BIFURCATION OF THE EQUIVARIANT MINIMAL INTERFACES IN A HYDROMECHANICS PROBLEM

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Abstract. In this work we study a deformation of the minimal interface of two fluids in a vertical tube under the presence of gravitation. We show that a symmetry of the base of tube let us to apply a method developed earlier by the first author and based on the Crandall-Rabinowitz bifurcation theorem. Using the natural symmetry of the corresponding variational problem defined by a symmetry of region and restricting the functional to spaces of invariant functions we show the existence of bifurcation, and describe its local picture, for interfaces parametrized by the square and disc.

0. Introduction

In this work we study a problem of hydromechanics connected with the Plateau problem. Our aim is to describe a bifurcation of interfaces between two fluids under a change a real parameter of a natural mechanical nature. Main difficulties in studying this bifurcation problem are:
1. The fact that the differential operator Plateau is not a selfmap of the Hilbert space $H^2(\Omega) = W_2^2(\Omega)$;
2. The fact that the operator Plateau is not of the form "linear Fredholm + completely continuous" and the degree theory is not applicable.

In works [B1], [B2] the first author introduced a nonlinear operator of Plateau type assigned to the discussed problem, which is Fredholm of index 0. It is defined on the Sobolev spaces $W_p^2(\Omega)$ for $p > 2$ and its construction is based on results of [KN]. The standard necessary condition led to simple

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and doubled (one, or two-dimensional kernel correspondingly) critical values of the bifurcation parameter.

In [B1], [B2] the existence of branching of solutions from single degeneracy point is shown. To prove this a finite-dimensional reduction of Liapunov-Schmidt is used and the statement follows from the Crandall-Rabinowitz theorem on bifurcation from simple eigenvalue ([CR]). Unfortunately the mentioned technics could be applied only in the case if the kernel and cokernel of the Fréchet differential are one-dimensional.

In this paper we discuss the same problem at doubled critical points. An assumption on symmetry of region parametrized given interface implies that the operator associated with problem is equivariant with respect to a linear symmetry induced on the functional spaces. This allows to restrict the operator to invariant subspaces of this symmetry and check that the restricted operator has simple degeneracy then which implies the existence and the same local behaviour of the bifurcation of minimal interfaces as that described in [B1] and [B2].

Describing the matter in more details, let us suppose that in a cylinder with a vertical section $\Omega \subset \mathbb{R}^2$ are two fluids with density ρ_1 and ρ_2 correspondingly, and set $\rho = \rho_2 - \rho_1 > 0$. Suppose next that separating them elastic interface $w = w(x, y)$, $(x, y) \in \overline{\Omega}$ is steady fixed on the boundary surface of cylinder $w|_{\partial\Omega} = 0$.

The history of equation of interface separating two fluids comes back to works of Laplace, Monge, Poisson and Young who already observed that the average curvature of it is proportional to the difference between pressures acting from the opposite sides. The average curvature $H(w)(x, y)$ is given by the formula

$$H(w) = -\text{div}T(w), \qquad T(w) = (1 + \nabla w^2)^{-1/2}\nabla w, \tag{1}$$

and the capillary interface $w(x, y)$ is given by the equation

$$-\text{div}T(w) = k\Delta p \tag{2}$$

where $\Delta p = p_2 - p_1$ is the difference of pressures acting from opposite sides, and the coefficient $k = 1/\sigma$ is the inverse of $\sigma > 0$ the membrane tension coefficient.

If additionally a gravitation g acts on the fluids the quantity $\lambda = \frac{\rho g}{\sigma}$, called the Bond parameter, is equal to 0.

In the presented work we study transformations of the minimal interfaces of two fluids with the presence of gravitation $g > 0$. In this case to a functional of membrane stress $E_\sigma(w)$ one have to add the functional $E_g(w)$ of potential energy of two substances contained in the cylinder. We have considered the potential energy functional of the following form

$$E_g(w) = E_0 - \frac{1}{2}\rho g \iint_\Omega w^2 dxdy. \tag{3}$$

Then an interface $w(x,y)$ is minimal if it is a critical point of the total functional $E_{complete}(w) = E_\sigma(w) + E_g(w)$ (cf. [B2]).

Complete surveys of various boundary problems of hydromechanics are included in books [DF], [FM], [DS], [DD], and [FN] (see also [BK1], [BK2], [RV1], [RV2], [PS], [BT], [FT], [TZ], and [BU] for an information on related problems).

We wish to emphasize that the discussed here scheme could be used as a mathematical model for many other problems of hydromechanics and theory of spring membrane (cf. [B2]).

1. General bifurcation problem

We shall study bifurcation of the capillary minimal interface in the case of the quadratic perturbation of the area functional

$$A(w) = \iint_\Omega (1 + \nabla w^2)^{1/2} dxdy.$$

Let us take the functional

$$E(w,\lambda) = \sigma A(w) + g\rho Q(w) = \sigma \iint_\Omega \left((1+\nabla w^2)^{1/2} - \frac{\lambda}{2} w^2 \right) dxdy, \tag{4}$$

with real parameter $\lambda = g\rho/\sigma$, and consider the following boundary-value problem

$$(\mathbf{BP}) \qquad \begin{cases} -\mathrm{div} T(w) - \lambda\, w = 0, & (x,y) \in \Omega, \\ w = 0, & (x,y) \in \partial\Omega. \end{cases} \tag{5}$$

Let $F(w) = -\operatorname{div} T(w)$ be the Euler - Lagrange operator for the area functional $A(w)$

$$\begin{aligned} F(w) &= -\mathrm{div}\left((1+\nabla w^2)^{-1/2} \nabla w \right) \\ &= -(1+\nabla w^2)^{-3/2} \left(\Delta w + w_y^2 w_{xx} - 2 w_x w_y w_{xy} + w_x^2 w_{yy} \right). \end{aligned} \tag{6}$$

In next we shall use the Jacobi operator of the area functional $A(w)$

$$\begin{aligned} J(w)h &= -\mathrm{div}\left((1+\nabla w^2)^{-3/2} \begin{bmatrix} 1+w_y^2 & w_x w_y \\ w_x w_y & 1+w_x^2 \end{bmatrix} \times \begin{bmatrix} h_x \\ h_y \end{bmatrix} \right) \\ &= -\,(1+\nabla w^2)^{-\frac{3}{2}} \Big((1+w_y^2) h_{xx} \;-\; 2 w_x w_y h_{xy} \;+\; (1+w_x^2) h_{yy} \\ &\quad +\; 2(w_x w_{yy} - w_y w_{xy}) h_x \;+\; 2(w_y w_{xx} - w_x w_{xy}) h_y \Big). \end{aligned} \tag{7}$$

Note that the minimal interface $w_0(x, y) = 0$ is a solution of the Problem (BP), called the trivial solution, for all $\lambda \in \mathbb{R}$. We shall study the necessary and sufficient conditions for bifurcation of a family of nontrivial solutions from a point $(0, \lambda_0)$ and the local behaviour of these branches of solutions with respect to the parameter λ.

We shall study Bifurcation Problem (BP) under the following assumptions:

- (A_1) The domain Ω is convex.
- (A_2) The boundary $\partial\Omega$ is a piecewise smooth C^2 - submanifold of $\mathbb{R}^2$ homeomorphic to the circle S^1.
- (A_3) The boundary has k corner points and the interior angle α_j at each corner point satisfies inequality $\frac{0<\alpha_j<\pi}{2}$ for all $j = 1, \ldots, k$.

Under assumptions $(A_1) - (A_3)$ the Bifurcation Problem (BP) is equivalent to the problem of branching of solutions for the following operator equation (see [B2])

$$P(w, \lambda) = (0, 0) \tag{8}$$

where a nonlinear operator P is defined by the formula

$$P(w, \lambda) = (F(w) - \lambda w, w|_{\partial\Omega}). \tag{9}$$

By $W_p^2(\Omega)$ we denote the Sobolev spaces and by $B_p^{2-1/p}(\partial\Omega)$ the Besov traces spaces. In next we shall use the following facts shown in [B1] and [B2].

Theorem 1. (see [B1, B2]) The nonlinear operator P as a map between the following spaces

$$P : W_p^2(\Omega) \times \mathbb{R} \to L_p(\Omega) \times B_p^{2-1/p}(\partial\Omega)\ , \qquad p > 2, \tag{10}$$

is of the class $C^a\Phi_0^{(w)}$, i.e. analytic operator with respect to all variables and its Frechét derivative $P'_w(w, \lambda)h = \big(\ J(w)h - \lambda h\ ,\ h|_{\partial\Omega}\big)$ with respect to main variable w

$$P'_w(w, \lambda) : W_p^2(\Omega) \to L_p(\Omega) \times B_p^{2-1/p}(\partial\Omega)\ , \qquad p > 2, \tag{11}$$

is a Fredholm linear map of index zero at each point $(w, \lambda) \in W_p^2(\Omega) \times \mathbb{R}$.

Let $N(\lambda) = \mathrm{Ker} P'_w(0, \lambda)$ be a finite-dimensional subspace of the Sobolev space $W_p^2(\Omega),\ p > 2$, consisting of solutions $h(x, y)$ of the following linear problem

$$P'_w(0, \lambda) = \big(-\Delta h - \lambda h,\ h|_{\partial\Omega}\ \big) = (0, 0). \tag{12}$$

Theorem 2. *For the bifurcation of solutions of the equation* $P(w, \lambda) = (0, 0)$ *at the point* $(0, \lambda_0)$ *it is necessary that*

$$\dim N(\lambda_0) \neq 0.$$

In the case of the one-dimensional degeneracy, i.e. when $\dim N(\lambda_0) = 1$, bifurcation of problem (BP), under the assumptions $(A_1) - (A_3)$ was studied in the ([B1, B2]), where bifurcation theorems were shown.

2. Bifurcation for multi-dimensional degeneracy in the presence of symmetry

In this part we shall study the Bifurcation Problem (BP) and the operator equation $P(w, \lambda) = (0, 0)$ in the case where domain $\Omega \subset \mathbb{R}^2$ has a symmetry with respect to a closed subgroup $H \subset O(2)$.

Assumption A_4. *Assume, that there exists a group* H *of orthogonal linear transformations* $h : \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $\Omega \subset \mathbb{R}^2$ *is an invariant set with respect to* H, *i.e.,*

$$(x, y) \in \Omega \quad h \in H \quad \Longrightarrow \quad h(x, y) \in \Omega.$$

A symmetry of domain Ω defines a structure of linear representation of the group H in the group of linear isomorphisms of the Banach space $L_p(\Omega)$ by a shift of argument. If $w(x, y) \in L_p(\Omega)$ and $h \in H$ then the correspondence isomorphism $\overline{h} : L_p(\Omega) \to L_p(\Omega)$ is defined by formula

$$\overline{h}\big(w(x, y)\big) = w(h(x, y)). \tag{13}$$

The map $H \times L_p(\Omega) \to L_p(\Omega)$ defines a representation $H \to GL(L_p(\Omega))$ which is continuous in the strong topology and continuous in operator topology if H is a finite group.

It is clear, that all Sobolev spaces $W_p^m(\Omega)$, imbedded into $L_p(\Omega)$, are invariant subspaces with respect to the action of the group H. For every $h \in H$ by the same letter $\overline{h}$ we denote its representation map

$$\overline{h} : W_p^m(\Omega) \to W_p^m(\Omega). \tag{14}$$

We will use the symbol $L_p(\Omega)^H$, or $W_p^m(\Omega)^H$, for the subspace of the H-invariant functions of the corresponding functional space, i.e. satisfying $\overline{h}(w) = w$ for every $h \in H$. In the same way we define a representation of H in the group of linear isomorphisms of the Besov traces space $B_p^{m-1/p}(\partial\Omega)$, since $\partial\Omega$ is also invariant with respect to H. We write $\overline{h}^* : B_p^{m-1/p}(\partial\Omega) \to B_p^{m-1/p}(\partial\Omega)$ for the linear isomorphism defined by $h \in H$ in this case.

Note that the defined above representation are orthogonal, since $H \subset O(2)$ (linear orthogonal substitution of variables does not change the value of integral). We consider the trivial representation of H in the group $GL(\mathbb{R})$.

Theorem 3. *Suppose that bifurcation problem (BP) satisfying the assumptions* $(A_1) - (A_3)$ *additionally satisfies assumption* (A_4) *of symmetry with respect a group* H. *Then the Plateau operator* P *(10) is* H-equivariant *and consequently* P *maps the spaces of* H-invariant *functions into spaces of* H-invariant *functions*

$$P : W_p^2(\Omega)^H \times \mathbb{R} \to L_p(\Omega)^H \times B_p^{2-1/p}(\partial\Omega)^H, \qquad p > 2, \tag{15}$$

and the restriction P^H *is an analytic operator with respect to the all variables.*

Proof. First note that the following equalities hold

$$\begin{aligned}
&1^\circ \overline{h}(f \circ w) = f \circ \overline{h}w,\\
&2^\circ \overline{h}(uv) = (\overline{h}u)(\overline{h}v),\\
&3^\circ \overline{h}(\nabla w) = \nabla(\overline{h}w) \times M_h,\\
&4^\circ \overline{h}(\nabla u \nabla v) = \nabla(\overline{h}u)\nabla(\overline{h}v),\\
&5^\circ \overline{h}(\Delta w) = \Delta(\overline{h}w),
\end{aligned}$$

for every orthogonal linear map $h \in H \subset O(2)$ and its matrix M_h in (x, y) - coordinates, every functions $w, u, v \in W_p^2(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$. We left it to the reader.

Let $h \in H$ be an element and $(g, \varphi) = P(w, \lambda)$. By $1^\circ - 5^\circ$, we have

$$\begin{aligned}
&\overline{h}(g) = \overline{h}\big(F(w) - \lambda w\big) = -\overline{h}\big(\mathrm{div}(1 + \nabla w^2)^{1/2}\nabla w\big) - \overline{h}(\lambda w) =\\
&= -\overline{h}\Big(\nabla(1 + \nabla w \nabla w)^{1/2}\nabla w + (1 + \nabla w \nabla w)^{1/2}\Delta w\Big) - \overline{h}(\lambda w) =\\
&= F(\overline{h}w) - \lambda(\overline{h}w) = F(\overline{h}w) - \overline{h}'(\lambda)(\overline{h}w),
\end{aligned}$$

Also $\overline{h}^*(\varphi) = \overline{h}^*(w|_{\partial\Omega}) = (\overline{h}w)|_{\partial\Omega}$.

This shows that $(\overline{h}g, \overline{h}^*\varphi) = P(\overline{h}w, \overline{h}\lambda)$ for every $h \in H$, which means that the nonlinear operator P is H-equivariant.

On the other hand the spaces of invariant functions are closed linear subspaces mapped into spaces of invariant functions by any equivariant map, which yields the second part of statement. The proof of Theorem 3 is complete. ∎

The next theorem says that the derivative of Plateau map, at a point $(0, \lambda)$, restricted to the space of invariant functions is a Fredholm operator of index zero.

Theorem 4. *With notation and assumptions of Theorem 3, for every point* $(0, \lambda)$ *the Frechét derivative* $P'_w(0, \lambda)u = (-\Delta u - \lambda u,\ u|_{\partial\Omega})$ *of the Plateau* *operator with respect to main variable w restricted to the subspace of invariant functions*

$$P'_w(0,\lambda) : W^2_p(\Omega)^H \to L_p(\Omega)^H \times B^{2-1/p}_p(\partial\Omega)^H, \qquad p > 2, \tag{16}$$

is a linear Fredholm map of index zero.

We begin with a lemma which states that the restriction of Laplacian to spaces of invariant functions is a linear isomorphism.

Lemma 4.1. *The restricted Laplace operator*

$$\Delta : W^2_{p,0}(\Omega)^H \to L_p(\Omega)^H, \qquad p > 2, \tag{17}$$

is a linear isomorphism.

Proof. The statement follows from the main theorem of [KN] on the linear isomorphism of Sobolev spaces given by the Laplacian

$$\Delta : W^2_{p,0}(\Omega) \to L_p(\Omega), \qquad p > 2, \tag{18}$$

and the fact that Δ is H-equivariant. It is known that every linear, equivariant isomorphism maps the all linear invariant subspaces corresponding to distinct irreducible representations of H into themselves and is an isomorphism between any such factors. In particular it is an isomorphism between the factors corresponding to the trivial representation.

In this special case one can show it by a direct argument. Indeed, since Δ is H - equivariant, it maps $W^2_{p,0}(\Omega)^H$ into $L_p(\Omega)^H$ and is a monomorphism by the mentioned Kondrat'ev theorem [KN]. We have to show that $\Delta^H = \Delta|_{W^2_{p,0}(\Omega)^H}$ is onto $L_p(\Omega)^H$. Let $v \in L_p(\Omega)^H$ and $u \in W^2_{p,0}(\Omega)$ be an element such that $\Delta u = v$. It is enough to show that $u \in W^2_{p,0}(\Omega)^H$. Using once more the mentioned theorem, for every $h \in H$ we have

$$\Delta u = \overline{h}(\Delta u) = \Delta(\overline{h}u) \qquad \Longrightarrow \qquad \overline{h}u = u. \tag{19}$$

This proves the Lemma. ∎

Proof of Theorem 4. Since $P(w,\lambda)$ is equivariant and $(0,\lambda) \in W^2_{p,0}(\Omega)^H \times \mathbb{R}$, the Frechét derivative $P'_w(0,\lambda)$ is also equivariant and consequently its restriction $P'_w(0,\lambda)^H$ may be written in the form

$$P'_w(0,\lambda)^H u = (-\Delta u,\ u|_{\partial\Omega}) + (-\lambda u,\ 0). \tag{20}$$

It is known that the boundary operator

$$B(w) = w|_{\partial\Omega}, \qquad B : W^2_p(\Omega)^H \to B^{2-1/p}_p(\partial\Omega)^H,$$

is onto by the same argument as in Lemma 4.1. From this and Lemma 4.1 it follows that the first operator of decomposition (20) is an isomorphism, thus the Fredholm operator of index zero. The second operator of decomposition (20) is compact, as follows from the corresponding theorem on embeddings of the Sobolev spaces. Consequently the total operator is a Fredholm operator of index zero as a compact perturbation of such a map. ∎

In the Sobolev space $W_p^2(\Omega)^H$, $p > 2$, consider the finite-dimensional subspace $N(\lambda)^H = \mathrm{Ker} P'_w(0,\lambda)^H$ of the H-invariant solutions $u(x,y)$ of the following linear problem in an H-invariant domain Ω

$$\left(-\Delta u - \lambda u,\ u|_{\partial\Omega}\right) = \left(0,\ 0\right) \tag{21}$$

By its definition, this subspace is the intersection

$$N(\lambda_0)^H = N(\lambda_0) \cap W_p^2(\Omega)^H, \tag{22}$$

where $N(\lambda_0) = \mathrm{Ker} P'_w(0,\lambda_0)$ is the subspace in $W_p^2(\Omega)$.

Assumption A_5**.** *Suppose that*

$$\dim N(\lambda_0)^H = 1$$

and denote by $e_s(x,y)$ *an invariant versor generating subspace* $N(\lambda_0)^H$*, such that* $\|e_s\| = 1$ *in the Hilbert space* $L_2(\Omega)$.

Suppose that the bifurcation problem (BP) satisfies assumptions $(A_1) - (A_5)$. Assumption A_5, together with remaining, ensures us that after restricting the problem to spaces of H-invariant functions we get a bifurcation problem with one-dimensional degeneracy at a critical point $(0,\lambda_0)$, and the problem in question reduces to the situation discussed in [B1] and [B2]. Consequently (cf. [B2]) our bifurcation problem (BP) at $(0,\lambda_0)$ reduces to the problem of branching of critical points of a "key" function $\Phi(\xi,\lambda)$

$$\nabla_\xi \Phi(\xi,\lambda) = 0, \tag{23}$$

which is a function of one real variable $\xi \in \mathbb{R}$ and the parameter λ, and is defined locally in some neighborhood of $(0,\lambda_0)$. This is a kind of Liapunov-Schmidt finite-dimensional reduction for the variational problems. As in [B2] we use a scheme of constructing a key function introduced by Yu. I. Sapronov [SP]. Following it the key function is given by the formula

$$\Phi(\xi,\lambda) = E(w(\xi,\lambda),\lambda) + \frac{1}{2}\left(\iint_\Omega w(\xi,\lambda) e_s dxdy - \xi\right)^2. \tag{24}$$

Here. a map $w(\xi,\lambda)$ is given in an implicit form by the equation $\widehat{P}(w,\lambda,\xi) = (0,0)$, where a nonlinear operator $\widehat{P}$ is given as

$$\widehat{P}(w,\lambda,\xi) \equiv \left(F(w,\lambda) - \left(\iint_\Omega w e_s dxdy - \xi\right) e_s, w|_{\partial\Omega}\right). \tag{25}$$

In such a situation (cf. [SP]) the problem of investigation of bifurcation of solutions of problem (BP) is equivalent to a description of transformation of the set of critical points of the function $\Phi_0(\xi) = \Phi(\xi, \lambda_0)$ under deformation $\Phi_0(\xi) + \delta\Phi(\xi, \lambda)$ with one-dimensional parameter λ, where

$$\delta\Phi(\xi, \lambda) = \Phi(\xi, \lambda) - \Phi(\xi, \lambda_0), \qquad \delta\Phi(\xi, \lambda_0) = 0. \tag{26}$$

It is also important to describe the stable (being in the generic position) transformations of the set of critical points of $\Phi_0(\xi)$ under all possible smooth deformations. For an answer it is necessary to derive the type of singularity of critical point $\xi_0 = 0$ of function $\Phi_0(\xi)$, the codimension of singularity μ (the Milnor number), and a form of miniversal deformation (see [AGV]) .

We are in position to formulate the main result of this work.

Theorem 5. *Suppose that the bifurcation problem (BP) satisfies Assumptions $(A_1) - (A_3)$, the symmetry assumption (A_4) and Assumption (A_5) of the one-dimensional H-invariant degeneracy at the point $(0, \lambda_0)$. Then*
1. The point $(0, \lambda_0)$ is a bifurcation point of the equation $P(w, \lambda) = (0, 0)$ in the space of H-invariant functions $W_p^2(\Omega)^H \times \mathbb{R}$ and in some neighborhood of this point the set of solutions consists of two smooth curves which intersect at the point $(0, \lambda_0)$ only.
2. These curves may be written in the following parametric form

$$\Gamma_1 = \{(0, \lambda) : \lambda \in \mathbb{R}\},$$
$$\Gamma_2 = \{(w_2^{\pm}(\lambda), \lambda) : \ \lambda \in [\lambda_0, \lambda_0 + \varepsilon)\},$$

where

$$w_2^{\pm}(\lambda) = \pm\frac{e_s}{\sqrt{\lambda_0^*}}(\lambda - \lambda_0)^{1/2} + o(\lambda - \lambda_0)^{1/2},$$
$$\lambda_0^* = \frac{1}{2}\sigma \iint_{\Omega} |\nabla e_s(x, y)|^4 dxdy > 0.$$

3. At the critical point $(0, \lambda_0)$ the key function $\Phi(\xi, \lambda_0)$ has a singularity of the type A_3 ("cusp"),

$$\Phi(0, \lambda_0) = \sigma\pi,$$
$$\Phi'_\xi(0, \lambda_0) = \Phi''_\xi(0, \lambda_0) = \Phi'''_\xi(0, \lambda_0) = 0,$$
$$\Phi_\xi^{(4)}(0, \lambda_0) = -3!\lambda_0^* \neq 0,$$

the Milnor number $\mu = 2$, while the miniversal deformation of the key function $\Phi(0, \lambda_0)$ has the form $\frac{1}{4}\xi^4 - \frac{1}{2}\xi^2(\lambda - \lambda_0) + \eta\xi$.

Proof. It is enough to check that all the arguments of proof of main theorem of [B2] (see also [B1]) hold in this case. This proof is based on the Crandall-Rabinowitz bifurcation theorem from simple eigenvalue and consists of technical computations checking the assumptions of that theorem. ∎

3. Applications. Bifurcations for fluid interfaces parametrized on the disc and the square.

At first we study the bifurcation problem (BP) of the equation $P(w, \lambda) = (0, 0)$ assuming that the region Ω is the two-dimensional disc.

$$\Omega = \{(x, y) : x^2 + y^2 < 1\}. \tag{26}$$

In this case the domain has a symmetry with respect to any axis l_θ defined by it's angle $\theta \in [0, \pi)$ and given by the equation $\cos\theta y = \sin\theta x$. Let $H \subset O(2)$ be two-elements group consisting of identity map and the reflection h_θ with respect to the axis l_θ. If the point (x_s, y_s) is a symmetric point to (x, y) with respect to axis l_θ, then it's coordinates are given by the formula

$$\begin{pmatrix} x_s & y_s \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \times \begin{pmatrix} \cos 2\theta & sin2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}, \tag{27}$$

or shortly $(x_s, y_s) = (x, y) \times M_\theta$. Remark that $M_\theta^{-1} = M_\theta^\top = M_\theta$.

On the other hand the subspace $N(\lambda)$ in $W_p^2(\Omega)$, $p > 2$, is defined by boundary problem (12). As follows from Theorem 2, a bifurcation can be only at these points λ of parameter space which are the eigenvalues of the operator $-\Delta$ on the space $W_{p0}^2(\Omega)$ of functions vanishing on the boundary. The eigenvalues of $-\Delta$ on two-dimensional disc are given as a double-indexed sequence $\{\lambda_{kj} : k = 0, 1, 2, \dots, j = 1, 2, \dots\}$, where λ_{kj} the j-th zero of the k-th Bessel function

$$J_k(\lambda) = \frac{1}{\pi} \int_0^\pi \cos(\lambda \sin t - kt) dt. \tag{28}$$

If $k = 0$ then for each $j = 1, 2, \dots$ the eigenspace $N(\lambda_{0j})$ is spanned by the function

$$e(x, y) = C J_0(\lambda_{0j} r), \tag{29}$$

where r the radius of point (x, y) and C a norming constants. Consequently f or $k = 0$ we have $\dim N(\lambda_{0j}) = 1$, and the existence and local description of bifurcation follows from the main theorem of [B2].

If $k, j \in \{1, 2, \dots\}$, then the corresponding eigenspace $N(\lambda_{kj})$ is spanned by two functions

$$\begin{aligned} e_1(x, y) &= C_1 J_k(\lambda_{kj} r) \cos k\varphi, \\ e_2(x, y) &= C_2 J_k(\lambda_{kj} r) \sin k\varphi, \end{aligned} \tag{30}$$

where (r, φ) are the polar coordinates of point (x, y) and $J_k(r)$ is k-th Bessel function. Consequently $\dim N(\lambda_{kj}) = 2$ and we can not apply the bifurcation theorem of [B2].

Theorem 6. *Let $P(w,\lambda) = (0,0)$ be the bifurcation problem (BP) parametrized by the two-dimensional disc. For every $\theta \in [0,\pi)$ let $H = H_\theta$ be the two-element subgroup generated by the reflection with respect to l_θ axis. Then $(0,\lambda_{kj})$ is a bifurcation point of the equation $P(w,\lambda) = (0,0)$ in the space of H-*invariant *functions $W_p^2(\Omega)^H \times \mathbb{R}$ and in some neighborhood of this point the set of solutions equation consists on two smooth curves with all properties stated in Theorem 5.*

Proof. It is sufficient to show that for every $k,\ j \in N$ we have $\dim N(\lambda_{kj})^H = 1$. Observe that the action of H on the function spaces is given by a change of variables throughout the matrix M_θ. It is easy to check that $N(\lambda_{kj})^H$ is of dimension 1 and spanned by the function

$$e(x,y)_\theta = CJ(\lambda_{kj}r)\cos k(\varphi-\theta). \tag{31}$$

This means that assumption (A_6) is satisfied and the statement follows from Theorem 5. ∎

Remark. Observe that the region Ω has the symmetry with respect every axis $l_\theta,\ \theta \in [0,\pi)$ by the reflection. Applying Theorem 6 to distinct $\theta \in [0,\frac{\pi}{k})$ we get different branches of solutions in general.

We now turn to the case when the region Ω is the square

$$\Omega = \{(x,y) : -1 < x < 1,\ -1 < y < 1\}. \tag{32}$$

As previously, a bifurcation can occur at these parameters for which the subspaces $N(\lambda)$ of $W_p^2(\Omega),\ p > 2$, given by (12) are nontrivial. In this case there are the eigenvalues of the operator $-\Delta$ on the space $W_{p0}^2(\Omega)$ of functions vanishing on the boundary.

It is well known that if

$$\lambda_{km} = (\frac{\pi}{2}k)^2 + (\frac{\pi}{2}m)^2, \qquad k,m \in N, \tag{33}$$

is an eigenvalue of the Dirichlet problem on square then $\dim N(\lambda_{k,m})$ is equal to the number of ordered pairs of natural numbers (p,q) such that $p^2+q^2 = k^2+m^2$. For example, $\dim N(\lambda_{1,7}) = 3$, since $1^2+7^2 = 7^2+1^2 = 5^2+5^2 = 50$. Moreover each eigenspace $N(\lambda_{k,m})$ has a natural structure of an orthogonal representation of the group of all symmetries of square (see [KrM] for a detailed discussion of this representation structure – also for the Dirichlet problem on the n - dimensional cube).

If $k = m$ and $\dim N(\lambda_{kk}) = 1$ then the eigenspace $N(\lambda_{kk})$ is spanned by the function

$$e(x,y) = Cv_k(x)v_k(y), \tag{34}$$

where

$$v_k(t) = \begin{cases} \cos\,(k\frac{\pi}{2}t), & \text{if } k \text{ is odd,} \\ \sin\,(k\frac{\pi}{2}t), & \text{if } k \text{ is even.} \end{cases}$$

Consequently, in this case the bifurcation problem reduces to that one studied in [B2].

If $k \neq m$ and $\dim N(\lambda_{km}) = 2$ then the space $N(\lambda_{km})$ is spanned by the functions

$$e_1(x,y) = C_1 v_k(x) v_m(y), \qquad e_2(x,y) = C_2 v_m(x) v_k(y). \tag{35}$$

Note that in this case there are four symmetries l_θ of the region Ω given by the following angles $\theta_1 = 0,\ \theta_2 = \frac{\pi}{4},\ \theta_3 = \frac{\pi}{2},\ \theta_4 = \frac{3\pi}{4}$.

Theorem 7. *Suppose that we have bifurcation problem (BP) on square satisfying assumptions* $(A_1) - (A_4)$*. Let* H *be the two-elements group generated by the reflection with respect the axis* $y = x(\theta = \frac{\pi}{4})$*. Then for every* $k,\ m \in N$*, such that* $\dim N(\lambda_{km}) = 2$*, the point* $(0, \lambda_{km})$ *is a bifurcation point of the equation* $P(w,\lambda) = (0,0)$ *in the space of* H*-invariant functions* $W_p^2(\Omega)^H \times \mathbb{R}$*, and in some neighborhood of this point the set of solutions consists of two smooth curves which intersect at the point* $(0, \lambda_{km})$ *only. Moreover the local bifurcation picture is as in Theorem 5.*

Proof. In view of Theorem 5, it is enough to check the assumption (A_6), i.e., that $\dim N(\lambda_{km})^H = 1$. It is easy to check that

$$N(\lambda_{km})^H = N(\lambda_{km}) \cap W_p^2(\Omega)^H$$

is spanned by the function

$$e(x,y)_\theta = C(v_k(x)v_m(y) + v_m(x)v_k(y)). \tag{36}$$

The statement follows from Theorem 5. ∎

Remark. *It is worth pointing out that for some* $k, m \in \mathbb{N}$*, such that* $k \neq m$ *and* $\dim N(\lambda_{km}) = 2$*, the invariant degeneracy space* $N(\lambda_{km})_\theta$ *is one-dimensional also for the reflections corresponding to other than the remaining angles* $\theta \in \{0, \frac{\pi}{2}, \frac{3\pi}{4}\}$*. Consequently, we get other branches of solutions that these of Theorem 7.*

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