ESTIMATES FOR THE GREEN FUNCTION AND SINGULAR SOLUTIONS FOR POLYHARMONIC NONLINEAR EQUATION

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We establish a new form of the 3G theorem for polyharmonic Green function on the unit ball of $\mathbb{R}^n$ $(n \geq 2)$ corresponding to zero Dirichlet boundary conditions. This enables us to introduce a new class of functions $K_{m,n}$ containing properly the classical Kato class K_n. We exploit properties of functions belonging to $K_{m,n}$ to prove an infinite existence result of singular positive solutions for nonlinear elliptic equation of order $2m$.

1. Introduction

In [2], Boggio gave an explicit expression for the Green function $G_{m,n}$ of $(-\triangle)^m$ on the unit ball B of $\mathbb{R}^n$ $(n \geq 2)$ with Dirichlet boundary conditions

$$u = \frac{\partial}{\partial \nu} u = \cdots = \frac{\partial^{m-1}}{\partial \nu^{m-1}} u = 0 \quad \text{on } \partial B, \tag{1.1}$$

where $\partial/\partial\nu$ is the outward normal derivate and m is a positive integer.

In fact, he proved that for each x, y in B, we have

$$G_{m,n}(x,y) = k_{m,n}|x-y|^{2m-n} \int_1^{[x,y]/|x-y|} \frac{\left(\nu^2-1\right)^{m-1}}{\nu^{n-1}} d\nu, \tag{1.2}$$

where $k_{m,n}$ is a positive constant and $[x,y]^2 = |x-y|^2 + (1-|x|^2)(1-|y|^2)$, for each x, y in B.

Hence, from its expression, it is clear that $G_{m,n}$ is positive in B^2, which does not hold for the Green function for the biharmonic or m-polyharmonic operator for an arbitrary bounded domain (see, e.g., [5]). Only for the case $m = 1$, we do not have this restriction.

In [7], using the Boggio formula (1.2), Grunau and Sweers have established some interesting estimates for the Green function $G_{m,n}$ in B. In particular, they

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obtained the following inequality called $3G$ theorem: there exists a constant $a_{m,n} > 0$ such that for each $x, y, z \in B$,

$$\frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} \leq a_{m,n} \begin{cases} |x-z|^{2m-n} + |z-y|^{2m-n}, & \text{for } 2m < n, \\ \log\left(\frac{3}{|x-z|}\right) + \log\left(\frac{3}{|z-y|}\right), & \text{for } 2m = n, \\ 1, & \text{for } 2m > n. \end{cases} \tag{1.3}$$

The Green function for the Laplacian ($m = 1$) satisfies the above inequality in an arbitrary bounded $C^{1,1}$ domain Ω in $\mathbb{R}^n$. In fact, for the case $n \geq 3$, Zhao proved in [19] the existence of a positive constant C_n such that for each x, y, z in Ω,

$$\frac{G_{1,n}(x,z)G_{1,n}(z,y)}{G_{1,n}(x,y)} \leq C_n\left(\frac{1}{|x-z|^{n-2}} + \frac{1}{|y-z|^{n-2}}\right). \tag{1.4}$$

Moreover, for the case $n = 2$, Chung and Zhao showed in [3] the existence of a positive constant C_2 such that for each x, y, z in Ω,

$$\frac{G_{1,2}(x,z)G_{1,2}(z,y)}{G_{1,2}(x,y)} \leq C_2\left[\max\left(1, \log\left(\frac{1}{|x-z|}\right)\right) + \max\left(1, \log\left(\frac{1}{|y-z|}\right)\right)\right]. \tag{1.5}$$

The $3G$ theorem related to $G_{1,n}$ has been exploited in the study of functions belonging to the Kato class $K_n(\Omega)$ (see Definition 1.1), which was widely used in the study of some nonlinear differential equations (see [15, 18]).

More properties pertaining to this class can be found in [1, 3].

Definition 1.1 (see [1, 3]). A Borel measurable function φ in Ω belongs to the Kato class $K_n(\Omega)$ if φ satisfies the following conditions:

$$\begin{aligned} \lim_{\alpha\to 0}\left(\sup_{x\in\Omega}\int_{\Omega\cap B(x,\alpha)} \frac{|\varphi(y)|}{|x-y|^{n-2}}dy\right) &= 0, \quad \text{if } n \geq 3, \\ \lim_{\alpha\to 0}\left(\sup_{x\in\Omega}\int_{\Omega\cap B(x,\alpha)} \log\left(\frac{1}{|x-y|}\right)|\varphi(y)|\,dy\right) &= 0, \quad \text{if } n = 2. \end{aligned} \tag{1.6}$$

The purpose of this paper is two-folded. One is to give a new form of the $3G$ theorem to the Green function $G_{m,n}$ in B^2 which improves (1.3) and enables us to introduce a new Kato class $K_{m,n} := K_{m,n}(B)$ in the sense of Definition 1.2. The second purpose is to investigate the existence of infinitely many singular positive solutions for the following nonlinear elliptic problem:

$$\begin{aligned} \Delta^m u = (-1)^m f(\cdot, u) \quad & \text{in } B \setminus \{0\} \text{ (in the sense of distributions)}, \\ u = \frac{\partial}{\partial \nu} u = \cdots = \frac{\partial^{m-1}}{\partial \nu^{m-1}} u = 0 \quad & \text{on } \partial B, \\ u(x) \sim c\rho(x), \quad \text{near } x = 0, \; & \text{for any sufficiently small } c > 0, \end{aligned} \tag{1.7}$$

where

$$\rho(x) = \begin{cases} \dfrac{1}{|x|^{n-2m}}, & \text{for } 2m < n, \\ \log\left(\dfrac{1}{|x|}\right), & \text{for } 2m = n, \\ 1, & \text{for } 2m > n, \end{cases} \tag{1.8}$$

and f is required to satisfy suitable assumptions related to the class $K_{m,n}$ which will be specified later.

The existence of infinitely many singular positive solutions for problem (1.7) in the case $m = 1$, for an arbitrary bounded $C^{1,1}$ domain Ω in $\mathbb{R}^n$ $(n \geq 3)$, has been established by Zhang and Zhao in [18] for the special nonlinearity

$$f(x,t) = p(x)t^{\mu}, \quad \mu > 1, \tag{1.9}$$

where the function p satisfies

$$x \longrightarrow \frac{p(x)}{|x|^{(n-2)(\mu-1)}} \in K_n(\Omega). \tag{1.10}$$

This result has been recently extended by Mâagli and Zribi in [14], where f satisfies some appropriate conditions related to the class $K_{1,n}(\Omega)$.

Here we extend these results to the high order.

The outline of the paper is as follows. In Section 2, we find again by a simpler argument some estimates on the Green function $G_{m,n}$ given by Grunau and Sweers in [7] and we give further ones, including the following:

$$\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y) \leq C \begin{cases} \dfrac{1}{|x-y|^{n-2m}}, & \text{for } 2m < n, \\ \log\left(\dfrac{3}{|x-y|}\right), & \text{for } 2m = n, \\ 1, & \text{for } 2m > n. \end{cases} \tag{1.11}$$

Next, we establish the 3G theorem in this form: there exists $C_{m,n} > 0$ such that for each $x, y, z \in B$,

$$\begin{aligned} &\frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} \\ &\qquad \le C_{m,n}\left[\left(\frac{\delta(z)}{\delta(x)}\right)^m G_{m,n}(x,z) + \left(\frac{\delta(z)}{\delta(y)}\right)^m G_{m,n}(y,z)\right], \end{aligned} \tag{1.12}$$

which improves (1.3). We note that, for $m = 1$, (1.12) holds for an arbitrary bounded domain Ω in $\mathbb{R}^n$. This was proved by Kalton and Verbitsky in [10] for $n \ge 3$ and by Selmi in [16] for the case $n = 2$.

In Section 3, we define and study some properties of functions belonging to the class $K_{m,n}$.

Definition 1.2. A Borel measurable function φ in B belongs to the class $K_{m,n}$ if φ satisfies the following condition:

$$\lim_{\alpha\to 0}\left(\sup_{x\in B}\int_{B\cap B(x,\alpha)}\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y)\,|\varphi(y)|\,dy\right) = 0. \tag{1.13}$$

In particular, we show that $K_{m,n}$ contains properly $K_{j,n}$, for $1 \le j \le m-1$, which contains properly $K_n(B)$. We close this section by giving a characterization of the radial functions belonging to the class $K_{m,n}$.

For the case $m = 1$, this class has been extensively studied for an arbitrary bounded $C^{1,1}$ domain in $\mathbb{R}^n$, in [14], for $n \ge 3$, and in [12, 17] for $n = 2$. To study problem (1.7) in Section 4, we assume that f satisfies the following hypotheses:

(H_1) f is a Borel measurable function on $B \times (0,\infty)$, continuous with respect to the second variable;

(H_2) $|f(x,t)| \le tq(x,t)$, where q is a nonnegative Borel measurable function in $B \times (0,\infty)$, such that the function $t \mapsto q(x,t)$ is nondecreasing on $(0,\infty)$ and $\lim_{t\to 0} q(x,t) = 0$;

(H_3) the function g, defined on B by $g(x) = q(x, G_{m,n}(x,0))$, belongs to the class $K_{m,n}$.

We point out that in the case $m = 1$ and $f(x,t) = p(x)t^\mu$, the assumption (1.10) implies (H_3).

In order to simplify our statements, we define some convenient notation.

Notation. (i) We denote $B = \{x \in \mathbb{R}^n; |x| < 1\}$ with $n \ge 2$.

(ii) We denote $s \wedge t = \min(s,t)$ and $s \vee t = \max(s,t)$ for $s, t \in \mathbb{R}$.

(iii) For $x, y \in B$,

$$\begin{gathered} [x,y]^2 = |x-y|^2 + (1-|x|^2)(1-|y|^2), \\ \delta(x) = 1 - |x|, \\ \theta(x,y) = [x,y]^2 - |x-y|^2 = (1-|x|^2)(1-|y|^2). \end{gathered} \tag{1.14}$$

Note that $[x,y]^2 \geq 1+|x|^2|y|^2-2|x||y| = (1-|x||y|)^2$. So we have

$$\delta(x) \leq [x,y], \qquad \delta(y) \leq [x,y]. \tag{1.15}$$

(iv) Let f and g be positive functions on a set S.
We call $f \sim g$ if there is $c>0$ such that

$$\frac{1}{c}g(x) \leq f(x) \leq cg(x) \quad \forall x \in S. \tag{1.16}$$

We call $f \preceq g$ if there is $c>0$ such that

$$f(x) \leq cg(x) \quad \forall x \in S. \tag{1.17}$$

The following properties will be used several times:

(i) for $s,t \geq 0$, we have

$$s \wedge t \sim \frac{st}{s+t}, \tag{1.18}$$

$$(s+t)^p \sim s^p+t^p, \quad p \in \mathbb{R}^+; \tag{1.19}$$

(ii) let $\lambda,\mu>0$ and $0<\gamma\leq 1$, then we have

$$1-t^\lambda \sim 1-t^\mu \quad \text{for } t \in [0,1], \tag{1.20}$$

$$\log(1+t) \preceq t^\gamma \quad \text{for } t \geq 0, \tag{1.21}$$

$$\log(1+\lambda t) \sim \log(1+\mu t) \quad \text{for } t \geq 0, \tag{1.22}$$

$$\log\left(1+t^\lambda\right) \sim t^\lambda \log(2+t) \quad \text{for } t \in [0,1]; \tag{1.23}$$

(iii) on B^2 (i.e., $(x,y) \in B^2$), we have

$$\theta(x,y) \sim \delta(x)\delta(y), \tag{1.24}$$

$$[x,y]^2 \sim |x-y|^2+\delta(x)\delta(y). \tag{1.25}$$

2. Inequalities for the Green function

We first find another expression of $G_{m,n}$ given by Hayman and Korenblum in [8], which will be used later.

Proposition 2.1. *The Green function $G_{m,n}$ satisfies*

$$G_{m,n}(x,y) = \alpha_{m,n}\sum_{k=0}^{\infty}\frac{\Gamma(n/2+k)\big(\theta(x,y)\big)^{m+k}}{(k+m)![x,y]^{n+2k}}, \tag{2.1}$$

where $\alpha_{m,n}$ is some fixed positive constant.

Proof. Using the transformation $v^2 = 1 + (\theta(x,y)/|x-y|^2)(1-t)$ in (1.2), $G_{m,n}$ becomes

$$G_{m,n}(x,y) = \frac{k_{m,n}}{2}\frac{(\theta(x,y))^m}{[x,y]^n}\int_0^1 \frac{(1-t)^{m-1}}{\left(1-t(\theta(x,y)/[x,y]^2)\right)^{n/2}}dt. \tag{2.2}$$

Since $0 < \theta(x,y)/[x,y]^2 \le 1$, and for each $t \in [0,1[$, we have

$$(1-t)^{-n/2} = \sum_{k=0}^{\infty}\frac{\Gamma(n/2+k)}{k!\Gamma(n/2)}t^k; \tag{2.3}$$

it follows that

$$G_{m,n}(x,y) = \frac{k_{m,n}}{2}\sum_{k=0}^{\infty}\frac{\Gamma(n/2+k)}{k!\Gamma(n/2)}\frac{(\theta(x,y))^{m+k}}{[x,y]^{n+2k}}B(k+1,m), \tag{2.4}$$

where $B(k+1,m) := \int_0^1 t^k(1-t)^{m-1}\,dt = k!(m-1)!/(k+m)!$.

That is,

$$G_{m,n}(x,y) = \alpha_{m,n}\sum_{k=0}^{\infty}\frac{\Gamma((n/2)+k)}{(k+m)!}\frac{(\theta(x,y))^{m+k}}{[x,y]^{n+2k}} \tag{2.5}$$

with $\alpha_{m,n} > 0$. □

Moreover, from formula (1.2), we may prove, by simpler argument, the following estimates on $G_{m,n}$ given in [7].

Proposition 2.2. *On B^2, the following estimates hold:*

(i) *for* $2m < n$,

$$G_{m,n}(x,y) \sim |x-y|^{2m-n}\left(1 \wedge \frac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}\right); \tag{2.6}$$

(ii) *for* $2m = n$,

$$G_{m,n}(x,y) \sim \log\left(1 + \frac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}\right); \tag{2.7}$$

(iii) *for* $2m > n$,

$$G_{m,n}(x,y) \sim (\delta(x)\delta(y))^{m-n/2}\left(1 \wedge \frac{(\delta(x)\delta(y))^{n/2}}{|x-y|^n}\right). \tag{2.8}$$

Proof. Using in (1.2) the transformation $t = (v^2-1)^m$, we obtain the following expression for $G_{m,n}$:

$$G_{m,n}(x,y) = C|x-y|^{2m-n}\int_0^{(\theta(x,y)/|x-y|^2)^m}\frac{dt}{\left(t^{1/m}+1\right)^{n/2}}. \tag{2.9}$$

Now, from (1.19) we have

$$G_{m,n}(x,y) \sim |x-y|^{2m-n}\int_0^{(\theta(x,y))^m/|x-y|^{2m}} \frac{dt}{(t^{n/2m}+1)}. \tag{2.10}$$

Next, we distinguish the following cases.
Case 1 $(2m=n)$. It follows from (2.10), (1.22), and (1.24) that

$$\begin{aligned} G_{m,n}(x,y) &\sim \log\left(1+\frac{(\theta(x,y))^m}{|x-y|^{2m}}\right) \\ &\sim \log\left(1+\frac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}\right). \end{aligned} \tag{2.11}$$

Case 2 $(2m<n)$. Using the fact that for each $a>0$ and $\lambda>1$, we have

$$\int_0^a \frac{1}{t^\lambda+1}dt \sim 1\wedge a, \tag{2.12}$$

hence, we deduce from (2.10) and (1.24) that

$$\begin{aligned} G_{m,n}(x,y) &\sim |x-y|^{2m-n}\left(1\wedge\frac{(\theta(x,y))^m}{|x-y|^{2m}}\right) \\ &\sim |x-y|^{2m-n}\left(1\wedge\frac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}\right). \end{aligned} \tag{2.13}$$

Case 3 $(2m>n)$. We recall that $0<\theta(x,y)/[x,y]^2\le 1$, which yields

$$\int_0^1 \frac{(1-t)^{m-1}}{(1-t(\theta(x,y)/[x,y]^2))^{n/2}}dt \sim 1. \tag{2.14}$$

This implies, with (2.2), that

$$G_{m,n}(x,y) \sim \frac{(\theta(x,y))^m}{[x,y]^n}, \tag{2.15}$$

which, together with (1.24), (1.18), and (1.19), gives that

$$G_{m,n}(x,y) \sim (\delta(x)\delta(y))^{m-n/2}\left(1\wedge\frac{(\delta(x)\delta(y))^{n/2}}{|x-y|^n}\right). \tag{2.16}$$

□

Corollary 2.3. *On B^2, the following estimates hold:*

(i) *if* $2m < n$,

$$G_{m,n}(x,y) \sim \frac{(\delta(x)\delta(y))^m}{|x-y|^{n-2m}\left(|x-y|^2+\delta(x)\delta(y)\right)^m} \sim \frac{(\delta(x)\delta(y))^m}{|x-y|^{n-2m}[x,y]^{2m}} \sim \frac{1}{|x-y|^{n-2m}} - \frac{1}{\left(|x-y|^{2m}+(\delta(x)\delta(y))^m\right)^{(n-2m)/2m}}; \tag{2.17}$$

(ii) *if* $2m = n$,

$$G_{m,n}(x,y) \sim \left(1 \wedge \frac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}\right)\log\left(2+\frac{\delta(x)\delta(y)}{|x-y|^2}\right) \sim \frac{(\delta(x)\delta(y))^m}{\left(|x-y|^2+\delta(x)\delta(y)\right)^m}\log\left(2+\frac{\delta(x)\delta(y)}{|x-y|^2}\right) \sim \frac{(\delta(x)\delta(y))^m}{[x,y]^{2m}}\log\left(1+\frac{[x,y]^2}{|x-y|^2}\right); \tag{2.18}$$

(iii) *if* $2m > n$,

$$G_{m,n}(x,y) \sim \frac{(\delta(x)\delta(y))^m}{\left(|x-y|^2+(\delta(x)\delta(y))\right)^{n/2}} \sim \frac{(\delta(x)\delta(y))^m}{[x,y]^n}. \tag{2.19}$$

Proof. The proof follows immediately from Proposition 2.2 and the statements (1.18), (1.19), (1.20), (1.22), (1.23), (1.24), and (1.25). □

From the above estimates, we derive some inequalities for the Green function $G_{m,n}$ including (1.11), which will be done in the following corollaries.

Corollary 2.4. *On B^2, the following estimates hold:*

$$\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y) \preceq \begin{cases} \dfrac{1}{|x-y|^{n-2m}}, & \text{for } 2m < n, \\ \log\left(\dfrac{3}{|x-y|}\right), & \text{for } 2m = n, \\ 1, & \text{for } 2m > n. \end{cases} \tag{2.20}$$

Proof. Using Corollary 2.3 and inequalities (1.15), we deduce that

(i) if $2m < n$,

$$\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y) \preceq \frac{1}{|x-y|^{n-2m}}\frac{(\delta(y))^{2m}}{[x,y]^{2m}} \preceq \frac{1}{|x-y|^{n-2m}}; \tag{2.21}$$

(ii) if $2m = n$,

$$\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y) \preceq \log\left(1+\frac{[x,y]^2}{|x-y|^2}\right)\frac{(\delta(y))^{2m}}{[x,y]^{2m}} \preceq \log\left(\frac{3}{|x-y|}\right); \tag{2.22}$$

(iii) if $2m > n$,

$$\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y) \preceq \frac{(\delta(y))^{2m}}{[x,y]^n} \preceq 1. \tag{2.23}$$

□

Corollary 2.5. *For each $x, y \in B$ such that $|x-y| \geq r$,*

$$G_{m,n}(x,y) \preceq \frac{(\delta(x)\delta(y))^m}{r^n}. \tag{2.24}$$

Moreover, on B^2, the following estimates hold:

$$(\delta(x)\delta(y))^m \preceq G_{m,n}(x,y), \tag{2.25}$$

$$G_{m,n}(x,y) \preceq (\delta(x))^m \wedge (\delta(y))^m \quad \text{if } m \geq n, \tag{2.26}$$

$$G_{m,n}(x,y) \preceq \frac{(\delta(x))^m \wedge (\delta(y))^m}{|x-y|^{n-m}} \quad \text{if } 1 \leq m < n. \tag{2.27}$$

Proof. Assertions (2.24) and (2.25) are obviously obtained using the estimates in Corollary 2.3 and the fact that $|x-y| \leq [x,y] \preceq 1$.

Now, if $m \geq n$, then we deduce from Corollary 2.3 and (1.15) that

$$G_{m,n}(x,y) \sim \frac{(\delta(x)\delta(y))^m}{[x,y]^n} \preceq (\delta(x))^m \wedge (\delta(y))^m. \tag{2.28}$$

Then (2.26) holds.

To prove (2.27), we suppose that $1 \leq m < n$. So we obtain, from Corollary 2.3, inequalities (1.15), and $|x-y| \leq [x,y]$ that

(i) if $2m < n$, then we have

$$\begin{aligned} G_{m,n}(x,y) &\sim \frac{(\delta(x)\delta(y))^m}{|x-y|^{n-2m}[x,y]^{2m}} \\ &\leq \frac{(\delta(x))^m}{|x-y|^{n-m}}\frac{(\delta(y))^m}{[x,y]^m} \\ &\leq \frac{(\delta(x))^m}{|x-y|^{n-m}}; \end{aligned} \tag{2.29}$$

(ii) if $2m = n$, then using further inequality (1.21), we deduce that

$$\begin{aligned} G_{m,n}(x,y) &\sim \log\left(1+\frac{[x,y]^2}{|x-y|^2}\right)\frac{(\delta(x)\delta(y))^m}{[x,y]^{2m}} \\ &\preceq \frac{[x,y]}{|x-y|}\frac{(\delta(x)\delta(y))^m}{[x,y]^{2m}} \\ &\leq \frac{(\delta(x))^m}{|x-y|^m}\frac{(\delta(y))^m}{[x,y]^m} \\ &\leq \frac{(\delta(x))^m}{|x-y|^m}; \end{aligned} \tag{2.30}$$

(iii) if $2m > n$, then we have

$$G_{m,n}(x,y) \sim \frac{(\delta(x)\delta(y))^m}{[x,y]^n} \leq \frac{(\delta(x))^m}{|x-y|^{n-m}}\frac{(\delta(y))^m}{[x,y]^m} \leq \frac{(\delta(x))^m}{|x-y|^{n-m}}. \tag{2.31}$$

Hence interchanging the roles of x and y, (2.27) is proved. □

In the sequel, for a nonnegative measurable function f on B, we put

$$V_{m,n}f(x) = \int_B G_{m,n}(x,y)f(y)dy \quad \text{for } x \in B. \tag{2.32}$$

Remark 2.6. Let $m \geq n$. Then there exists a positive constant C_1 such that, for each $f \in L^1_+(B)$ and $x \in B$, we have

$$\frac{1}{C_1}\left(\int_B (\delta(y))^m f(y)dy\right)(\delta(x))^m \leq V_{m,n}f(x) \leq C_1\|f\|_1(\delta(x))^m. \tag{2.33}$$

In particular, we have $V_{m,n}1(x) \sim (\delta(x))^m$.

Moreover, let $1 \leq m < n$. Then there exists a positive constant C_2 such that for each $f \in L^p_+(B)$ with $p > n/m$ and $x \in B$, we have

$$\frac{1}{C_2}\left(\int_B (\delta(y))^m f(y)dy\right)(\delta(x))^m \leq V_{m,n}f(x) \leq C_2\|f\|_p(\delta(x))^m. \tag{2.34}$$

Indeed, (2.33) holds by (2.25) and (2.26). To prove (2.34), we use (2.25) and (2.27) and we apply the Hölder inequality, so we obtain that, for $x \in B$,

$$\begin{aligned} \left(\int_B (\delta(y))^m f(y)dy\right)(\delta(x))^m &\preceq V_{m,n}f(x) \\ &\preceq (\delta(x))^m\|f\|_p\left(\int_B \frac{dy}{|x-y|^{(n-m)p/(p-1)}}\right)^{(p-1)/p}. \end{aligned} \tag{2.35}$$

Now, for each $x \in B$, we have

$$\int_B \frac{dy}{|x-y|^{(n-m)p/(p-1)}} \le \int_{B(0,2)} \frac{d\xi}{|\xi|^{(n-m)p/(p-1)}}, \tag{2.36}$$

and this last integral is finite if and only if $p > n/m$, which gives (2.34).

Next, we aim to prove inequality (1.12). So, we need the following key lemma.

Lemma 2.7 (see [11, 13]). *Let $x, y \in B$. Then the following properties are satisfied:*

(1) *if $\delta(x)\delta(y) \le |x-y|^2$, then $(\delta(x) \vee \delta(y)) \le ((\sqrt{5}+1)/2)|x-y|$;*
(2) *if $|x-y|^2 \le \delta(x)\delta(y)$, then $((3-\sqrt{5})/2)\delta(x) \le \delta(y) \le ((3+\sqrt{5})/2)\delta(x)$.*

Proof. (1) We may assume that $(\delta(x)\vee\delta(y)) = \delta(y)$. Then the inequalities $\delta(y) \le \delta(x) + |x-y|$ and $\delta(x)\delta(y) \le |x-y|^2$ imply that

$$(\delta(y))^2 - \delta(y)|x-y| - |x-y|^2 \le 0, \tag{2.37}$$

that is,

$$\left(\delta(y) + \frac{(\sqrt{5}-1)}{2}|x-y|\right)\left(\delta(y) - \frac{(\sqrt{5}+1)}{2}|x-y|\right) \le 0. \tag{2.38}$$

It follows that

$$(\delta(x) \vee \delta(y)) \le \frac{(\sqrt{5}+1)}{2}|x-y|. \tag{2.39}$$

(2) For each $z \in \partial B$, we have $|y-z| \le |x-y| + |x-z|$ and since $|x-y|^2 \le \delta(x)\delta(y)$, we obtain

$$|y-z| \le \sqrt{\delta(x)\delta(y)} + |x-z| \le \sqrt{|x-z|\,|y-z|} + |x-z|, \tag{2.40}$$

that is,

$$\left(\sqrt{|y-z|} + \frac{(\sqrt{5}-1)}{2}\sqrt{|x-z|}\right)\left(\sqrt{|y-z|} - \frac{(\sqrt{5}+1)}{2}\sqrt{|x-z|}\right) \le 0. \tag{2.41}$$

It follows that

$$|y-z| \le \frac{(3+\sqrt{5})}{2}|x-z|. \tag{2.42}$$

Thus, interchanging the roles of x and y, we have

$$\left(\frac{3-\sqrt{5}}{2}\right)|x-z| \le |y-z| \le \left(\frac{3+\sqrt{5}}{2}\right)|x-z|, \tag{2.43}$$

which gives

$$\left(\frac{3-\sqrt{5}}{2}\right)\delta(x) \leq \delta(y) \leq \left(\frac{3+\sqrt{5}}{2}\right)\delta(x). \tag{2.44}$$

□

THEOREM 2.8 (3G theorem). *There exists a constant $C_{m,n} > 0$ such that, for each $x, y, z \in B$,*

$$\begin{aligned} &\frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} \\ &\quad \leq C_{m,n}\left[\left(\frac{\delta(z)}{\delta(x)}\right)^m G_{m,n}(x,z) + \left(\frac{\delta(z)}{\delta(y)}\right)^m G_{m,n}(y,z)\right]. \end{aligned} \tag{2.45}$$

Proof. To prove the inequality, we denote $A(x,y) := (\delta(x)\delta(y))^m/G_{m,n}(x,y)$ and we claim that A is a quasimetric, that is, for each $x, y, z \in B$,

$$A(x,y) \preceq A(x,z) + A(y,z). \tag{2.46}$$

To show the claim, we separate the proof into three cases.
Case 1. For $2m < n$, using Proposition 2.2, we have

$$A(x,y) \sim |x-y|^{n-2m}\left(|x-y|^2 \vee (\delta(x)\delta(y))\right)^m. \tag{2.47}$$

We distinguish the following subcases:

(i) if $\delta(x)\delta(y) \leq |x-y|^2$, then we have

$$A(x,y) \sim |x-y|^n \preceq |x-z|^n + |y-z|^n \preceq A(x,z) + A(y,z); \tag{2.48}$$

(ii) the inequality $|x-y|^2 \leq \delta(x)\delta(y)$ implies, from Lemma 2.7, that $\delta(x) \sim \delta(y)$. So we deduce the following:

(a) if $|x-z|^2 \leq \delta(x)\delta(z)$ or $|y-z|^2 \leq \delta(y)\delta(z)$, then it follows from Lemma 2.7 that $\delta(x) \sim \delta(y) \sim \delta(z)$. Hence,

$$\begin{aligned} A(x,y) &\sim |x-y|^{n-2m}(\delta(x)\delta(y))^m \\ &\preceq (\delta(x)\delta(y))^m\left(|x-z|^{n-2m} + |y-z|^{n-2m}\right) \\ &\preceq |x-z|^{n-2m}(\delta(x)\delta(z))^m + |y-z|^{n-2m}(\delta(y)\delta(z))^m \\ &\preceq A(x,z) + A(y,z); \end{aligned} \tag{2.49}$$

(b) if $|x-z|^2 \geq \delta(x)\delta(z)$ and $|y-z|^2 \geq \delta(y)\delta(z)$, then using Lemma 2.7, we have

$$(\delta(x) \vee \delta(z)) \preceq |x-z|, \qquad (\delta(y) \vee \delta(z)) \preceq |y-z|. \tag{2.50}$$

So, we have

$$\begin{aligned} A(x,y) &\sim |x-y|^{n-2m}\left(\delta(x)\delta(y)\right)^m \\ &\preceq \left(|x-z|^{n-2m}+|y-z|^{n-2m}\right)\left(\delta(x)\delta(y)\right)^m \\ &\preceq |x-z|^{n-2m}\left(\delta(x)\right)^{2m}+|y-z|^{n-2m}\left(\delta(y)\right)^{2m} \\ &\preceq |x-z|^{n}+|y-z|^{n} \\ &\preceq A(x,z)+A(y,z). \end{aligned} \tag{2.51}$$

Case 2. For $2m=n$, using Proposition 2.2, we have

$$A(x,y) \sim \frac{\left(\delta(x)\delta(y)\right)^m}{\log\left(1+\left(\delta(x)\delta(y)\right)^m/|x-y|^{2m}\right)}. \tag{2.52}$$

Then, since for each $t \geq 0$,

$$\frac{t}{1+t} \leq \log(1+t) \leq t, \tag{2.53}$$

we deduce that

$$|x-y|^{2m} \preceq A(x,y) \preceq |x-y|^{2m}+\left(\delta(x)\delta(y)\right)^m. \tag{2.54}$$

So we distinguish the following subcases:

(i) if $\delta(x)\delta(y) \leq |x-y|^2$, then by (1.19), we have

$$A(x,y) \preceq |x-y|^{2m} \preceq |x-z|^{2m}+|y-z|^{2m} \preceq A(x,z)+A(y,z); \tag{2.55}$$

(ii) if $|x-y|^2 \leq \delta(x)\delta(y)$, it follows by Lemma 2.7 that $\delta(x) \sim \delta(y)$.

So, we distinguish the following two subcases:

(a) if $|x-z|^2 \leq \delta(x)\delta(z)$ or $|y-z|^2 \leq \delta(y)\delta(z)$, so from Lemma 2.7, we deduce that $\delta(x) \sim \delta(y) \sim \delta(z)$.

Now, since

$$|x-y|^{2m} \preceq |x-z|^{2m}+|y-z|^{2m} \preceq \left(|x-z|^{2m} \vee |y-z|^{2m}\right), \tag{2.56}$$

then we obtain that

$$\begin{aligned} &\left(\log\left(1+\frac{\left(\delta(x)\delta(z)\right)^m}{|x-z|^{2m}}\right) \wedge \log\left(1+\frac{\left(\delta(y)\delta(z)\right)^m}{|y-z|^{2m}}\right)\right) \\ &\qquad \preceq \log\left(1+\frac{\left(\delta(x)\delta(y)\right)^m}{|x-y|^{2m}}\right), \end{aligned} \tag{2.57}$$

which, together with (2.52), implies that

$$A(x,y) \preceq A(x,z)+A(y,z); \tag{2.58}$$

(b) if $|x-z|^2 \geq \delta(x)\delta(z)$ and $|y-z|^2 \geq \delta(y)\delta(z)$, then by Lemma 2.7, it follows that

$$(\delta(x) \vee \delta(z)) \preceq |x-z|, \qquad (\delta(y) \vee \delta(z)) \preceq |y-z|. \tag{2.59}$$

Hence, by (2.54), we have

$$\begin{aligned} A(x,y) &\preceq (\delta(x)\delta(y))^m \\ &\preceq (\delta(x))^{2m} + (\delta(y))^{2m} \\ &\preceq |x-z|^{2m} + |y-z|^{2m} \\ &\preceq A(x,z) + A(y,z). \end{aligned} \tag{2.60}$$

Case 3. For $2m > n$, from Proposition 2.2, we have

$$A(x,y) \sim \left(|x-y|^2 \vee (\delta(x)\delta(y))\right)^{n/2}. \tag{2.61}$$

Then the result holds by arguments similar to that of Case 2(i). □

3. The Kato class $K_{m,n}$

In this section, we will study properties of functions belonging to the class $K_{m,n}$. We first compare the classes $K_{j,n}$ for $j \geq 1$.

Proposition 3.1. *For each $m \geq 1$, the following estimate is satisfied on B^2:*

$$\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y) \preceq (\delta(y))^{2(m-1)} \left(\frac{\delta(y)}{\delta(x)}\right) G_{1,n}(x,y). \tag{3.1}$$

In particular, $K_{1,n} \subset (\delta(\cdot))^{2(m-1)} K_{m,n}$.

Proof. Using (1.2), we have

$$G_{m,n}(x,y) \preceq |x-y|^{2m-n} \left(\frac{[x,y]^2}{|x-y|^2} - 1\right)^{m-1} \int_1^{[x,y]/|x-y|} \frac{dv}{v^{n-1}}. \tag{3.2}$$

Now, we remark by (1.25) that

$$\frac{[x,y]^2}{|x-y|^2} - 1 \sim \frac{\delta(x)\delta(y)}{|x-y|^2}. \tag{3.3}$$

So we deduce that

$$G_{m,n}(x,y) \preceq (\delta(x)\delta(y))^{m-1} G_{1,n}(x,y), \tag{3.4}$$

which implies (3.1). The proof is complete by (1.13). □

Remark 3.2. Let $j, m \in \mathbb{N}$ such that $1 \leq j < m$, then we have

$$K_n(B) \subset K_{j,n} \subset K_{m,n}. \tag{3.5}$$

Indeed, by a similar argument as above, we prove that, on B^2,

$$\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y) \preceq (\delta(y))^{2(m-j)} \left(\frac{\delta(y)}{\delta(x)}\right)^j G_{j,n}(x,y), \tag{3.6}$$

which implies that $K_{j,n} \subset K_{m,n}$. The first inclusion in (3.5) holds by putting $m = 1$ in Corollary 2.4.

Lemma 3.3. *Let φ be a function in $K_{m,n}$. Then the function*

$$x \longrightarrow (\delta(x))^{2m} \varphi(x) \tag{3.7}$$

is in $L^1(B)$.

Proof. Let $\varphi \in K_{m,n}$, then by (1.13), there exists $\alpha > 0$ such that for each $x \in B$,

$$\int_{B(x,\alpha)\cap B} \left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y) |\varphi(y)| \, dy \leq 1. \tag{3.8}$$

Let $x_1, \dots, x_p$ be in B such that $B \subset \cup_{1 \leq i \leq p} B(x_i, \alpha)$. Then by (2.25), there exists $C > 0$ such that for all $i \in \{1, \dots, p\}$ and $y \in B(x_i, \alpha) \cap B$, we have

$$(\delta(y))^{2m} \leq C \left(\frac{\delta(y)}{\delta(x_i)}\right)^m G_{m,n}(x_i, y). \tag{3.9}$$

Hence, we have

$$\begin{aligned} \int_B (\delta(y))^{2m} |\varphi(y)| \, dy &\leq C \sum_{1 \leq i \leq p} \int_{B(x_i,\alpha)\cap B} \left(\frac{\delta(y)}{\delta(x_i)}\right)^m G_{m,n}(x_i,y) |\varphi(y)| \, dy \\ &\leq Cp < \infty. \end{aligned} \tag{3.10}$$

This completes the proof. □

In the sequel, we use the notation

$$\|\varphi\|_B := \sup_{x \in B} \int_B \left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y) |\varphi(y)| \, dy. \tag{3.11}$$

Proposition 3.4. *Let φ be a function in $K_{m,n}$, then $\|\varphi\|_B < \infty$.*

Proof. Let $\varphi \in K_{m,n}$ and $\alpha > 0$. Then we have

$$\begin{aligned}
\int_B \left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y)|\varphi(y)|\,dy
&\le \int_{B\cap|x-y|\le\alpha} \left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y)|\varphi(y)|\,dy \\
&\quad + \int_{B\cap|x-y|\ge\alpha} \left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y)|\varphi(y)|\,dy.
\end{aligned} \tag{3.12}$$

Now, since by (2.24), we have

$$\int_{B\cap|x-y|\ge\alpha} \left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y)|\varphi(y)|\,dy \preceq \frac{1}{\alpha^n}\int_B (\delta(y))^{2m}|\varphi(y)|\,dy, \tag{3.13}$$

then the result follows from (1.13) and Lemma 3.3. □

Proposition 3.5. *There exists a constant $C > 0$ such that, for all $\varphi \in K_{m,n}$ and h a nonnegative harmonic function in B,*

$$\int_B G_{m,n}(x,y)(\delta(y))^{m-1}h(y)|\varphi(y)|\,dy \le C\|\varphi\|_B(\delta(x))^{m-1}h(x) \tag{3.14}$$

for all x in B.

Proof. Let h be a nonnegative harmonic function in B. So by Herglotz representation theorem (see [9, page 29]), there exists a nonnegative measure μ on ∂B such that

$$h(y) = \int_{\partial B} P(y,\xi)\mu(d\xi), \tag{3.15}$$

where $P(y,\xi) = (1-|y|^2)/|y-\xi|^n$, for $y \in B$ and $\xi \in \partial B$. So we need only to verify (3.14) for $h(y) = P(y,\xi)$ uniformly in $\xi \in \partial B$.

By (2.1) we have for each $x, y \in B$,

$$G_{m,n}(x,y) = \alpha_{m,n}\frac{(\theta(x,y))^m}{[x,y]^n}\left(1+o(1-|y|^2)\right). \tag{3.16}$$

Hence, for x, y, z in B,

$$\frac{G_{m,n}(y,z)}{G_{m,n}(x,z)} = \frac{(1-|y|^2)^m[x,z]^n}{(1-|x|^2)^m[y,z]^n}\left(1+o(1-|z|^2)\right), \tag{3.17}$$

which implies that

$$\lim_{z\to\xi}\frac{G_{m,n}(y,z)}{G_{m,n}(x,z)}=\frac{(1-|y|^2)^m}{(1-|x|^2)^m}\frac{|x-\xi|^n}{|y-\xi|^n}\sim\left(\frac{\delta(y)}{\delta(x)}\right)^{m-1}\frac{P(y,\xi)}{P(x,\xi)}. \tag{3.18}$$

Thus by Fatou's lemma and (1.12), we deduce that

$$\begin{aligned}\int_B G_{m,n}(x,y)&\left(\frac{\delta(y)}{\delta(x)}\right)^{m-1}\frac{P(y,\xi)}{P(x,\xi)}|\varphi(y)|\,dy\\ &\preceq\liminf_{z\to\xi}\int_B G_{m,n}(x,y)\frac{G_{m,n}(y,z)}{G_{m,n}(x,z)}|\varphi(y)|\,dy\\ &\preceq\liminf_{z\to\xi}\left[\int_B\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y)|\varphi(y)|\,dy\right.\\ &\qquad\qquad\left.+\int_B\left(\frac{\delta(y)}{\delta(z)}\right)^m G_{m,n}(z,y)|\varphi(y)|\,dy\right]\\ &\preceq\|\varphi\|_B,\end{aligned} \tag{3.19}$$

which completes the proof. □

Corollary 3.6. *Let φ be in $K_{m,n}$. Then*

$$\sup_{x\in B}\int_B G_{m,n}(x,y)(\delta(y))^{m-1}|\varphi(y)|\,dy<\infty. \tag{3.20}$$

Moreover, the function $x\mapsto(\delta(x))^{2m-1}\varphi(x)$ is in $L^1(B)$.

Proof. Put $h\equiv 1$ in (3.14) and using Proposition 3.4, we get (3.20).
Moreover, by (2.25), it follows that

$$\int_B(\delta(y))^{2m-1}|\varphi(y)|\,dy\preceq\int_B G_{m,n}(0,y)(\delta(y))^{m-1}|\varphi(y)|\,dy. \tag{3.21}$$

Hence the result follows from (3.20). □

Remark 3.7. We recall (see [1]) that for $m=1$ and $n\geq 3$, a radial function φ is in the classical Kato class $K_n(B)$ if and only if

$$\int_0^1 r|\varphi(r)|\,dr<\infty. \tag{3.22}$$

Similarly, we will give in the sequel a characterization of the radial functions belonging to $K_{m,n}$, which asserts, in particular, that inclusions (3.5) are proper. More precisely, we will prove in the next proposition that a radial function φ is in $K_{m,n}$ if and only if (3.20) is satisfied.

Proposition 3.8. *Let φ be a radial function in B, then the following assertions are equivalent:*

(1) $\varphi \in K_{m,n}$;

(2) $\sup_{x\in B}\int_B G_{m,n}(x,y)(\delta(y))^{m-1}|\varphi(y)|dy<\infty$;

(3) *for* $2m<n$,

$$\int_0^1 r^{2m-1}(1-r)^{2m-1}|\varphi(r)|dr<\infty. \tag{3.23}$$

For $2m=n$,

$$\int_0^1 r^{n-1}(1-r)^{n-2}\log\left(\frac{1}{r}\right)|\varphi(r)|dr<\infty. \tag{3.24}$$

For $2m>n$,

$$\int_0^1 r^{n-1}(1-r)^{2m-1}|\varphi(r)|dr<\infty. \tag{3.25}$$

Proof. Since the function $x\to\int_{S^{n-1}}G_{m,n}(x,r\omega)d\sigma(\omega)$ is radial in B, then we denote that $t=|x|$ and

$$\psi_{m,n}(t,r)=\int_{S^{n-1}}G_{m,n}(x,r\omega)d\sigma(\omega), \tag{3.26}$$

where σ is the normalized measure on the unit sphere S^{n-1} of $\mathbb{R}^n$.

Now, using Corollary 2.3 and the fact that for each $y\in B$, $[0,y]=1$, we deduce that

$$\psi_{m,n}(0,r)\sim\begin{cases} r^{2m-n}(1-r)^m, & \text{for } 2m<n,\\ (1-r)^m\log\left(1+\dfrac{1}{r^2}\right)\sim(1-r)^{m-1}\log\left(\dfrac{1}{r}\right), & \text{for } 2m=n,\\ (1-r)^m, & \text{for } 2m>n. \end{cases} \tag{3.27}$$

So, assertion (3) is equivalent to

(3′) $\int_0^1 r^{n-1}(1-r)^{m-1}\psi_{m,n}(0,r)|\varphi(r)|dr<\infty$.

We now prove the equivalences.

(1)⇒(2) follows from Corollary 3.6.

(2)⇔(3′). By virtue of [4, Theorem 2.4], we have that $t\to\psi_{m,n}(t,r)$ is a nonincreasing map on $[0,1]$, so that

$$\begin{aligned}\sup_{x\in B}&\int_B G_{m,n}(x,y)(\delta(y))^{m-1}|\varphi(y)|dy\\ &=\sup_{t\in[0,1]}\int_0^1 r^{n-1}(1-r)^{m-1}\psi_{m,n}(t,r)|\varphi(r)|dr\\ &=\int_0^1 r^{n-1}(1-r)^{m-1}\psi_{m,n}(0,r)|\varphi(r)|dr.\end{aligned} \tag{3.28}$$

$(3')\Rightarrow(1)$. Let $0<\alpha<1/4$, then we have

$$\begin{aligned}\sup_{x\in B}\int_{B\cap B(x,\alpha)}&\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y)|\varphi(y)|\,dy\\ &\leq \sup_{0\leq t\leq 1}\int_{(t-\alpha)\vee 0}^{(t+\alpha)\wedge 1} r^{n-1}\frac{(1-r)^m}{(1-t)^m}\psi_{m,n}(t,r)|\varphi(r)|\,dr\\ &\leq \sup_{0\leq t\leq 1/2}\int_{(t-\alpha)\vee 0}^{(t+\alpha)\wedge 1} r^{n-1}\frac{(1-r)^m}{(1-t)^m}\psi_{m,n}(t,r)|\varphi(r)|\,dr\\ &\quad+\sup_{1/2\leq t\leq 1}\int_{(t-\alpha)\vee 0}^{(t+\alpha)\wedge 1} r^{n-1}\frac{(1-r)^m}{(1-t)^m}\psi_{m,n}(t,r)|\varphi(r)|\,dr\\ &= I_1+I_2.\end{aligned}\tag{3.29}$$

Using [4, Theorem 2.4], we have

$$\begin{aligned}I_1&\leq \sup_{0\leq t\leq 1/2}\int_{(t-\alpha)\vee 0}^{(t+\alpha)\wedge 1} r^{n-1}\frac{(1-r)^m}{(1-t)^m}\psi_{m,n}(0,r)|\varphi(r)|\,dr\\ &\preceq \sup_{0\leq t\leq 1/2}\int_{(t-\alpha)\vee 0}^{(t+\alpha)\wedge 1} r^{n-1}(1-r)^{m-1}\psi_{m,n}(0,r)|\varphi(r)|\,dr.\end{aligned}\tag{3.30}$$

On the other hand, by (3.1), we have

$$I_2\leq \sup_{1/2\leq t\leq 1}\int_{(t-\alpha)\vee 0}^{(t+\alpha)\wedge 1} r^{n-1}\frac{(1-r)^{2m-1}}{(1-t)}\psi_{1,n}(t,r)|\varphi(r)|\,dr.\tag{3.31}$$

Now, by elementary calculus, we obtain that

$$\psi_{1,n}(t,r)=\begin{cases}\dfrac{1}{n-2}(t\vee r)^{2-n}\left(1-(t\vee r)^{n-2}\right), & \text{for } n\geq 3,\\ \log\left(\dfrac{1}{t\vee r}\right), & \text{for } n=2.\end{cases}\tag{3.32}$$

So, using (1.20) and the fact that $\log(1/s)\preceq(1-s)$ for $s\geq 1/2$, we have for each $n\geq 2$ and $t\geq 1/2$,

$$\psi_{1,n}(t,r)\preceq(1-t\vee r).\tag{3.33}$$

Hence, from (3.27), we have

$$\begin{aligned}I_2&\preceq \sup_{1/2\leq t\leq 1}\int_{(t-\alpha)\vee 0}^{(t+\alpha)\wedge 1} r^{n-1}(1-r)^{2m-1}\frac{(1-t\vee r)}{1-t}|\varphi(r)|\,dr\\ &\preceq \sup_{1/2\leq t\leq 1}\int_{(t-\alpha)\vee 0}^{(t+\alpha)\wedge 1} r^{n-1}(1-r)^{m-1}\psi_{m,n}(0,r)|\varphi(r)|\,dr.\end{aligned}\tag{3.34}$$

Thus, $I_1+I_2\preceq \sup_{0\leq t\leq 1}\int_{(t-\alpha)\vee 0}^{(t+\alpha)\wedge 1} r^{n-1}(1-r)^{m-1}\psi_{m,n}(0,r)|\varphi(r)|\,dr$.

Let $\phi(s) = \int_0^s r^{n-1}(1-r)^{m-1}\psi_{m,n}(0,r)|\varphi(r)|dr$ for $s \in [0,1]$.

Then using (3′), we deduce that ϕ is a continuous function on $[0,1]$, which implies that

$$\int_{(t-\alpha)\vee 0}^{(t+\alpha)\wedge 1} r^{n-1}(1-r)^{m-1}\psi_{m,n}(0,r)\,|\,\varphi(r)\,|\,dr = \phi((t+\alpha)\wedge 1) - \phi((t-\alpha)\vee 0) \tag{3.35}$$

converges to zero as $\alpha \to 0$ uniformly for $t \in [0,1]$. So, $\lim_{\alpha\to 0}(I_1 + I_2) = 0$, that is, $\varphi \in K_{m,n}$. □

Example 3.9. let q be the function defined in B by

$$q(x) = \frac{1}{(\delta(x))^{\lambda}}. \tag{3.36}$$

By Proposition 3.8, $q \in K_{m,n}$ if and only if $\lambda < 2m$ and $V_{m,n}q$ is bounded if and only if $\lambda < m+1$. In fact, we give in the next proposition more precise estimates on the m-potential $V_{m,n}q$.

Proposition 3.10. *On B, the following estimates hold:*

(i) $(\delta(x))^m \preceq V_{m,n}q(x) \preceq (\delta(x))^{2m-\lambda}$ *if* $m < \lambda < m+1$;

(ii) $(\delta(x))^m \preceq V_{m,n}q(x) \preceq (\delta(x))^m \log(2/\delta(x))$ *if* $\lambda = m$;

(iii) $V_{m,n}q(x) \sim (\delta(x))^m$ *if* $\lambda < m$.

Proof. Let $\lambda < m+1$. Then from (2.25), we have

$$(\delta(x))^m \int_B \frac{dy}{(\delta(y))^{\lambda-m}} \preceq V_{m,n}q(x), \tag{3.37}$$

which implies the lower estimates.

For the upper estimates, we have, from (3.1),

$$\begin{aligned} V_{m,n}q(x) &\preceq \int_B (\delta(x))^{m-1}(\delta(y))^{m-1}G_{1,n}(x,y)q(y)dy \\ &\preceq (\delta(x))^{m-1}\int_0^1 \frac{r^{n-1}}{(1-r)^{\lambda+1-m}}\psi_{1,n}(|x|,r)dr. \end{aligned} \tag{3.38}$$

On the other hand, using (1.20) and the inequality $t\log(1/t) \leq (1-t)$, for $t \in [0,1]$, we deduce from (3.32) that $r^{n-1}\psi_{1,n}(|x|,r) \preceq (1-|x|\vee r)$ for each $n \geq 2$.

This implies that

$$
\begin{aligned}
V_{m,n}q(x) &\preceq (\delta(x))^{m-1}\int_0^1 \frac{1-(|x|\vee r)}{(1-r)^{\lambda+1-m}}\,dr\\
&\preceq (\delta(x))^{m}\int_0^{|x|} \frac{dr}{(1-r)^{\lambda+1-m}} + (\delta(x))^{m-1}\int_{|x|}^1 \frac{dr}{(1-r)^{\lambda-m}}\\
&= I_1 + I_2.
\end{aligned}
\tag{3.39}
$$

So, by elementary calculus, we obtain that

$$
\begin{gathered}
I_1 \preceq (\delta(x))^{m}\begin{cases}(\delta(x))^{m-\lambda}, & \text{if } m<\lambda<m+1,\\ \log\dfrac{2}{\delta(x)}, & \text{if } \lambda=m,\\ 1, & \text{if } \lambda<m,\end{cases}\\
I_2 \preceq (\delta(x))^{2m-\lambda}.
\end{gathered}
\tag{3.40}
$$

This completes the proof. □

Remark 3.11. By Proposition 3.10, we find again the result of Gilbarg and Trudinger in [6, Theorem 4.9] for the case $m=1$ and $1<\lambda<2$.

4. Positive singular solutions of the equation $\Delta^m u = (-1)^m f(\cdot,u)$

In this section, we are interested in the existence of positive singular solutions for problem (1.7). We present in the next theorem the main result of this section.

THEOREM 4.1. *Assume (H_1), (H_2), and (H_3). Then problem (1.7) has infinitely many solutions. More precisely, there exists $b_0>0$ such that for each $b\in(0,b_0]$, there exists a solution u of (1.7) continuous on $B\backslash\{0\}$ and satisfying for all $x\in B$,*

$$
\frac{b}{2}G_{m,n}(x,0) \le u(x) \le \frac{3b}{2}G_{m,n}(x,0) \tag{4.1}
$$

and, for $2m\le n$,

$$
\lim_{|x|\to 0}\frac{u(x)}{G_{m,n}(x,0)} = b. \tag{4.2}
$$

For the proof, we need the following lemmas.

LEMMA 4.2. *Let $\varphi\in K_{m,n}$ and $x_0\in\overline{B}$. Then*

$$
\lim_{\alpha\to 0}\left(\sup_{x,z\in B}\frac{1}{G_{m,n}(x,z)}\int_{B\cap B(x_0,\alpha)} G_{m,n}(x,y)G_{m,n}(y,z)\,|\varphi(y)|\,dy\right) = 0. \tag{4.3}
$$

Proof. Let $\varepsilon > 0$. Then by (1.13), there exists $r > 0$ such that

$$\sup_{\xi\in B}\int_{B\cap B(\xi,r)}\left(\frac{\delta(y)}{\delta(\xi)}\right)^m G_{m,n}(\xi,y)|\varphi(y)|\,dy \le \varepsilon. \tag{4.4}$$

Let $\alpha > 0$. Then it follows, from Theorem 2.8, that for each $x, z \in B$,

$$\begin{aligned}
&\frac{1}{G_{m,n}(x,z)}\int_{B\cap B(x_0,\alpha)} G_{m,n}(x,y)G_{m,n}(y,z)|\varphi(y)|\,dy\\
&\quad\le C_{m,n}\int_{B\cap B(x_0,\alpha)}\left[\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y)+\left(\frac{\delta(y)}{\delta(z)}\right)^m G_{m,n}(y,z)\right]|\varphi(y)|\,dy\\
&\quad\le 2C_{m,n}\sup_{\xi\in B}\int_{B\cap B(x_0,\alpha)}\left(\frac{\delta(y)}{\delta(\xi)}\right)^m G_{m,n}(\xi,y)|\varphi(y)|\,dy.
\end{aligned}\tag{4.5}$$

On the other hand, by (2.24), we have

$$\begin{aligned}
&\int_{B\cap B(x_0,\alpha)}\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y)|\varphi(y)|\,dy\\
&\quad\le \int_{B\cap(|x-y|\le r)}\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y)|\varphi(y)|\,dy\\
&\qquad+\int_{B\cap B(x_0,\alpha)\cap(|x-y|\ge r)}\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y)|\varphi(y)|\,dy\\
&\quad\preceq \sup_{\xi\in B}\int_{B\cap B(\xi,r)}\left(\frac{\delta(y)}{\delta(\xi)}\right)^m G_{m,n}(\xi,y)|\varphi(y)|\,dy\\
&\qquad+\int_{B\cap B(x_0,\alpha)}(\delta(y))^{2m}|\varphi(y)|\,dy.
\end{aligned}\tag{4.6}$$

Now, using Lemma 3.3 and (4.4), the result holds by letting $\alpha \to 0$. □

Put $F := \{\omega \in C^+(\overline{B}) : \|\omega\|_\infty \le 1\}$, where $\|\cdot\|_\infty$ is the uniform norm. So we have the following result.

Lemma 4.3. *Assume* (H_1), (H_2), *and* (H_3). *Define the operator T on F by*

$$T\omega(x) = \frac{1}{G_{m,n}(x,0)}\int_B G_{m,n}(x,y)f(y,\omega(y)G_{m,n}(y,0))\,dy, \quad x\in B. \tag{4.7}$$

Then the family of functions $T(F)$ is relatively compact in $C(\overline{B})$.

Proof. By (H_2), we have for all $\omega \in F$,

$$|T\omega(x)| \le \frac{1}{G_{m,n}(x,0)}\int_B G_{m,n}(x,y)G_{m,n}(y,0)g(y)\,dy. \tag{4.8}$$

Since $g(x) = q(x, G_{m,n}(x,0)) \in K_{m,n}$, then, by Theorem 2.8, we deduce that

$$\begin{aligned} \|T\omega\|_\infty &\le 2C_{m,n} \sup_{\xi\in B} \int_B \left(\frac{\delta(y)}{\delta(\xi)}\right)^m G_{m,n}(\xi,y)g(y)dy \\ &\preceq \|g\|_B. \end{aligned} \tag{4.9}$$

Hence, the family $T(F)$ is uniformly bounded. Now, we will prove the equicontinuity of $T(F)$ in $\overline{B}$. Let $x_0 \in \overline{B}$ and $\alpha > 0$. Let $x, x' \in B(x_0,\alpha) \cap B$ and $\omega \in F$, then

$$\begin{aligned} &\left| T\omega(x) - T\omega(x') \right| \\ &\quad \le \int_B \left| \frac{G_{m,n}(x,y)}{G_{m,n}(x,0)} - \frac{G_{m,n}(x',y)}{G_{m,n}(x',0)} \right| G_{m,n}(y,0)g(y)dy \\ &\quad \le 2\sup_{\xi\in B} \frac{1}{G_{m,n}(\xi,0)} \int_{B\cap B(0,2\alpha)} G_{m,n}(\xi,y)G_{m,n}(y,0)g(y)dy \\ &\qquad + 2\sup_{\xi\in B} \frac{1}{G_{m,n}(\xi,0)} \int_{B\cap B(x_0,2\alpha)} G_{m,n}(\xi,y)G_{m,n}(y,0)g(y)dy \\ &\qquad + \int_{B\cap B^c(0,2\alpha)\cap B^c(x_0,2\alpha)} \left| \frac{G_{m,n}(x,y)}{G_{m,n}(x,0)} - \frac{G_{m,n}(x',y)}{G_{m,n}(x',0)} \right| G_{m,n}(y,0)g(y)dy. \end{aligned} \tag{4.10}$$

If $|x_0 - y| \ge 2\alpha$, then $|x-y| \ge \alpha$ and $|x'-y| \ge \alpha$. So (1.12) and (2.24) imply that, for all $x \in B(x_0,\alpha)\cap B$ and $y \in \Omega := B^c(0,2\alpha)\cap B^c(x_0,2\alpha)\cap B$,

$$\frac{G_{m,n}(x,y)}{G_{m,n}(x,0)} G_{m,n}(y,0) \preceq (\delta(y))^{2m}. \tag{4.11}$$

Moreover, using (3.18), we deduce, when $y \in \Omega$, that the function $x \to G_{m,n}(x,y)/G_{m,n}(x,0)$ is continuous in $B(x_0,\alpha)\cap B$. Then, by Lemma 3.3 and the dominated convergence theorem, we obtain that

$$\int_\Omega \left| \frac{G_{m,n}(x,y)}{G_{m,n}(x,0)} - \frac{G_{m,n}(x',y)}{G_{m,n}(x',0)} \right| G_{m,n}(y,0)g(y)dy \longrightarrow 0 \tag{4.12}$$

as $|x - x'| \to 0$.

By Lemma 4.2, we deduce that

$$\left| T\omega(x) - T\omega(x') \right| \longrightarrow 0, \quad \text{as } |x-x'| \longrightarrow 0, \tag{4.13}$$

uniformly for all $\omega \in F$. The result follows by Ascoli's theorem. □

Remark 4.4. Let $\alpha > 0$. Then for $2m \le n$ and $y \in B^c(0, 2\alpha) \cap B$, we have

$$\lim_{|x| \to 0} \frac{G_{m,n}(x, y)}{G_{m,n}(x, 0)} = 0. \tag{4.14}$$

So, using the same argument as in the proof of Lemma 4.3, we deduce that for $2m \le n$,

$$|T\omega(x)| \longrightarrow 0, \quad \text{as } |x| \longrightarrow 0, \tag{4.15}$$

uniformly for all $\omega \in F$.

Proof of Theorem 4.1. We aim to show that there exists $b_0 > 0$ such that for each $b \in (0, b_0]$, there exists a continuous function u in $B \backslash \{0\}$ satisfying the following integral equation:

$$u(x) = bG_{m,n}(x, 0) + \int_B G_{m,n}(x, y) f(y, u(y)) dy, \quad x \in B \backslash \{0\}. \tag{4.16}$$

Let $\beta \in (0, 1)$. Then by Lemma 4.3, the function

$$T_\beta(x) = \frac{1}{G_{m,n}(x, 0)} \int_B G_{m,n}(x, y) G_{m,n}(y, 0) q(y, \beta G_{m,n}(y, 0)) dy \tag{4.17}$$

is continuous in $\overline{B}$. Moreover, using (1.12), (H_2), and (H_3), we have

$$\sup_{\zeta \in B} \int_B \left(\frac{\delta(y)}{\delta(\zeta)} \right)^m G_{m,n}(\zeta, y) g(y) dy \preceq \|g\|_B. \tag{4.18}$$

So, we deduce by the dominated convergence theorem and (H_2) that

$$\lim_{\beta \to 0} T_\beta(x) = 0 \quad \forall x \in \overline{B}. \tag{4.19}$$

Since the function $\beta \to T_\beta(x)$ is nondecreasing in $(0, 1)$, it follows by Dini's lemma that

$$\lim_{\beta \to 0} \left(\sup_{x \in B} \frac{1}{G_{m,n}(x, 0)} \int_B G_{m,n}(x, y) G_{m,n}(y, 0) q(y, \beta G_{m,n}(y, 0)) dy \right) = 0. \tag{4.20}$$

Thus, there exists $\beta \in (0,1)$ such that for each $x \in \overline{B}$,

$$\frac{1}{G_{m,n}(x,0)} \int_B G_{m,n}(x,y) G_{m,n}(y,0) q(y, \beta G_{m,n}(y,0)) dy \le \frac{1}{3}. \tag{4.21}$$

Let $b_0 = (2/3)\beta$ and $b \in (0, b_0]$. We will use a fixed-point argument. Let

$$S = \left\{ \omega \in C(\overline{B}) : \frac{b}{2} \le \omega(x) \le \frac{3b}{2} \right\}. \tag{4.22}$$

Then, S is a nonempty, closed, bounded, and convex set in $C(\overline{B})$. We define the operator Γ on S by

$$\Gamma\omega(x) = b + \frac{1}{G_{m,n}(x,0)} \int_B G_{m,n}(x,y) f(y, \omega(y) G_{m,n}(y,0)) dy, \quad x \in B. \tag{4.23}$$

By Lemma 4.3, $\Gamma S \subset C(\overline{B})$. Moreover, let $\omega \in S$, then for any $x \in B$, we have

$$\begin{aligned} |\Gamma\omega(x) - b| &\le \frac{3b}{2} \frac{1}{G_{m,n}(x,0)} \int_B G_{m,n}(x,y) G_{m,n}(y,0) q(y, \beta G_{m,n}(y,0)) dy \\ &\le \frac{b}{2}. \end{aligned} \tag{4.24}$$

It follows that $b/2 \le \Gamma\omega(x) \le 3b/2$ and so $\Gamma S \subset S$.

Next, we will prove the continuity of Γ in the uniform norm. Let $(\omega_k)_k$ be a sequence in S which converges uniformly to $\omega \in S$. Then since f is continuous with respect to the second variable, we deduce by the dominated convergence theorem that

$$\Gamma\omega_k(x) \longrightarrow \Gamma\omega(x) \quad \text{as } k \longrightarrow \infty, \ \forall x \in B. \tag{4.25}$$

Now, since ΓS is a relatively compact family in $C(\overline{B})$, then

$$\|\Gamma\omega_k - \Gamma\omega\|_\infty \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{4.26}$$

So the Schauder fixed-point theorem implies the existence of $\omega \in S$ such that $\Gamma\omega = \omega$.

For all $x \in B$, put $u(x) = \omega(x) G_{m,n}(x,0)$. Then, u is a continuous function in $B \backslash \{0\}$ satisfying (4.16).

Furthermore, if $2m \le n$, then by Remark 4.4, we obtain that $\lim_{|x|\to 0} \omega(x) = b$, that is, $\lim_{|x|\to 0} u(x)/G_{m,n}(x,0) = b$. This ends the proof. □

Example 4.5. Let $p > 0$, $\lambda < 2m$, and $\mu < n \wedge 2m$. Let V be a measurable function in B such that for each $x \in B$,

$$|V(x)| \le \frac{1}{(\delta(x))^\lambda |x|^\mu (G_{m,n}(x,0))^p}. \tag{4.27}$$

Then there exists $b_0 > 0$ such that for each $b \in (0, b_0]$, the nonlinear problem

$$\Delta^m u = (-1)^m V(x) u^{p+1}(x) \quad \text{in } B \setminus \{0\} \text{ (in the sense of distributions)},$$
$$u = \frac{\partial}{\partial \nu} u = \cdots = \frac{\partial^{m-1}}{\partial \nu^{m-1}} u = 0 \quad \text{on } \partial B, \tag{4.28}$$

has a positive solution u, continuous on $B \setminus \{0\}$ and satisfying for all $x \in B$,

$$\frac{b}{2} G_{m,n}(x,0) \le u(x) \le \frac{3b}{2} G_{m,n}(x,0) \tag{4.29}$$

and for $2m \le n$, we have

$$\lim_{|x| \to 0} \frac{u(x)}{G_{m,n}(x,0)} = b. \tag{4.30}$$

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