LOWER BOUNDS FOR EIGENVALUES OF THE ONE-DIMENSIONAL p-LAPLACIAN

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Received 14 April 2003

We present sharp lower bounds for eigenvalues of the one-dimensional p-Laplace operator. The method of proof is rather elementary, based on a suitable generalization of the Lyapunov inequality.

1. Introduction

In [9], Krein obtained sharp lower bounds for eigenvalues of weighted second-order Sturm-Liouville differential operators with zero Dirichlet boundary conditions. In this paper, we give a new proof of this result and we extend it to the one-dimensional p-Laplacian

$$\begin{gathered} -\left(\left|u'(x)\right|^{p-2}u'(x)\right)' = \lambda r(x)\left|u(x)\right|^{p-2}u(x), \quad x \in (a,b), \\ u(a) = 0, \qquad u(b) = 0, \end{gathered} \tag{1.1}$$

where λ is a real parameter, $p > 1$, and r is a bounded positive function. The method of proof is based on a suitable generalization of the Lyapunov inequality to the nonlinear case, and on some elementary inequalities. Our main result is the following theorem.

Theorem 1.1. *Let λ_n be the nth eigenvalue of problem (1.1). Then,*

$$\frac{2^p n^p}{(b-a)^{p-1}\int_a^b r(x)dx} \leq \lambda_n. \tag{1.2}$$

We also prove that the lower bound is sharp.

Eigenvalue problems for quasilinear operators of p-Laplace type like (1.1) have received considerable attention in the last years (see, e.g., [1, 2, 3, 5, 8, 13]). The asymptotic behavior of eigenvalues was obtained in [6, 7].

Lyapunov inequalities have proved to be useful tools in the study of qualitative nature of solutions of ordinary linear differential equations. We recall the classical Lyapunov's inequality.

Abstract and Applied Analysis 2004:2 (2004) 147–153
2000 Mathematics Subject Classification: 34L15, 34L30
URL: http://dx.doi.org/10.1155/S108533750431002X

THEOREM 1.2 (Lyapunov). *Let $r : [a,b] \to \mathbb{R}$ be a positive continuous function. Let u be a solution of*

$$\begin{aligned} -u''(x) = r(x)u(x), \quad x \in (a,b), \\ u(a) = 0, \qquad u(b) = 0. \end{aligned} \tag{1.3}$$

Then, the following inequality holds:

$$\int_a^b r(x)dx \geq \frac{4}{b-a}. \tag{1.4}$$

For the proof, we refer the interested reader to [10, 11, 12]. We wish to stress the fact that those proofs are based on the linearity of (1.3), by direct integration of the differential equation. Also, in [12], the special role played by the Green function $g(s,t)$ of a linear differential operator $L(u)$ was noted, by reformulating the Lyapunov inequality for

$$L(u)(x) - r(x)u(x) = 0 \tag{1.5}$$

as

$$\int_a^b r(x)dx \geq \frac{1}{\operatorname{Max}\{g(s,s) : s \in (b-a)\}}. \tag{1.6}$$

The paper is organized as follows. Section 2 is devoted to the Lyapunov inequality for the one-dimensional p-Laplace equation. In Section 3, we focus on the eigenvalue problem and we prove Theorem 1.1.

2. The Lyapunov inequality

We consider the following quasilinear two-point boundary value problem:

$$-\left(|u'|^{p-2}u'\right)' = r|u|^{p-2}u, \qquad u(a) = 0 = u(b), \tag{2.1}$$

where r is a bounded positive function and $p > 1$. By a solution of problem (2.1), we understand a real-valued function $u \in W_0^{1,p}(a,b)$, such that

$$\int_a^b |u'|^{p-2}u'v' = \int_a^b r|u|^{p-2}uv \quad \text{for each } v \in W_0^{1,p}(a,b). \tag{2.2}$$

The regularity results of [4] imply that the solutions u are at least of class $C^{1,\alpha}_{\text{loc}}$ and satisfy the differential equation almost everywhere in (a,b).

Our first result provides an estimation of the location of the maxima of a solution in (a,b). We need the following lemma.

Lemma 2.1. *Let $r:[a,b]\to\mathbb{R}$ be a bounded positive function, let u be a solution of problem (2.1), and let c be a point in (a,b) where $|u(x)|$ is maximized. Then, the following inequalities hold:*

$$\int_a^c r(x)dx \geq \left(\frac{1}{c-a}\right)^{p/q}, \qquad \int_c^b r(x)dx \geq \left(\frac{1}{b-c}\right)^{p/q}, \tag{2.3}$$

where q is the conjugate exponent of p, that is, $1/p+1/q=1$.

Proof. Clearly, by using Hölder's inequality,

$$u(c)=\int_a^c u'(x)dx \leq (c-a)^{1/q}\left(\int_a^c |u'(x)|^p dx\right)^{1/p}. \tag{2.4}$$

We note that $u'(c)=0$. So, integrating by parts in (2.1) after multiplying by u gives

$$\int_a^c |u'(x)|^p dx = \int_a^c r(x)|u(x)|^p dx. \tag{2.5}$$

Thus,

$$\begin{aligned} u(c) &\leq (c-a)^{1/q}\left(\int_a^c r(x)|u(x)|^p dx\right)^{1/p} \\ &\leq (c-a)^{1/q}|u(c)|\left(\int_a^c r(x)dx\right)^{1/p}. \end{aligned} \tag{2.6}$$

Then, the first inequality follows after cancelling $u(c)$ in both sides while the second is proved in a similar fashion. □

Remark 2.2. The sum of both inequalities shows that c cannot be too close to a or b. We have $\int_a^b r(x)dx<\infty$, but

$$\lim_{c\to a^+}\left[\left(\frac{1}{c-a}\right)^{p/q}+\left(\frac{1}{b-c}\right)^{p/q}\right]=\lim_{c\to b^-}\left[\left(\frac{1}{c-a}\right)^{p/q}+\left(\frac{1}{b-c}\right)^{p/q}\right]=\infty. \tag{2.7}$$

Our next result restates the Lyapunov inequality.

Theorem 2.3. *Let $r:[a,b]\to\mathbb{R}$ be a bounded positive function, let u be a solution of problem (2.1), and let q be the conjugate exponent of $p\in(1,+\infty)$. The following inequality holds:*

$$\frac{2^p}{(b-a)^{p/q}} \leq \int_a^b r(x)dx. \tag{2.8}$$

Proof. For every $c\in(a,b)$, we have

$$2|u(c)| = \left|\int_a^c u'(x)dx\right| + \left|\int_c^b u'(x)dx\right| \leq \int_a^b |u'(x)|dx. \tag{2.9}$$

By using Hölder's inequality,

$$2|u(c)| \le (b-a)^{1/q}\left(\int_a^b |u'(x)|^p dx\right)^{1/p} = (b-a)^{1/q}\left(\int_a^b r(x)|u(x)|^p dx\right)^{1/p}. \tag{2.10}$$

We now choose c in (a,b) such that $|u(x)|$ is maximized. Then,

$$2|u(c)| \le (b-a)^{1/q}|u(c)|\left(\int_a^b r(x)dx\right)^{1/p}. \tag{2.11}$$

After cancelling, we obtain

$$\frac{2^p}{(b-a)^{p/q}} \le \int_a^b r(x)dx, \tag{2.12}$$

and the theorem is proved. □

Remark 2.4. We note that, for $p = 2 = q$, inequality (2.8) coincides with inequality (1.4).

3. Eigenvalues bounds

In this section, we focus on the following eigenvalue problem:

$$-\left(|u'|^{p-2}u'\right)' = \lambda r|u|^{p-2}u, \qquad u(a) = 0 = u(b), \tag{3.1}$$

where $r \in L^\infty(a,b)$ is a positive function, λ is a real parameter, and $p > 1$.

Remark 3.1. The eigenvalues could be characterized variationally:

$$\lambda_k(\Omega) = \inf_{F\in C_k^\Omega}\sup_{u\in F}\frac{\int_\Omega |u'|^p}{\int_\Omega r|u|^p}, \tag{3.2}$$

where

$$C_k^\Omega = \{C \subset M^\Omega : C \text{ compact},\ C = -C,\ \gamma(C) \ge k\},$$
$$M^\Omega = \left\{u \in W_0^{1,p}(\Omega) : \int_\Omega |u'|^p = 1\right\}, \tag{3.3}$$

and $\gamma : \Sigma \to \mathbb{N}\cup\{\infty\}$ is the Krasnoselskii genus,

$$\gamma(A) = \min\{k \in \mathbb{N}, \text{ there exist } f \in C(A,\mathbb{R}^k\setminus\{0\}),\ f(x) = -f(-x)\}. \tag{3.4}$$

The spectrum of problem (1.1) consists of a countable sequence of nonnegative eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$, and coincides with the eigenvalues obtained by Ljusternik-Schnirelmann theory.

Now, we prove the lower bound for the eigenvalues of problem (3.1) for every $p \in (1,+\infty)$. We now prove our main result, Theorem 1.1.

Proof of Theorem 1.1. Let λ_n be the nth eigenvalue of problem (3.1) and let u_n be an associate eigenfunction. As in the linear case, u_n has n nodal domains in $[a,b]$ (see [2, 13]).

Applying inequality (2.8) in each nodal domain, we obtain

$$\sum_{k=1}^{n} \frac{2^p}{(x_k - x_{k-1})^{p/q}} \le \lambda_n \sum_{k=1}^{n} \left(\int_{x_{k-1}}^{x_k} r(x)dx \right) \le \lambda_n \int_a^b r(x)dx, \tag{3.5}$$

where $a = x_0 < x_1 < \cdots < x_n = b$ are the zeros of u_n in $[a,b]$.

Now, the sum on the left-hand side is minimized when all the summands are the same, which gives the lower bound

$$2^p n \left(\frac{n}{b-a} \right)^{p/q} \le \lambda_n \int_a^b r(x)dx. \tag{3.6}$$

The theorem is proved. □

Finally, we prove that the lower bound is sharp.

Theorem 3.2. *Let $\varepsilon \in \mathbb{R}$ be a positive number. There exist a family of weight functions $r_{n,\varepsilon}$ such that*

$$\lim_{\varepsilon \to 0^+} \left(\lambda_{n,\varepsilon} - \frac{2^p n^p}{(b-a)^{p-1} \int_a^b r_{n,\varepsilon}} \right) = 0, \tag{3.7}$$

where $\lambda_{n,\varepsilon}$ is the nth eigenvalue of

$$-\left(|u'|^{p-2}u'\right)' = \lambda r_{n,\varepsilon}|u|^{p-2}u, \qquad u(a) = 0 = u(b). \tag{3.8}$$

Proof. We begin with the first eigenvalue λ_1. We fix $\int_a^b r(x)dx = M$, and let c be the midpoint of the interval (a,b).

Let r_1 be the delta function $M\delta_c(x)$. We obtain

$$\lambda_1 = \min_{u \in W_0^{1,p}} \frac{\int_a^b |u'|^p}{\int_a^b \delta_c u^p} = \min_{u \in W_0^{1,p}} \frac{2\int_a^c |u'|^p}{M u^p(c)} = \frac{2\mu_1}{M}, \tag{3.9}$$

where μ_1 is the first Steklov eigenvalue in $[a,c]$,

$$\begin{gathered} -\left(|u'(x)|^{p-2} u'(x) \right)' = 0, \\ |u'(c)|^{p-2} u'(c) = \mu |u(c)|^{p-2} u(c), \qquad u(a) = 0. \end{gathered} \tag{3.10}$$

A direct computation gives

$$\mu_1 = \frac{2^{p-1}}{(b-a)^{p-1}}. \tag{3.11}$$

Now, we define the functions $r_{1,\varepsilon}$:

$$r_{1,\varepsilon}=\begin{cases}0 & \text{for } x\in\left[a,\frac{a+b}{2}-\varepsilon\right],\\ \frac{M}{2\varepsilon} & \text{for } x\in\left[\frac{a+b}{2}-\varepsilon,\frac{a+b}{2}+\varepsilon\right],\\ 0 & \text{for } x\in\left[\frac{a+b}{2}+\varepsilon,b\right],\end{cases}\tag{3.12}$$

and the result follows by testing, in the variational formulation (3.2), the first Steklov eigenfunction

$$u(x)=\begin{cases}x-a & \text{if } x\in\left[a,\frac{a+b}{2}\right],\\ b-x & \text{if } x\in\left[\frac{a+b}{2},b\right].\end{cases}\tag{3.13}$$

Thus, the inequality is sharp for $n=1$.

We now consider the case $n\geq 2$. We divide the interval $[a,b]$ in n subintervals I_i of equal length, and let c_i be the midpoint of the ith subinterval.

By using a symmetry argument, the nth eigenvalue corresponding to the weight

$$r_n(x)=\frac{M}{n}\sum_{i=1}^{n}\delta_{c_i}(x),\tag{3.14}$$

restricted to I_i, is the first eigenvalue in this interval, that is,

$$\lambda_n=\frac{2n\mu_1}{M}=\frac{2^p n^p}{M(b-a)^{p-1}}.\tag{3.15}$$

The proof is now completed. □

Acknowledgments

This work has been supported by Fundacion Antorchas and ANPCyT PICT Grant 03-05009. We would like to thank Prof. R. Duran and Prof. N. Wolanski for interesting conversations.

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