is said to be proper if the inverse image of each compact set is also compact.

Corollary 7.1. *Let D be a bounded domain in X and let $f \in HR(D)$ be continuous and proper on $\overline{D}$. Then*
1) f has a null point in $\overline{D}$;
2)If f has no null points on ∂D, then it has a unique null point in D and it is regular.

Proof. Once again, let J_r be the resolvent of f, $r > 0$. As we saw above, for each $x \in D$, $f(J_r(x))$ converges to zero, as r tends to infinity. Since f is proper, the net $\{J_r(x)\}$ must be precompact. Its limit point is a null point of f since f is continuous on $\overline{D}$.

If f has no null points on ∂D, then it satisfies condition (7.1) for some $K \subset\subset D$ because f is assumed to be proper. Hence assertion 2) is a consequence of Theorem 7.1. □

Corollary 7.2. *Let D be a bounded domain in X and let $F\colon D \mapsto D$ be a holomorphic mapping which has a uniformly continuous extension to $\overline{D}$. If F satisfies the condition*

$$\|x - Fx\| \geq \varepsilon > 0 \textit{ for all } x \in \partial D, \tag{7.3}$$

then it has a unique fixed point a in D, and the spectrum of the linear operator $I - F'(a)$ lies strictly inside the right half plane.

For the proof it is sufficient to note that $f = I - F$ in this case belongs to HR(D) and satisfies the condition (7.1).

Remark 7.1. The last assertion of Corollary 7.2 implies, by Theorem 6.3, that the fixed point of F is the strong limit, as $t \to \infty$, of the semigroup $\{\Phi_t\} = S_{I-F}$ generated by $I - F$.

Example 7.1. Let X be a complex Banach algebra with a unit e. Let a be an invertible element of X such that $\|a^{-1}\| = \|a\| = 1$. Consider the mapping $F\colon D \mapsto X$, where D is the open unit ball of X, defined as follows:

$$F(x) = (e + ixa)(3a - ia^2x)^{-1}.$$

The equation $x = Fx$ is equivalent to the algebraic Riccati equation

$$(3 - i)xa - ixa^2x - e = 0.$$

(Note that in general the element a^2 does not commute with all $x \in X$.)

The mapping F is clearly a self-mapping of $\overline{D}$. But it does not map $\overline{D}$ strictly inside D because $F(-ia^{-1}) = a^{-1}$, so we cannot apply Theorem B (§1). In addition, generally speaking, F is not compact in the case of an infinite dimensional X.

Nevertheless, it is easy to see that $\|x - F(x)\| \geq \frac{|3-i|-2}{4}$ when $\|x\| = 1$. Thus F has a fixed point $x^* \in D$ by Corollary 7.2.

Corollary 7.3. *Let D be a ball in X. Suppose that $f\colon D \mapsto X$ is a holomorphic mapping which has a uniformly continuous extension to $\overline{D}$ and satisfies the following boundary condition:*

$$\inf\{\operatorname{Re}\langle f(x), x^*\rangle \colon x^* \in Jx\} \geq \varepsilon > 0$$

for all $x \in \partial D$, where J is the duality mapping of X. Then f has a unique null point in D and it is regular.

This corollary can be proved by combining Corollary 4.5 with Theorem 7.1.

7.2. **The Hilbert ball and its powers.**

Theorem 7.2. *Let B be the open unit ball in a complex Hilbert space H, and let $f \in \mathrm{HG}\,(B)$ be bounded on B and continuous on $\overline{B}$. Then f has a null point in $\overline{B}$.*

Proof. Consider the resolvent $J = J_1 = (I + f)^{-1} \colon B \mapsto B$, which is holomorphic on B. If J has a fixed point in B, the problem is solved because $f(z) = 0$. Suppose now that J has no fixed point in B. Then by Theorem 27.3 in [23] the approximating curve $z(t)$, defined implicitly by the equation

$$z(t) = (1-t)x + tJ(z(t)), \quad x \in B, \tag{7.4}$$

on the interval $[0, 1)$, converges strongly, as $t \to 1^-$, to the point $z^* \in \partial B$. (The problem is that in general we don't know if J is also continuous on $\overline{B}$.)

However, $y(t) = J(z(t)) = \frac{t-1}{t}\, x + \frac{1}{t}\, z(t) \in B$, and $y(t)$ converges to z^* when t tends to 1^-. Since f in continuous on $\overline{D}$, it follows that $f(y(t))$ converges to $f(z^*)$ when t tends to 1^-. But on the other hand,

$$\|f(y(t))\| = \|f\Big(J(z(t))\Big)\| = \|z(t) - J(z(t))\| \leq 2(1-t),$$

and hence it converges to zero when t tends to 1^-. Thus $f(z^*) = 0$ and the theorem is proved. □

Remark 7.2. Another proof of this result can be based on Theorem 30.8 in [23]. Theorem 7.2 is a generalization of Theorem 15 in [24] (see also [23]) by Proposition 4.3. As a matter of fact, there is another generalization of this theorem due to T. Kuczumov and A. Stachura [47, 48], which provides the existence of a fixed point for a holomorphic self-mapping of the unit ball $D = B^n$ in H^n which is continuous on $\overline{D}$. But unfortunately we don't know if the approximating curve (7.4) strongly converges in this case too. Nevertheless, if $f \in \mathrm{HG}\,(D)$ is Lipschitzian on $\overline{D}$, then we can prove that for sufficiently small $r > 0$ the resolvent J_r is also Lipschitzian on $\overline{D}$.

Since in this case $\mathrm{Null}_{\overline{D}} f = \mathrm{Fix}_{\overline{D}} J_r$, we obtain the following result.

Proposition 7.1. *Let $D = B^n$, where B is the open unit ball in a Hilbert space H, and let $f \in \mathrm{HG}\,(D)$ be Lipschitzian on $\overline{D}$. Then f has a null point in $\overline{D}$.*

Now let D be a bounded convex domain in $\mathbb{C}^n$, and let F be a holomorphic self-mapping of D. It follows by Lemma 4.2 and the compactness of $\overline{D}$ that F has a fixed point in D if and only if the approximating curve $z(t) = (1-t)y + tF(z(t))$ is strictly inside D for a fixed $y \in D$ (see also [76]). Thus if F is fixed point free, then there is a sequence $\{t_n\} \to 1$ such that $\{z(t_n)\}$ converges to a point on the boundary of D. Therefore the same arguments as in Theorem 7.2 lead to the following result.

Proposition 7.2. *Let D be a bounded convex domain in $\mathbb{C}^n$, and let $f \in HG(D, \mathbb{C}^n)$ be continuous on $\overline{D}$. Then f has a null point in $\overline{D}$.*

This assertion is also a direct consequence of Corollary 7.1 and Theorem 4.1.

8. Continuation by complex parameter

In this section we consider a family of semi-plus complete vector fields which depend holomorphically on a complex parameter. We show that if for at least one value of the parameter the semi-plus complete vector field has a null point, then each element of the family has a null point. Moreover, each such point belongs to a holomorphic "branch" of null points.

These results improve upon those in [37] where the vector fields were assumed to be Lipschitzian. Using the resolvent method of §4 we are able to eliminate this strong assumption.

More precisely, we have the following result.

Theorem 8.1. *Let D be a convex bounded domain in a reflexive Banach space X, and let Δ be a domain in a reflexive Banach space Λ. Suppose that $f(\cdot,\cdot)\colon D \times \Delta \mapsto X$ is a bounded holomorphic mapping on $D \times \Delta$ such that $f(\cdot,\lambda) \in \mathrm{HG}\,(D)$ for each $\lambda \in \Delta$.*

Assume that for some $\lambda_0 \in \Delta$, $f(\cdot,\lambda)$ has a null point x_0:

$$f(x_0, \lambda_0) = 0. \tag{8.1}$$

Then

1) $f(\cdot,\lambda)$ has a null point $x(\lambda)$ for all $\lambda \in \Delta$, i.e.

$$f(x(\lambda), \lambda) = 0. \tag{8.2}$$

2) The sets $\mathcal{N}_\lambda = \{x \in D\colon f(x,\lambda) = 0, \quad \lambda \in \Delta\}$ are complex connected submanifolds of D with the same dimension, i.e.

$$\dim \mathcal{N}_\lambda = \text{const.}, \quad \lambda \in \Delta.$$

3) There is a holomorphic mapping $\rho(\cdot,\cdot)\colon D \times \Delta \mapsto X$, such that for each $\lambda \in \Delta$, $\rho(\cdot,\lambda)$ is a retraction onto $\mathcal{N}_\lambda$, i.e. for each $x \in D$ and $\lambda \in \Delta$, $\rho(x,\lambda) \in X$ is a solution of (8.2) and $\rho(\rho(x,\lambda),\lambda) = \rho(x,\lambda)$.

Proof. **Step 1.** First we note that $f(\cdot,\lambda) \in \mathrm{HR}\,(D)$ for all $\lambda \in \Delta$, by Theorem 4.1. Now recall that for each $r > 0$,

$$\mathrm{Null}_D f(\cdot,\lambda) = \mathrm{Fix}_D J_r(\cdot,\lambda)$$

and therefore it is sufficient to prove our assertion for the equation

$$x(\lambda) = J(x(\lambda), \lambda), \tag{8.2$'$}$$

where $J = J_1$, under the condition

$$x_0 = J(x_0, \lambda_0). \tag{8.1$'$}$$

Step 2. Consider the domain $\Omega = D \times \Delta$ which is bounded and convex in the complex Banach space $Z = X \times \Lambda$, equipped, for example, with the max norm. Define the mapping $T\colon \Omega \mapsto \Omega$ via the formula

$$T(x, \lambda) = (J(x, \lambda), \lambda).$$

By assumption (8.1$'$), T has a fixed point $z_0 = (x_0, \lambda_0) \in \Omega$. Without loss of generality, assume that $z_0 = 0$, and set $S = T'(0)$. Since Z is reflexive, by Theorem C (§1) the fixed point set of T in Ω ($\mathrm{Fix}_\Omega T$) is a complex connected submanifold of Ω tangent to $L = \mathrm{Ker}\ (I|_Z - S)$. Let $B = J'_x(x_0, \lambda_0)$. Our claim is that L is isomorphic to $N_0 \times \Lambda$, where $N_0 = \mathrm{Ker}\ (I|_X - B)$ in the tangent space of $\mathcal{N}_0 = \mathrm{Fix}_D J(\cdot, 0)$ at the origin. This will prove assertions 1) and 2) of the theorem.

Step 3. Proof of the claim. It follows by the chain rule that $B^n = (J^n)'_x(0,0)$, where B^n are the iterates of the linear operator B, and J^n are the iterates of the resolvent $J\colon D \mapsto D$. Since D is bounded, $\{B^n\}$ is uniformly bounded by the Cauchy inequalities. Therefore the reflexivity of X and the mean ergodic theorem imply that

$$\mathrm{Ker}\ (I|_X - B) \oplus \overline{\mathrm{Im}\,(I|_X - B)} = X.$$

By the same token we also have

$$\mathrm{Ker}\ (I|_Z - S) \oplus \overline{\mathrm{Im}\,(I|_Z - S)} = Z, \tag{$*$}$$

where $S = T'(0)$. We want to show that

$$\overline{\mathrm{Im}\,(I|_X - B) \times \{0\}} = \overline{\mathrm{Im}\,(I|_X - S)}. \tag{8.3}$$

This will prove our claim.

Let P be a linear projection of X onto $N = \mathrm{Ker}\ (I|_X - B)$, and let $K = (J_f)'_\lambda(0,0)\colon \Lambda \mapsto X$. First we prove that

$$PK = 0. \tag{8.4}$$

Indeed, let P_1 be the linear projection in Z, defined by the formula

$$P_1 = \begin{bmatrix} P|_{X\mapsto N} & O|_{\Lambda\mapsto X} \\ & \\ O|_{X\mapsto\Lambda} & I|_{\Lambda\mapsto\Lambda} \end{bmatrix}.$$

There is $0 \le M < \infty$ such that

$$\|P_1 S^n\| \le M < \infty. \tag{8.5}$$

On the other hand, by direct calculation, we have

$$S^n = \begin{bmatrix} B^n & (I|_X + B + \cdots + B^{n-1})K \\ & \\ O & I|_{\Lambda\mapsto\Lambda} \end{bmatrix}.$$

In addition, $PB^n = P$ for all $n = 1, 2, \dots$. Hence we obtain the following explicit form of P_1S^n:

$$P_1S^n = \begin{bmatrix} P & nPK \\ \\ O|_{X\mapsto\Lambda} & I|_{\Lambda\mapsto\Lambda} \end{bmatrix}.$$

This contradicts (8.5) unless (8.4) holds. So we have now $P_1(I|_Z - S)z = 0$ for all $z \in Z$, and hence

$$\begin{aligned}\overline{\mathrm{Im}\,(I|_Z - S)} \subset \mathrm{Ker}\ P_1 = (I|_Z - P_1)Z &= \{(I|_X - P)X\} \times \{0\} \\ &= \overline{\mathrm{Im}\,(I|_X - B)} \times \{0\}.\end{aligned}$$

Conversely, let $z \in \overline{\mathrm{Im}\,(I|_X - B)} \times \{0\}$. Since the equation $(I|_Z - S)y = z$, $y = (x, \lambda)$, is equivalent to $(I|_X - B)x + K\lambda = z$, we have $z \in \overline{\mathrm{Im}\,(I|_Z - S)}$. Thus (8.3) holds and our claim is proved.

Step 4. Proof of assertion 3. By step 1, for each $r > 0$ and each $\lambda \in \Delta$ there exists the resolvent $J_r(\cdot, \lambda) \colon D \mapsto D$.

Since X is reflexive, one can find a subsequence $r_n \to \infty$ such that $J_{r_n}(\cdot, \lambda)$ weakly converges to a holomorphic mapping $h(\cdot, \cdot) \colon D \times \Delta \mapsto \overline{D}$. But because $J_{r_n}(x(\lambda), \lambda) = x(\lambda)$, $\lambda \in \Delta$, and $h(x(\lambda), \lambda) = x(\lambda) \in D$, $h(\cdot, \cdot)$ maps $D \times \Delta$ into D. In addition, for each $\lambda \in \Delta$, by the mean ergodic theorem, $(J_{r_n})'_x(x(\lambda), \lambda)$ strongly converges to a projection P_λ onto the set N_λ tangent to $\mathcal{N}_\lambda$, as $r_n \to \infty$, i.e. $h'_x(x(\lambda), \lambda) = P_\lambda$. By Vesentini's theorem, [70, 71], the sequence of iterates $h^n(\cdot, \lambda) \colon D \mapsto D$ converges to a mapping $\rho(\cdot, \lambda) \colon D \mapsto D$ which evidently satisfies the requirements of assertion 3. □

Corollary 8.1. *Suppose that under the conditions of Theorem 8.1, x_0 is an isolated null point of $f(\cdot, \lambda_0)$. Then the equation (8.2) has a unique solution $x(\lambda)$ for all $\lambda \in \Delta$, and $x(\cdot) \colon \Delta \mapsto D$ is holomorphic on Δ.*

Remark 8.1. Our theorem and corollary no longer hold when X is an arbitrary complex Banach space. Indeed, let

$$X = c_0 = \{x = (x_1, x_2, \dots, x_n, \dots) : x_n \in \mathbb{C},\ n \in N,\ x_n \to 0 \text{ as } n \to \infty\}$$

with $\|x\| = \sup\limits_{n \in N} |x_n|$. It is easy to see that $f(\cdot, \cdot)$ defined by

$$f(x, \lambda) = (x_1 - \lambda, x_2 - x_1, x_3 - x_2 \dots, x_{n+1} - x_n, \dots),$$

where $x \in c_0$, $\|x\| < 1$, $\lambda \in \mathbb{C}$, $|\lambda| < 1$, belongs to $\mathrm{HG}\,(D)$, for each $\lambda \in \Delta$, where D is the unit ball in X. In addition, $f(0, 0) = 0$, but $f(\lambda, \cdot)$ has no null point in X for all $\lambda \in \Delta$, $\lambda \neq 0$.

Nevertheless, as we saw in the proof, our theorem is still true under the additional condition

$$(*) \qquad \mathrm{Ker}\ f'(x_0, \lambda_0) \oplus \mathrm{Im}\, f'(x_0, \lambda_0) = X$$

for an arbitrary Banach space.

This implies a global implicit function theorem of a classical type.

Corollary 8.2. *Let X be an arbitrary Banach space and let D, Δ and f be as above. If for at least one $\lambda_0 \in \Delta$ there exists a null point $x_0 \in D$ of the mapping $f(\cdot, \lambda_0)$ such that $f'(x_0, \lambda_0)$ is invertible, then for all $\lambda \in \Delta$ there exists a unique solution of the equation (8.2) which is holomorphic in $\lambda \in \Delta$ and regular, i.e. $f'_x(x(\lambda), \lambda)$ is invertible for all $\lambda \in \Delta$.*

Combining this corollary with the results of Section 4 we obtain the following two assertions.

Corollary 8.3. *Let X, D, Δ and f be as above. Suppose that for some $\lambda_0 \in \Delta$, $f(\cdot, \lambda_0)$ admits a uniformly continuous extension to $\overline{D}$ and satisfies the condition*

$$\|f(x, \lambda_0)\| \geq \varepsilon > 0 \text{ for all } x \in \partial D.$$

Then the equation (8.2) has a unique solution $x(\lambda)$ for all $\lambda \in \Delta$ and this solution is regular, i.e. $f'_x(x(\lambda), \lambda)$ is invertible.

Corollary 8.4. *Let D be a ball in a complex Banach space X, and let Δ be the unit disk in $\mathbb{C}$. Suppose that f is a holomorphic mapping on $D \times \Delta$ which satisfies the following conditions: for each $\lambda \in \Delta$, $f(\cdot, \lambda)$ has a uniformly continuous extension to ∂D, and for all $x \in \partial D$ the following inequality holds:*

$$\inf_{x^*} \operatorname{Re} \langle f(x, \lambda), x^* \rangle \geq 0$$

where x^* is a selection of the duality mapping at x.

Then if for some $\lambda_0 \in \Delta$ and $\varepsilon > 0$,

$$\inf_{x^*} \operatorname{Re} \langle f(x, \lambda_0), x^* \rangle \geq \varepsilon$$

for all $x \in \partial D$, the equation (8.2) has a unique solution $x(\lambda) \in D$, which is holomorphic in $\lambda \in \Delta$ and regular.

Example 8.1. Consider the following question on perturbations of a differential equation by parameters. Let X be a complex Banach space and $A \colon X \mapsto X$ a bounded linear, strongly accretive operator. Then the equation

$$\frac{dx}{dt} + Ax = 0$$

has an asymptotically stable solution $x = x(t)$ on $[0, \infty)$ for all initial values $x(0)$ in X.

Consider now the perturbed equation

$$\frac{dx}{dt} + Ax + \lambda B^{(k)} x + \mu C = 0, \tag{8.6}$$

where $B^{(k)}$ is a homogeneous polynomial operator in X of order k, C is a given element of X and $(\lambda, \mu) \in \mathbb{C}^2$.

Examples of this kind may be given by the very important Riccati type flows in a Banach algebra X governed by the equation

$$\frac{dx}{dt} + ax + \lambda b_1 x b_2 x + \mu c = 0, \tag{8.7}$$

where a, b_1, b_2 and c are elements of X (see [32]).

Other examples include the Abel equations

$$\frac{dx}{dt} = bx^k + ax + c,$$

where $k = 2, 3, \dots,$ as well as certain integro-differential equations (see for example, [68]).

The general question is for which values of λ and μ the equation (8.6) has a stable solution on $[0, \infty)$ with respect to a stationary point (if it exists) of this equation.

In other words, the problem is to find a set $\Omega \subset \mathbb{C}^2$ such that the equation (8.6) has a stationary solution $x_0(\lambda, \mu)$ and a bounded solution $x(t, \lambda, \mu)$ for all $t \geq 0$, $(\lambda, \mu) \in \Omega$, and all initial values $x(0, \lambda, \mu)$ in a neighborhood of $x_0(\lambda, \mu)$. When $\mu = 0$ the equation (8.6) is said to be quasilinear. Its stability was established in [15] for sufficiently small $|\lambda|$.

If $B^{(k)}$ is compact, some estimates for Ω in the form of a bidisk $\{|\lambda| < \rho_1, \quad |\mu| < \rho_2\}$ or a triangle $\{|\lambda| + |\mu| < \rho\}$ may be found in [45] and [66]. We will see below that in our case Ω may be chosen as a logarithmic convex domain $\{|\lambda| \cdot |\mu| < \ell\}$. This allows us to increase one of these parameters while decreasing the other.

Indeed, let A satisfy the condition

$$\inf\{\mathrm{Re}\langle Ax, x^*\rangle : x^* \in Jx\} \geq \delta \|x\|^2, \tag{8.8}$$

for all $x \in X$, where J is the duality mapping of X. We want to show that there exist $\rho > 0$ and $\ell > 0$ such that for all $(\lambda, \mu) \in \Omega = \{|\lambda| \cdot |\mu| < \ell\}$ and for all x with $\|x\| = \rho$, the mapping $f(\lambda, \mu, x) = Ax + \lambda B^{(k)}x + \mu C$ satisfies the inequality

$$\inf\{\mathrm{Re}\langle f(\lambda, \mu, x), x^*\rangle : x^* \in Jx\} \geq 0. \tag{8.9}$$

Indeed, for any $\rho > 0$ and $\|x\| = \rho$ we have

$$\mathrm{Re}\,\langle f(\lambda, \mu, x), x^*\rangle : x^* \in Jx\} \geq \delta\rho^2 - (|\lambda|\,\|B^{(k)}\|\rho^{k+1} + |\mu|\,\|C\|\rho).$$

Consider the function $\varphi(\rho) = |\lambda|\,\|B^{(k)}\|\rho^k + |\mu|\,\|C\| - \delta\rho$. The inequality (8.9) holds for some ρ if the minimum of $\varphi(\rho)$ is negative. This function reaches its minimum at the point

$$\rho^* = \sqrt[k-1]{\frac{\delta}{k|\lambda|\,\|B^{(k)}\|}} \tag{8.10}$$

and this minimum is negative if

$$\mu\,\|C\| \leq \frac{k-1}{k} \sqrt[k-1]{\frac{\delta^k}{k|\lambda|\,\|B^k\|}}. \tag{8.11}$$

Thus let $\Omega \in \mathbb{C}^2$ consists of all points for which (8.11) holds. Then the equation $\varphi(\rho) = 0$ has two solutions $0 < \rho_1 \leq \rho_2$ and for each $\rho \in [\rho_1, \rho_2]$ the condition (8.9) is satisfied. But $0 \in \Omega$ and if $\lambda = \mu = 0$, the equation $f(x, 0, 0) = 0$ has a unique solution $x = 0$. Hence by Corollary 8.4 the equation $f(x, \lambda, \mu) = 0$ also has a unique solution $x^* = x^*(\lambda, \mu)$ for all

$(\lambda, \mu) \in \Omega$ such that $\|x^*(\lambda, \mu)\| < \rho$. In addition, the differential equation (8.6) has a global solution $x(t, \lambda, \mu)$ on $[0, \infty)$ with $x(0, \lambda, \mu) = x_0$ for all x_0 with $\|x_0\| < \rho$, such that $\|x(t, \lambda, \mu)\| < \rho$ for all $t \in [0, \infty)$ and $\lambda, \mu \in \Omega$.

The point $x^*(\lambda, \mu)$ is the unique stationary point of this equation. In particular, for Riccati's equation (8.7) the condition (8.11) is

$$4|\lambda|\,|\mu| \leq \delta^2(\|b_1\|\,\|b_2\|\,\|C\|)^{-1}.$$

9. Some open problems

In this final section we collect several questions related to the results in the previous sections which remain open.

1. Let $D = B^n,\ n > 1$, where B is the open unit ball of a complex Hilbert space H, and let $f \in$HG$\,(D)$ be bounded on D and continuous on $\overline{D}$. Does f have a null point in $\overline{D}$?

Note that the answer is affirmative if f is Lipschitzian (Proposition 7.1) or if $n = 1$ (Theorem 7.2).

This problem is closely related to the following one.

2. If F is fixed point free, does the approximating curve $\{z_t \colon 0 \leq t < 1\}$, defined implicitly by

$$z_t = (1 - t)a + tFz_t,$$

strongly converge, as $t \to 1^-$, to a point on the boundary of D, at least for one $a \in D$?

For $n = 1$ the answer is again known to be positive [23].

3. In this connection, it would also be of interest to determine the asymptotic behavior of the semigroups generated by null point free generators.

4. If D is a finite-dimensional taut complex manifold and $\{F_t \colon t \geq 0\}$ is a continuous semigroup of holomorphic self-mappings of D, then it is known [2] that $\{F_t\}$ has a generator. This is no longer true in the infinite-dimensional case. Therefore it would be of great interest to find sufficient conditions for the existence of a generator of a given semigroup. For example, does a semigroup which is continuous with respect to the topology of local uniform convergence have a generator?

5. Let D be a bounded convex domain is a complex Banach space X and let $f \in$ Hol (D, X) be bounded.

According to Corollary 4.1, if there exists a positive δ and a T-continuous curve $G_t \colon [0, \delta) \mapsto$ Hol (D, D) such that $G_0 = I$ and T-$\lim_{t\to 0^+}(I - G_t)/t = f$, then f is a generator of a semigroup $\{F_t\}$. Is it true that in this case

$$F_t = \text{T-}\lim_{n\to\infty} G^n_{t/n} \tag{9.1}$$

for all $0 \leq t < \delta$?

This would be an analog of Chernoff's product formula for linear semigroups. For the nonlinear case see, for example, [11] and [57]. Note also that in the special case when $G_t = J_t$, (9.1) is indeed valid by Theorem 4.2.

6. Another interesting special case of (9.1) is the following one. Let f and g belong to HG$\,(D)$. If f and g are bounded, then their sum $h = f + g$ also

belongs to HG (D) by Corollary 4.4. Denote the semigroups generated by f, g and h by $\{F_t\}, \{G_t\}$ and $\{H_t\}$, respectively.

Is it true that

$$H_t = \text{T-}\lim_{n\to\infty} (F_{t/n} \circ G_{t/n})^n \tag{9.2}$$

for all $t \geq 0$?

Acknowledgments. The research of the first author was partially supported by the Fund for the Promotion of Research at the Technion and by the Technion VPR Fund — M. and M.L. Bank Mathematics Research Fund.

References

[1] Abate, M., Horospheres and iterates of holomorphic maps, *Math. Z.* **198** (1988), 225–238.

[2] Abate, M., The infinitesimal generators of semigroups of holomorphic maps, *Ann. Mat. Pura Appl.* (4) **161** (1992), 167–180.

[3] Abd-Alla, M., L'ensemble des points fixes d'une application holomorphe dans un produit fini de boules-unités d'espaces de Hilbert et une sous-variété banachique complexe, *Ann. Mat. Pura Appl.* (4) **153** (1988), 63–75.

[4] Abts, D., On injective holomorphic Fredholm mappings of index 0 in complex Banach spaces, *Comment. Math. Univ. Carolinae* **21** (1980), 513–525.

[5] Aizenberg, L., Reich, S., and Shoikhet, D., One-sided estimates for the existence of null points of holomorphic mappings in Banach spaces, Technion Preprint Series No. MT-1015, 1995, J. Math. Anal. Appl. (to appear).

[6] Angenent, S.B., Nonlinear analytic semiflows, *Proc. Roy. Soc. Edinburgh* **115** (1990), 91–107.

[7] Arazy, J., An application of infinite dimensional holomorphy to the geometry of Banach spaces, *Lecture Notes in Math.* **1267**, Springer, Berlin, 1987, 122–150.

[8] Azizov, T. Y., Khatskevich, V. and Shoikhet, D., On the number of fixed points of a holomorphism, *Siberian Math. J.* **31** (1990), 192–195.

[9] Barbu, V., *Nonlinear Semigroups and Differential Equations in Banach spaces*, Noordhoff, Leyden, 1976.

[10] Brézis, H., *Opérateurs Maximaux Monotones*, North Holland, Amsterdam, 1973.

[11] Brézis, H., and Pazy, A., Convergence and approximation of nonlinear operators in Banach spaces, *J. Funct. Anal.* **9** (1972), 63–64.

[12] Bruck, R.E., Structure of the approximate fixed-point sets of nonexpansive mappings in general Banach spaces, *Pitman Res. Notes Math.* **252** (1991), 91–96.

[13] Cartan, H., *Calcul Différentiel*, Hermann, Paris, 1967.

[14] Chirka, E., *Complex Analytic Sets*, Nauka, Moscow, 1985.

[15] Daletskii, Yu. L. and Krein, M. G., *Stability of Solutions of Differential Equations in a Banach Space*, Nauka, Moscow, 1970.

[16] Dieudonné, J., *Foundations of Modern Analysis*, Academic Press, New York and London, 1960.

[17] Dineen, S., *The Schwarz Lemma*, Clarendon Press, Oxford, 1989.

[18] Dineen, S., Timoney, P.M. and Vigué, J.P., Pseudodistances invariantes sur les domaines d'un espace localement convexe, *Ann. Scuola Norm. Sup. Pisa* **12** (1985), 515–529.

[19] Do Duc Thai, The fixed points of holomorphic maps on a convex domain, *Ann. Polon. Math.* **56** (1992), 143–148.

[20] Dorroh, J. R. and Neuberger, J. W., Lie generators for semigroups of transformations on a Polish space, *Electron. J. Differential Equations* **1** (1993), 1–7.

[21] Earle, C. J. and Hamilton, R. S., A fixed-point theorem for holomorphic mappings, *Proc. Sympos. Pure Math.* Vol. **16**, Amer. Math. Soc., Providence, R. I., 1970, pp. 61–65.
[22] Franzoni, T. and Vesentini, E., *Holomorphic Maps and Invariant Distances,* Amsterdam, North-Holland 1980.
[23] Goebel, K. and Reich, S., *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Dekker, New York, 1984.
[24] Goebel, K., Sekowski, T. and Stachura, A., Uniform convexity of the hyperbolic metric and fixed points of holomorphic mappings in the Hilbert ball, *Nonlinear Anal.* **4** (1980), 1011–1021.
[25] Harris, L. A., Schwarz's lemma in normed linear spaces, *Proc. Nat. Acad. Sci. U.S.A.* **62** (1969), 1014–1017.
[26] Harris, L. A., A continuous form of Schwarz's lemma in normed linear spaces, *Pacific J. Math.* **38** (1971), 635–639.
[27] Harris, L. A., The numerical range of holomorphic functions in Banach spaces, *Amer. J. Math.* **93** (1971), 1005–1019.
[28] Harris, T. E., *The Theory of Branching Processes*, Springer, Berlin, 1963.
[29] Hayden, T. L. and Suffridge, T. J., Biholomorphic maps in Hilbert space have a fixed point, *Pacific J. Math.* **38** (1971) 419–422.
[30] Hayden, T. L. and Suffridge, T. J., Fixed points of holomorphic maps in Banach spaces, *Proc. Amer. Math. Soc.* **60** (1976), 95–105.
[31] Heath, L. F. and Suffridge, T. J., Holomorphic retracts in complex n-space, *Illinois J. Math.* **25** (1981), 125–135.
[32] Helmke, U. and Moore, J. B., *Optimization and Dynamical Systems*, Springer, London, 1994.
[33] Hervé, M., *Analyticity in Infinite Dimensional Spaces*, De Gruyter, Berlin 1989.
[34] Hille, E. and Phillips, R., *Functional Analysis and Semigroups*, Amer. Math. Soc., Providence, R. I., 1957.
[35] Isidro, J. M. and Stacho, L. L., *Holomorphic Automorphism Groups in Banach Spaces: An Elementary Introduction*, North-Holland, Amsterdam, 1984.
[36] Kato, T., *Perturbation Theory for Linear Operators*, Springer, Berlin, 1966.
[37] Khatskevich, V., Reich, S. and Shoikhet, D., Global implicit function and fixed point theorems for holomorphic mappings and semigroups, *Complex Variables* (to appear).
[38] Khatskevich, V., Reich, S. and Shoikhet, D., Fixed point theorems for holomorphic mappings and operator theory in indefinite metric spaces, *Integral Equations Operator Theory* **22** (1995), 305–316.
[39] Khatskevich, V., Reich, S. and Shoikhet, D., Ergodic type theorems for nonlinear semigroups with holomorphic generators, *Recent Developments in Evolution Equations, Pitman Res. Notes Math.* **324** (1995), 191-200.
[40] Khatskevich, V. and Shoikhet, D., Fixed points of analytic operators in a Banach space and applications, *Siberian Math. J.* **25** (1984), 188-200.
[41] Khatskevich V. and Shoikhet D., *Differentiable Operators and Nonlinear Equations*, Birkhäuser, Basel, 1994.
[42] Khatskevich, V. and Shoikhet, D., One version of implicit function theorem for holomorphic mappings, *C. R. Acad. Sci. Paris*, **319** (1994), 599–604.
[43] Khatskevich, V. and Shoikhet, D., Null-point sets of holomorphic generators of one-parameter semigroups, *Dynam. Systems Appl.* **4** (1995), 611 - 629.
[44] Kobayashi, Y. and Oharu, S., Semigroups of locally Lipschitzian operators in Banach spaces, *Hiroshima Math. J.* **20** (1990), 573–611.
[45] Krasnoselski, M.A., Vainikko, G.M., Zabreiko, P.P., Ruticki, Ya.B. and Stecenko, V.Ya., *Approximate Solution of Operator Equations*, Nauka, Moscow, 1969.
[46] Krasnoselski, M. A. and Zabreiko, P. P., *Geometric Methods of Nonlinear Analysis*, Springer, Berlin, 1984.

[47] Kuczumow, T. and Stachura, A., Convexity and fixed points of holomorphic mappings in Hilbert ball and polydisk, *Bull. Polish Acad. Sci.* **34** (1986), 189–193.
[48] Kuczumow, T. and Stachura, A., Fixed points of holomorphic mappings in Cartesian products of n unit Hilbert balls, *Canad. Math. Bull.* **29** (1986), 281–286.
[49] Lempert, L., Holomorphic retracts and intrinsic metrics in convex domains, *Anal. Math.* **8** (1982), 257–261.
[50] Martin, R.H., Jr., Differential equations on closed subsets of a Banach space, *Trans. Amer. Math. Soc.*, **179** (1973), 399-414.
[51] Mazet, P., Les points fixes d'une application holomorphe d'un domaine borné dans lui-même admettent une base de voisinages convexes stable, *C. R. Acad. Sci. Paris* **314** (1992), 197–199.
[52] Mazet, P. and Vigué, J. P., Points fixes d'une application holomorphe d'un domaine borné dans lui-même, *Acta Math.* **166** (1991), 1–26.
[53] Oharu, S. and Takahashi, T., Locally Lipschitz continuous perturbations of linear dissipative operators and nonlinear semigroups, *Proc. Amer. Math. Soc.* **100** (1987), 187–194.
[54] Oharu, S. and Takahashi, T., Characterization of nonlinear semigroups associated with semilinear evolution equations, *Trans. Amer. Math. Soc.* **311** (1989), 593–619.
[55] Pazy A., Semigroups of nonlinear contractions and their asymptotic behavior, *Nonlinear Analysis and Mechanics, Pitman Res. Notes Math.* **30** (1979), 36-134.
[56] Reich, S., On fixed point theorems obtained from existence theorems for differential equations, *J. Math. Anal. Appl.* **54** (1976), 26–36.
[57] Reich, S., Product formulas, nonlinear semigroups and accretive operators, *J. Funct. Anal.* **36** (1980), 147–168.
[58] Reich, S., A nonlinear Hille-Yosida theorem in Banach spaces, *J. Math. Anal. Appl.* **84** (1981), 1–5.
[59] Reich, S., The asymptotic behavior of a class of nonlinear semigroups in the Hilbert ball, *J. Math. Anal. Appl.* **157** (1991), 237–242.
[60] Reich, S., Approximating fixed points of holomorphic mappings, *Math. Japon.* **37** (1992), 457–459.
[61] Reich, S., and Shafrir, I., Nonexpansive iterations in hyperbolic spaces. *Nonlinear Anal.* **15** (1990), 537–558.
[62] Reich, S. and Torrejon, R., Zeros of accretive operators, *Comment. Math. Univ. Carolin.* **21** (1980), 619–625.
[63] Rudin, W., The fixed-point set of some holomorphic maps, *Bull. Malaysian Math. Soc.* **1** (1978), 25–28.
[64] Sevastyanov, B. A., *Branching Processes*, Nauka, Moscow, 1971.
[65] Shoikhet, D., Some properties of Fredholm mappings in Banach analytic manifolds, *Integral Equations Operator Theory* **16** (1993), 430–451.
[66] Trenogin, V.A., *Functional Analysis*, Nauka, Moscow, 1980.
[67] Upmeier H., *Jordan Algebras in Analysis, Operator Theory and Quantum Mechanics*, CBMS Regional Conf. Ser. in Math., 67, Amer. Math. Soc., Providence, R.I., 1986.
[68] Vainberg, M. M. and Trenogin, V. A., *Theory of Bifurcation of Solutions of Nonlinear Equations*, Moscow, Nauka 1969.
[69] Vesentini, E., Complex geodesics and holomorphic maps, *Sympos. Math.* **26** (1982), 211–230.
[70] Vesentini, E., Iterates of holomorphic mappings, *Uspekhi Mat. Nauk* **40** (1985), 13–16.
[71] Vesentini, E., Su un teorema di Wolff e Denjoy, *Rend. Sem. Mat. Fis. Milano* **53** (1983), 17–26 (1986).
[72] Vesentini, E., Krein spaces and holomorphic isometries of Cartan domains, *Geometry and Complex Variables*, Marcel Dekker, 1991, pp. 409–443.
[73] Vesentini, E., Semigroups of holomorphic isometries, *Complex Potential Theory*, Kluwer, Dordrecht, 1994, pp. 475–548.

[74] Vigué, J. P., Points fixes d'applications holomorphes dans un produit fini de boules-unités d'espaces de Hilbert, *Ann Mat. Pura Appl.* (4) **137** (1984), 245–256.
[75] Vigué, J. P., Sur les points fixes d'applications holomorphes, *C. R. Acad. Sci. Paris* **303** (1986), 927–930.
[76] Vigué, J. P., Fixed points of holomorphic mappings in a bounded convex domain in C^n, *Proc. Sympos. Pure Math.* Vol. **52** (1991), Part 2, pp. 579–582.
[77] Yosida, K., *Functional Analysis*, Springer, Berlin, 1965.

Department of Mathematics, The Technion - Israel Institute of Technology, 32000 Haifa, Israel
E-mail address: sreich@tx.technion.ac.il

Department of Applied Mathematics, International College of Technology, 20101 Karmiel, Israel
E-mail address: davs@tx.technion.ac.il