MULTIVALUED SEMILINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH NONCONVEX-VALUED RIGHT-HAND SIDE

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We investigate the existence of mild solutions on a compact interval to some classes of semilinear neutral functional differential inclusions. We will rely on a fixed-point theorem for contraction multivalued maps due to Covitz and Nadler and on Schaefer's fixed-point theorem combined with lower semicontinuous multivalued operators with decomposable values.

1. Introduction

This paper is concerned with the existence of mild solutions defined on a compact real interval for first- and second-order semilinear neutral functional differential inclusions (NFDIs).

In Section 3, we consider the following class of semilinear NFDIs:

$$\begin{gathered}\frac{d}{dt}\left[y(t)-f(t,y_t)\right] \in Ay(t)+F(t,y_t), \quad \text{a.e. } t \in J := [0,b],\\ y(t)=\phi(t), \quad t\in[-r,0],\end{gathered} \tag{1.1}$$

where $F : J \times C([-r,0],E) \to \mathcal{P}(E)$ is a multivalued map, A is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geq 0$, $\phi \in C([-r,0],E)$, $f : J \times C([-r,0],E) \to E$, $\mathcal{P}(E)$ is the family of all subsets of E, and E is a real separable Banach space with norm $|\cdot|$.

Section 4 is devoted to the study of the following second-order semilinear NFDIs:

$$\begin{gathered}\frac{d}{dt}\left[y'(t)-f(t,y_t)\right] \in Ay(t)+F(t,y_t), \quad t \in J,\\ y(t)=\phi(t), \quad t\in[-r,0],\ y'(0)=\eta,\end{gathered} \tag{1.2}$$

where F, ϕ, f, $\mathcal{P}(E)$, and E are as in problem (1.1), A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, and $\eta \in E$.

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For any continuous function y defined on the interval $[-r,b]$ and any $t \in J$, we denote by y_t the element of $C([-r,0],E)$, defined by

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r,0]. \tag{1.3}$$

Here, $y_t(\cdot)$ represents the history of the state from time $t-r$, up to the present time t.

In the last two decades, several authors have paid attention to the problem of existence of mild solutions to initial and boundary value problems for semilinear evolution equations. We refer the interested reader to the monographs by Goldstein [11], Heikkilä and Lakshmikantham [13], and Pazy [19], and to the paper of Heikkilä and Lakshmikantham [14]. In [17, 18], existence theorems of mild solutions for semilinear evolution inclusions are given by Papageorgiou. Recently, by means of a fixed-point argument and the semigroup theory, existence theorems of mild solutions on compact and noncompact intervals for first- and second-order semilinear NFDIs with a convex-valued right-hand side were obtained by Benchohra and Ntouyas in [1, 4]. Similar results for the case $A = 0$ are given by Benchohra and Ntouyas in [2]. Here, we will extend the above results to semilinear NFDIs with a nonconvex-valued right-hand side. The method we are going to use is to reduce the existence of solutions to problems (1.1) and (1.2) to the search for fixed points of a suitable multivalued map on the Banach space $C([-r,b],E)$. For each intial value problem (IVP), we give two results. In the first one, we use a fixed-point theorem for contraction multivalued maps due to Covitz and Nadler [7] (see also Deimling [8]). This method was applied recently by Benchohra and Ntouyas in [3], in the case when $A = 0$ and $f \equiv 0$. In the second one, we use Schaefer's theorem combined with a selection theorem of Bressan and Colombo [5] for lower semicontinuous (l.s.c) multivalued operators with decomposable values.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis, which are used throughout this paper.

We denote by $\mathcal{P}(E)$ the set of all subsets of E normed by $\|\cdot\|_{\mathcal{P}}$. Let $C([-r,0],E)$ be the Banach space of all continuous functions from $[-r,0]$ into E with the norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}. \tag{2.1}$$

By $C([-r,b],E)$ we denote the Banach space of all continuous functions from $[-r,b]$ into E with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in [-r,b]\}. \tag{2.2}$$

A measurable function $y : J \to E$ is Bochner-integrable if and only if $|y|$ is Lebesgue-integrable. (For properties of the Bochner-integral, see, e.g., Yosida [24].) By $L^1(J,E)$ we denote the Banach space of functions $y : J \to E$ which are Bochner-integrable and normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|\,dt, \tag{2.3}$$

and by $B(E)$ the Banach space of bounded linear operators from E to E with norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}. \tag{2.4}$$

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is a strongly continuous cosine family if

(i) $C(0) = I$ (I is the identity operator in E),
(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s,t \in \mathbb{R}$,
(iii) the map $t \mapsto C(t)y$ is strongly continuous for each $y \in E$.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)y\,ds, \quad y \in E,\ t \in \mathbb{R}. \tag{2.5}$$

The infinitesimal generator $A : E \to E$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ay = \frac{d^2}{dt^2}C(t)y|_{t=0}. \tag{2.6}$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books of Fattorini [9], Goldstein [11], and to the papers of Travis and Webb [22, 23]. For properties of semigroup theory, we refer the interested reader to the books of Goldstein [11] and Pazy [19].

Let (X,d) be a metric space. We use the following notations:

$$\begin{aligned} P(X) &= \{Y \in \mathcal{P}(X) : Y \neq \emptyset\}, \\ P_{cl}(X) &= \{Y \in P(X) : Y \text{ closed}\}, \\ P_b(X) &= \{Y \in P(X) : Y \text{ bounded}\}, \\ P_{cp}(X) &= \{Y \in P(X) : Y \text{ compact}\}. \end{aligned} \tag{2.7}$$

Consider $H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\}, \tag{2.8}$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$.

Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized (complete) metric space (see [16]).

A multivalued map $N : J \to P_{cl}(X)$ is said to be measurable if, for each $x \in X$, the function $Y : J \to \mathbb{R}$, defined by

$$Y(t) = d(x, N(t)) = \inf\{d(x,z) : z \in N(t)\}, \tag{2.9}$$

is measurable.

Definition 2.1. A multivalued operator $N : X \to P_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y), \quad \text{for each } x, y \in X; \tag{2.10}$$

(b) contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

The multivalued operator N has *a fixed point* if there is $x \in X$ such that $x \in N(x)$. The fixed-point set of the multivalued operator N will be denoted by $\operatorname{Fix} N$.

We recall the following fixed-point theorem for contraction multivalued operators given by Covitz and Nadler [7] (see also Deimling [8, Theorem 11.1]).

Theorem 2.2. *Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \emptyset$.

Let $\mathcal{A}$ be a subset of $J \times C([-r, 0], E)$. The set $\mathcal{A}$ is $\mathcal{L} \otimes \mathcal{B}$ measurable if $\mathcal{A}$ belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue-measurable in J and $\mathcal{D}$ is Borel-measurable in $C([-r, 0], E)$. A subset B of $L^1(J, E)$ is decomposable if, for all $u, v \in B$ and $\mathcal{J} \subset J$ measurable, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in B$, where $\chi_{\mathcal{J}}$ denotes the characteristic function for $\mathcal{J}$.

Let E be a Banach space, X a nonempty closed subset of E, and $G : X \to \mathcal{P}(E)$ a multivalued operator with nonempty closed values. The operator G is l.s.c. if the set $\{x \in X : G(x) \cap C \neq \emptyset\}$ is open for any open set C in E. For more details on multivalued maps, we refer to the books of Deimling [8], Górniewicz [12], Hu and Papageorgiou [15], and Tolstonogov [21].

Definition 2.3. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1(J, E))$ be a multivalued operator. The operator N has property (BC) if it satisfies the following conditions:

(1) N is l.s.c.;
(2) N has nonempty, closed, and decomposable values.

Let $F : J \times C([-r, 0], E) \to \mathcal{P}(E)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : C([-r, b], E) \longrightarrow \mathcal{P}(L^1(J, E)) \tag{2.11}$$

by letting

$$\mathcal{F}(y) = \{w \in L^1(J, E) : w(t) \in F(t, y_t) \text{ for a.e. } t \in J\}. \tag{2.12}$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F.

Definition 2.4. Let $F : J \times C([-r, 0], E) \to \mathcal{P}(E)$ be a multivalued function with nonempty compact values. We say that F is of l.s.c. type if its associated Niemytzki operator $\mathcal{F}$ is l.s.c. and has nonempty closed and decomposable values.

Next we state a selection theorem due to Bressan and Colombo.

Theorem 2.5 (see [5]). *Let Y be separable metric space and let $N : Y \to \mathcal{P}(L^1(J,E))$ be a multivalued operator which has property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1(J,E)$ such that $g(y) \in N(y)$ for every $y \in Y$.*

3. First-order semilinear NFDIs

Now, we are able to state and prove our first theorem for the IVP (1.1). Before stating and proving this result, we give the definition of a mild solution of the IVP (1.1).

Definition 3.1. A function $y \in C([-r,b],E)$ is called a mild solution of (1.1) if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t,y_t)$, a.e. on J, $y_0 = \phi$, and

$$\begin{aligned} y(t) = T(t)[\phi(0) - f(0,\phi)] + f(t,y_t) + \int_0^t AT(t-s)f(s,y_s)\,ds \\ + \int_0^t T(t-s)v(s)\,ds, \quad t \in J. \end{aligned} \tag{3.1}$$

Theorem 3.2. *Assume that*

(H1) *A is the infinitesimal generator of a semigroup of bounded linear operators $T(t)$ in E such that $\|T(t)\|_{B(E)} \le M_1$, for some $M_1 > 0$ and $\|AT(t)\|_{B(E)} \le M_2$, for each $t > 0$, $M_2 > 0$;*

(H2) *$F : J \times C([-r,0],E) \to P_{cp}(E)$ has the property that $F(\cdot,u) : J \to P_{cp}(E)$ is measurable for each $u \in C([-r,0],E)$;*

(H3) *there exists $l \in L^1(J,\mathbb{R})$ such that*

$$H_d(F(t,u),F(t,\overline{u})) \le l(t)\|u - \overline{u}\|, \tag{3.2}$$

for each $t \in J$ and $u,\overline{u} \in C([-r,0],E)$, and

$$d(0,F(t,0)) \le l(t), \quad \text{for almost each } t \in J; \tag{3.3}$$

(H4) *$|f(t,u) - f(t,\overline{u})| \le c\|u - \overline{u}\|$ for each $t \in J$ and $u,\overline{u} \in C([-r,0],E)$, where c is a nonnegative constant;*

(H5) *$c + M_2cb + M_1\ell^* < 1$, where $\ell^* = \int_0^b l(s)ds$.*

Then the IVP (1.1) has at least one mild solution on $[-r,b]$.

Proof. Transform problem (1.1) into a fixed-point problem. Consider the multivalued operator $N : C([-r,b],E) \to \mathcal{P}(C([-r,b],E))$, defined by

$$N(y) := \{h \in C([-r,b],E)\} \tag{3.4}$$

such that

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ T(t)[\phi(0) - f(0,\phi)] + f(t,y_t) + \displaystyle\int_0^t AT(t-s)f(s,y_s)\,ds & \\ + \displaystyle\int_0^t T(t-s)v(s)\,ds, & \text{if } t \in J, \end{cases} \tag{3.5}$$

where

$$v \in S_{F,y} = \{v \in L^1(J,E) : v(t) \in F(t,y_t) \text{ for a.e. } t \in J\}. \tag{3.6}$$

We remark that the fixed points of N are solutions to (1.1). Also, for each $y \in C([-r,b],E)$, the set $S_{F,y}$ is nonempty since, by (H2), F has a measurable selection (see [6, Theorem III.6]).

We will show that N satisfies the assumptions of Theorem 2.2. The proof will be given in two steps.

Step 1. We prove that $N(y) \in P_{cl}(C([-r,b],E))$ for each $y \in C([-r,b],E)$.

Indeed, let $(y_n)_{n\geq 0} \in N(y)$ such that $y_n \to \tilde{y}$ in $C([-r,b],E)$. Then $\tilde{y} \in C([-r,b],E)$ and there exists $g_n \in S_{F,y}$ such that

$$\begin{aligned} y_n(t) = T(t)[\phi(0) - f(0,\phi)] + f(t,y_t) + \int_0^t AT(t-s)f(s,y_s)\,ds \\ + \int_0^t T(t-s)g_n(s)\,ds, \quad t \in J. \end{aligned} \tag{3.7}$$

Using the fact that F has compact values and from (H3), we may pass to a subsequence if necessary to get that g_n converges to g in $L^1(J,E)$ and hence $g \in S_{F,y}$. Then, for each $t \in J$,

$$\begin{aligned} y_n(t) \longrightarrow \tilde{y}(t) = T(t)[\phi(0) - f(0,\phi)] + f(t,y_t) + \int_0^t AT(t-s)f(s,y_s)\,ds \\ + \int_0^t T(t-s)g(s)\,ds, \quad t \in J. \end{aligned} \tag{3.8}$$

So, $\tilde{y} \in N(y)$.

Step 2. We prove that $H_d(N(y_1),N(y_2)) \leq \gamma\|y_1 - y_2\|_\infty$ for each $y_1,y_2 \in C([-r,b],E)$, where $\gamma < 1$.

Let $y_1,y_2 \in C([-r,b],E)$ and $h_1 \in N(y_1)$. Then there exists $g_1(t) \in F(t,y_{1t})$ such that

$$\begin{aligned} h_1(t) = T(t)[\phi(0) - f(0,\phi)] + f(t,y_{1t}) + \int_0^t AT(t-s)f(s,y_{1s})\,ds \\ + \int_0^t T(t-s)g_1(s)\,ds, \quad t \in J. \end{aligned} \tag{3.9}$$

From (H3), it follows that

$$H_d(F(t,y_{1t}),F(t,y_{2t})) \leq l(t)\|y_{1t} - y_{2t}\|, \quad t \in J. \tag{3.10}$$

Hence, there is $w \in F(t,y_{2t})$ such that

$$|g_1(t) - w| \leq l(t)\|y_{1t} - y_{2t}\|, \quad t \in J. \tag{3.11}$$

Consider $U : J \to \mathcal{P}(E)$ given by

$$U(t) = \{w \in E : |g_1(t) - w| \leq l(t)\|y_{1t} - y_{2t}\|\}. \tag{3.12}$$

Since the multivalued operator $V(t) = U(t) \cap F(t, y_{2t})$ is measurable (see [6, Proposition III.4]), there exists $g_2(t)$, a measurable selection for V. So, $g_2(t) \in F(t, y_{2t})$ and

$$|g_1(t) - g_2(t)| \leq l(t)\|y_{1t} - y_{2t}\|, \quad \text{for each } t \in J. \tag{3.13}$$

We define, for each $t \in J$,

$$\begin{aligned} h_2(t) = T(t)[\phi(0) - f(0,\phi)] + f(t, y_{2t}) + \int_0^t AT(t-s)f(s, y_{2s})ds \\ + \int_0^t T(t-s)g_2(s)ds. \end{aligned} \tag{3.14}$$

Then we have

$$\begin{aligned} &|h_1(t) - h_2(t)| \\ &\quad \leq |f(t, y_{1t}) - f(t, y_{2t})| + M_2\int_0^t |f(t, y_{1s}) - f(t, y_{2s})|ds + M_1\int_0^t |g_1(s) - g_2(s)|ds \\ &\quad \leq c\|y_{1t} - y_{2t}\| + M_2 c\int_0^t \|y_{1s} - y_{2s}\|ds + M_1\int_0^t l(s)\|y_{1s} - y_{2s}\|ds \\ &\quad \leq c\|y_1 - y_2\|_\infty + M_2 cb\|y_1 - y_2\|_\infty + M_1\|y_1 - y_2\|_\infty \int_0^b l(s)ds \\ &\quad \leq [c + M_2 cb + M_1\ell^*]\|y_1 - y_2\|_\infty. \end{aligned} \tag{3.15}$$

Then

$$\|h_1 - h_2\|_\infty \leq [c + M_2 cb + M_1\ell^*]\|y_1 - y_2\|_\infty. \tag{3.16}$$

By the analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$H_d(N(y_1), N(y_2)) \leq [c + M_2 cb + M_1\ell^*]\|y_1 - y_2\|_\infty. \tag{3.17}$$

Since $\gamma := c + M_2 cb + M_1\ell^* < 1$, N is a contraction, and thus, by Theorem 2.2, it has a fixed point y which is a mild solution to (1.1). □

Remark 3.3. Recall that, in the proof of Theorem 3.2, we have assumed that $\gamma < 1$. Since this assumption is hard to verify, we would like point out that using the well-known Bielecki's renorming method, it can be simplified. The technical details are omitted here.

By the help of Schaefer's fixed-point theorem, combined with the selection theorem of Bressan and Colombo for l.s.c. maps with decomposable values, we will present the second existence result for problem (1.1). Before this, we introduce the following hypotheses which are assumed hereafter:

(C1) $F : J \times C([-r,0], E) \to \mathcal{P}(E)$ is a nonempty compact-valued multivalued map such that
 (a) $(t,u) \mapsto F(t,u)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,
 (b) $u \mapsto F(t,u)$ is l.s.c. for a.e. $t \in J$;

(C2) for each $\rho > 0$, there exists a function $h_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t,u)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t,u)\} \le h_\rho(t) \tag{3.18}$$

for a.e. $t \in J$, $u \in C([-r,0],E)$ with $\|u\| \le \rho$.

In the proof of our following theorem, we will need the next auxiliary result.

Lemma 3.4 [10]. *Let $F : J \times C([-r,0],E) \to \mathcal{P}(E)$ be a multivalued map with nonempty compact values. Assume that (C1) and (C2) hold. Then F is of l.s.c. type.*

Theorem 3.5. *Assume that hypotheses (C1), (C2), and the following ones are satisfied.*

(A0) *A is the infinitesimal generator of a compact semigroup $T(t)$, $t > 0$, such that $\|T(t)\|_{B(E)} \le M_1$, $M_1 > 0$, and $\|AT(t)\|_{B(E)} \le M_2$, for each $t \ge 0$, $M_2 > 0$.*

(A1) *There exist constants $0 \le c_1 < 1$ and $c_2 \ge 0$ such that*

$$|f(t,u)| \le c_1\|u\| + c_2, \quad t \in J,\ u \in C([-r,0],E). \tag{3.19}$$

(A2) *The function f is completely continuous and, for any bounded set $\mathcal{A} \subseteq C([-r,b],E)$, the set $\{t \to f(t,y_t) : y \in \mathcal{A}\}$ is equicontinuous in $C(J,E)$.*

(A3) *There exist $p \in L^1(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\psi : \mathbb{R}^+ \to (0,\infty)$ such that*

$$\|F(t,u)\|_{\mathcal{P}} \le p(t)\psi(\|u\|) \tag{3.20}$$

for a.e. $t \in J$ and each $u \in C([-r,0],E)$ with

$$\int_0^b \widehat{M}(s)\,ds < \int_{c_0}^{\infty} \frac{du}{u + \psi(u)}, \tag{3.21}$$

where

$$\begin{gathered} c_0 = \frac{1}{1-c_1}\left[M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_2 + bc_2M_2\right], \\ \widehat{M}(t) = \max\left\{\frac{1}{1-c_1}c_1M_2, \frac{1}{1-c_1}M_1p(t)\right\}. \end{gathered} \tag{3.22}$$

(A4) *For each $t \in J$, the multivalued map $F(t,\cdot) : C([-r,0],E) \mapsto \mathcal{P}(E)$ maps bounded sets into relatively compact sets.*

Then problem (1.1) has at least one solution.

Proof. Hypotheses (C1) and (C2) imply, by Lemma 3.4, that F is of l.s.c. type. Then, from Theorem 2.5, there exists a continuous function $g : C([-r,b],E) \to L^1([0,b],E)$ such that $g(y) \in \mathcal{F}(y)$ for all $y \in C([-r,b],E)$.

Consider the operator $N_1 : C([-r,b],E) \to \mathcal{P}(C([-r,b],E))$ defined by

$$N_1(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ T(t)[\phi(0) - f(0,\phi)] + f(t,y_t) & \\ +\int_0^t AT(t-s)f(s,y_s)\,ds + \int_0^t T(t-s)g(y)(s)\,ds, & \text{if } t \in J. \end{cases} \tag{3.23}$$

We will show that N_1 is completely continuous. The proof will be given in several steps.

Step 1. The operator N_1 sends bounded sets into bounded sets in $C([-r,b],E)$.

Indeed, it is enough to show that for any $q > 0$, there exists a positive constant l such that, for each $y \in B_q := \{y \in C([-r,b],E) : \|y\|_\infty \le q\}$, one has $\|N_1(y)\|_\infty \le l$. Let $y \in B_q$, then

$$\begin{aligned} N_1(y)(t) = {} & T(t)[\phi(0) - f(0,\phi)] + f(t,y_t) + \int_0^t AT(t-s)f(s,y_s)\,ds \\ & + \int_0^t T(t-s)g(y)(s)\,ds, \quad t \in J. \end{aligned} \tag{3.24}$$

From (A0), (C2), (A1), and (A3), we have, for each $t \in J$,

$$\begin{aligned} &|N_1(y)(t)| \\ &\quad \le M_1[\|\phi\| + |f(0,\phi)|] + |f(t,y_t)| \\ &\qquad + \int_0^t ||AT(t-s)||_{B(E)}|f(s,y_s)|\,ds + \int_0^t ||T(t-s)||_{B(E)}|g(y)(s)|\,ds \\ &\quad \le M_1[\|\phi\| + c_1q + c_2] + c_1q + c_2 + bM_2c_1q + bM_2c_2 + M_1||h_q||_{L^1}. \end{aligned} \tag{3.25}$$

Then, for each $h \in N(B_q)$, we have

$$||N_1(y)||_\infty \le M_1[\|\phi\| + c_1q + c_2] + c_1q + c_2 + bM_2c_1q + bM_2c_2 + M_1||h_q||_{L^1} := l. \tag{3.26}$$

Step 2. The operator N_1 sends bounded sets in $C([-r,b],E)$ into equicontinuous sets.

Using (A2), it suffices to show that the operator $N_2 : C([-r,b],E) \to C([-r,b],E)$, defined by

$$N_2(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0]; \\ T(t)\phi(0) + \int_0^t AT(t-s)f(s,y_s)\,ds & \\ +\int_0^t T(t-s)g(y)(s)\,ds, & \text{if } t \in J, \end{cases} \tag{3.27}$$

maps bounded sets into equicontinuous sets of $C([-r,b],E)$. Let $u_1,u_2 \in J$, $u_1 < u_2$, let $B_q := \{y \in C([-r,b],E) : \|y\|_\infty \le q\}$ be a bounded set in $C([-r,b],E)$, and $y \in B_q$. Then we have

$$
\begin{aligned}
&|N_2(y)(u_2)-N_2(y)(u_1)| \\
&\quad\le |T(u_1)\phi(0)-T(u_2)\phi(0)| + (c_1q+c_2)\int_0^{u_1}||AT(u_2-s)-AT(u_1-s)||_{B(E)}ds \\
&\qquad + (c_1q+c_2)\int_{u_1}^{u_2}||AT(u_2-s)||_{B(E)}ds + \int_0^{u_1}||T(u_2-s)-T(u_1-s)||_{B(E)}h_q(s)ds \\
&\qquad + \int_{u_1}^{u_2}||T(u_2-s)||_{B(E)}h_q(s)ds.
\end{aligned}
\tag{3.28}
$$

As $u_2 \to u_1$, the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $u_1 < u_2 \le 0$ and $u_1 \le 0 \le u_2$ is obvious.

Step 3. The operator N_2 is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([-r,b],E)$. Then

$$
\begin{aligned}
&|N_2(y_n)(t)-N_2(y)(t)| \\
&\quad\le M_2\int_0^t |f(s,y_{n,s})-f(s,y_s)|ds + M_1\int_0^b |g(y_n)(s)-g(y)(s)|ds.
\end{aligned}
\tag{3.29}
$$

Since the function g is continuous and f is completely continuous, then

$$
\begin{aligned}
&||N_2(y_n)-N_2(y)||_\infty \\
&\quad\le M_2\sup_{t\in J}\int_0^t |f(s,y_{ns})-f(s,y_s)|ds + M_1||g(y_n)-g(y)||_{L^1} \longrightarrow 0.
\end{aligned}
\tag{3.30}
$$

As a consequence of Steps 1, 2, and 3 and (A2), (A4), together with the Arzelá-Ascoli theorem, we can conclude that $N_2 : C([-r,b],E) \to C([-r,b],E)$ is completely continuous.

Step 4. The set $\mathscr{E}(N_1) = \{y \in C([-r,b],E) : y = \lambda N_1(y) \text{ for some } \lambda \in (0,1)\}$ is bounded.

Let $y \in \mathscr{E}(N_1)$. Then $y = \lambda N_1(y)$ for some $0<\lambda<1$, and for $t \in [0,b]$, we have

$$
\begin{aligned}
y(t) = \lambda\bigg[&T(t)(\phi(0)-f(0,\phi)) + f(t,y_t) \\
&+\int_0^t AT(t-s)f(s,y_s)ds + \int_0^t T(t-s)g(y)(s)ds\bigg].
\end{aligned}
\tag{3.31}
$$

This implies, by (A0), (A1), and (A3), that for each $t \in J$, we have

$$
\begin{aligned}
|y(t)| \le\ & M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_1||y_t|| + c_2 + bc_2M_2 + c_1M_2\int_0^t ||y_s||ds \\
&+ M_1\int_0^t p(s)\psi(||y_s||)ds.
\end{aligned}
\tag{3.32}
$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq b. \tag{3.33}$$

Let $t^* \in [-r,b]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by inequality (3.32), we have, for $t \in J$,

$$\begin{aligned}\mu(t) \leq{}& M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_1\mu(t) + c_2 + bc_2M_2 + c_1M_2\int_0^t \mu(s)ds \\ &+ M_1\int_0^t p(s)\psi(\mu(s))ds.\end{aligned} \tag{3.34}$$

Thus

$$\begin{aligned}\mu(t) \leq \frac{1}{1-c_1}\Big[&M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_2 + bc_2M_2 + c_1M_2\int_0^t \mu(s)ds \\ &+ M_1\int_0^t p(s)\psi(\mu(s))ds\Big], \quad t \in J.\end{aligned} \tag{3.35}$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\phi\|$ and inequality (3.35) holds. We take the right-hand side of inequality (3.35) as $v(t)$; then we have

$$\begin{aligned}v(0) &= \frac{1}{1-c_1}[M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_2 + bc_2M_2], \quad \mu(t) \leq v(t),\ t \in J, \\ v'(t) &= \frac{1}{1-c_1}\{c_1M_2\mu(t) + M_1p(t)\psi(\mu(t))\}, \quad t \in J.\end{aligned} \tag{3.36}$$

Since ψ is nondecreasing, we have

$$v'(t) \leq \widehat{M}(t)\{v(t) + \psi(v(t))\}, \quad t \in J. \tag{3.37}$$

From this inequality, it follows that

$$\int_0^t \frac{v'(s)}{v(s) + \psi(v(s))}ds \leq \int_0^t \widehat{M}(s)ds. \tag{3.38}$$

We then have

$$\int_{v(0)}^{v(t)} \frac{du}{u + \psi(u)} \leq \int_0^t \widehat{M}(s)ds \leq \int_0^b \widehat{M}(s)ds < \int_{v(0)}^{\infty} \frac{du}{u + \psi(u)}. \tag{3.39}$$

This inequality implies that there exists a constant K_1 such that $v(t) \leq K_1$, $t \in J$, and hence $\mu(t) \leq K_1$, $t \in J$. Since for every $t \in J$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_\infty \leq K_1' := \max\{\|\phi\|, K_1\}, \tag{3.40}$$

where K_1' depends only on b, M_1, and M_2 and on the functions p and ψ. This shows that $\mathscr{E}(N_1)$ is bounded. As a consequence of Schaefer's theorem (see [20]), we deduce that N_1 has a fixed point y which is a solution to problem (1.1). □

4. Second-order semilinear NFDIs

Definition 4.1. A function $y \in C([-r,b],E)$ is called a mild solution of (1.2) if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t,y_t)$ a.e. on J, $y_0 = \phi$, $y'(0) = \eta$, and

$$y(t) = C(t)\phi(0) + S(t)(\eta - f(0,\phi)) + \int_0^t C(t-s)f(s,y_s)\,ds + \int_0^t S(t-s)v(s)\,ds. \tag{4.1}$$

Theorem 4.2. *Assume that hypotheses (H2), (H3), and (H4) and the following ones are satisfied:*

(H6) *A is an infinitesimal generator of a given strongly continuous bounded and compact cosine family $\{C(t) : t > 0\}$ with $\|C(t)\|_{B(E)} \le \overline{M}$;*
(H7) $\overline{M}b(c+\ell^*) < 1$.

Then the IVP (1.2) has at least one mild solution on $[-r,b]$.

Proof. Transform problem (1.2) into a fixed-point problem. Consider the multivalued operator $N_3 : C([-r,b],E) \to \mathcal{P}(C([-r,b],E))$ defined by

$$N_3(y) := \{h \in C([-r,b],E)\} \tag{4.2}$$

such that

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ C(t)\phi(0) + S(t)(\eta - f(0,\phi)) & \\ +\displaystyle\int_0^t C(t-s)f(s,y_s)\,ds & \\ +\displaystyle\int_0^t S(t-s)g(s)\,ds, & \text{if } t \in J, \end{cases} \tag{4.3}$$

where $g \in S_{F,y}$.

We will show that N_3 satisfies the assumptions of Theorem 2.2. The proof will be given in two steps.

Step 1. We prove that $N_3(y) \in P_{cl}(C([-r,b],E))$ for each $y \in C([-r,b],E)$.

Indeed, let $(y_n)_{n\ge 0} \in N_3(y)$ such that $y_n \to \tilde{y}$ in $C([-r,b],E)$. Then $\tilde{y} \in C([-r,b],E)$ and there exists $g_n \in S_{F,y}$ such that

$$\begin{aligned} y_n(t) &= C(t)\phi(0) + S(t)(\eta - f(0,\phi)) \\ &\quad + \int_0^t C(t-s)f(s,y_s)\,ds + \int_0^t S(t-s)g_n(s)\,ds. \end{aligned} \tag{4.4}$$

Using the fact that F has compact values and from (H3), we may pass to a subsequence, if necessary, to get that g_n converges to g in $L^1(J,E)$ and hence $g \in S_{F,y}$. Then, for each $t \in J$,

$$y_n(t) \longrightarrow \tilde{y}(t) = C(t)\phi(0) + S(t)(\eta - f(0,\phi)) + \int_0^t C(t-s)f(s,y_s)\,ds + \int_0^t S(t-s)g(s)\,ds. \tag{4.5}$$

So, $\tilde{y} \in N_3(y)$.

Step 2. We prove that $H_d(N_3(y_1),N_3(y_2)) \leq \gamma \|y_1 - y_2\|_\infty$ for each $y_1, y_2 \in C([-r,b],E)$, where $\gamma < 1$.

Let $y_1, y_2 \in C([-r,b],E)$ and $h_1 \in N_1(y_1)$. Then there exists $g_1(t) \in F(t,y_{1t})$ such that

$$h_1(t) = C(t)\phi(0) + S(t)(\eta - f(0,\phi)) + \int_0^t C(t-s)f(s,y_{1s})\,ds + \int_0^t S(t-s)g_1(s)\,ds, \quad t \in J. \tag{4.6}$$

From (H3), it follows that $H_d(F(t,y_{1t}),F(t,y_{2t})) \leq l(t)\|y_{1t} - y_{2t}\|$, $t \in J$. Hence, there is $w \in F(t,y_{2t})$ such that $|g_1(t) - w| \leq l(t)\|y_{1t} - y_{2t}\|$, $t \in J$. Consider $U : J \to \mathcal{P}(E)$ given by $U(t) = \{w \in E : |g_1(t) - w| \leq l(t)\|y_{1t} - y_{2t}\|\}$. Since the multivalued operator $V(t) = U(t) \cap F(t,y_{2t})$ is measurable (see [6, Proposition III.4]), there exists $g_2(t)$, a measurable selection for V. So, $g_2(t) \in F(t,y_{2t})$ and $|g_1(t) - g_2(t)| \leq l(t)\|y_{1t} - y_{2t}\|$, for each $t \in J$. We define, for each $t \in J$,

$$h_2(t) = C(t)\phi(0) + S(t)(\eta - f(0,\phi)) + \int_0^t C(t-s)f(s,y_{2s})\,ds + \int_0^t S(t-s)g_2(s)\,ds. \tag{4.7}$$

Then we have

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \overline{M}\int_0^t |f(s,y_{1s}) - f(s,y_{2s})|\,ds + \overline{M}b\int_0^t |g_1(s) - g_2(s)|\,ds \\ &\leq \overline{M}c\int_0^t ||y_{1s} - y_{2s}||\,ds + \overline{M}b\int_0^t l(s)||y_{1s} - y_{2s}||\,ds \\ &\leq \overline{M}cb||y_1 - y_2||_\infty + \overline{M}b||y_1 - y_2||_\infty \int_0^t l(s)\,ds \\ &\leq \overline{M}b(c + \ell^*)||y_1 - y_2||_\infty. \end{aligned} \tag{4.8}$$

Then

$$||h_1 - h_2||_\infty \leq \overline{M}b(c + \ell^*)||y_1 - y_2||_\infty. \tag{4.9}$$

By the analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$H_d(N_3(y_1),N_3(y_2)) \leq \overline{M}b(c + \ell^*)||y_1 - y_2||_\infty. \tag{4.10}$$

Since $\overline{M}b(c + \ell^*) < 1$, N_1 is a contraction, and thus, by Theorem 2.2, it has a fixed point y which is a mild solution to (1.2). □

THEOREM 4.3. *Assume that hypotheses (H6), (C1), (C2), (A1) (with $c_1, c_2 \geq 0$), (A2), and (A4) and the following one are satisfied:*

(A5) *there exist $p \in L^1(J, \mathbb{R}^+)$ and a continuous and nondecreasing function $\psi : \mathbb{R}^+ \to (0, \infty)$ such that, for a.e. $t \in J$ and each $u \in C([-r, 0], E)$,*

$$\|F(t,u)\|_{\mathcal{P}} \leq p(t)\psi(\|u\|), \qquad \int_0^b M(s)\,ds < \int_{\bar{c}}^{\infty} \frac{d\tau}{\tau + \psi(\tau)}, \tag{4.11}$$

where

$$\begin{gathered} \bar{c} = \overline{M}\|\phi\| + b\overline{M}[|\eta| + c_1\|\phi\| + 2c_2], \\ M(t) = \max\{1, c_1\overline{M}, b\overline{M}p(t)\}. \end{gathered} \tag{4.12}$$

Then the IVP (1.2) has at least one solution on $[-r, b]$.

Proof. Hypotheses (C1) and (C2) imply, by Lemma 3.4, that F is of l.s.c. type. Then, from Theorem 2.5, there exists a continuous function $g : C([-r,b],E) \to L^1([0,b],E)$ such that $g(y) \in \mathcal{F}(y)$ for all $y \in C([-r,b],E)$. Consider the operator $N_4 : C([-r,b],E) \to C([-r,b],E)$ defined by

$$N_4(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ C(t)\phi(0) + S(t)[\eta - f(0,\phi)] & \\ +\displaystyle\int_0^t C(t-s)f(s,y_s)\,ds + \int_0^t S(t-s)g(y)(s)\,ds, & \text{if } t \in J. \end{cases} \tag{4.13}$$

As in Theorem 3.5, we can show that N_4 is completely continuous.

Now, we only prove that the set

$$\mathcal{E}(N_4) := \{y \in C([-r,b],E) : y = \lambda N_4(y) \text{ for some } 0 < \lambda < 1\} \tag{4.14}$$

is bounded.

Let $y \in \mathcal{E}(N_4)$. Then $y = \lambda N_4(y)$ for some $0 < \lambda < 1$. Thus

$$\begin{aligned} y(t) = {} & \lambda C(t)\phi(0) + \lambda S(t)[\eta - f(0,\phi)] + \lambda\int_0^t C(t-s)f(s,y_s)\,ds \\ & + \lambda\int_0^t S(t-s)g(y)(s)\,ds, \quad t \in J. \end{aligned} \tag{4.15}$$

This implies by (H4), (H6), (A1), and (A5) that, for each $t \in J$,

$$\begin{aligned} |y(t)| \leq {} & \overline{M}\|\phi\| + b\overline{M}(|\eta| + c_1\|\phi\| + c_2) + c_1\overline{M}\int_0^t \|y_s\|\,ds + bc_2\overline{M} \\ & + b\overline{M}\int_0^t p(s)\psi(\|y_s\|)\,ds. \end{aligned} \tag{4.16}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq b. \tag{4.17}$$

Let $t^* \in [-r,b]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by (4.16) we have, for $t \in J$,

$$\begin{aligned}\mu(t) \leq \overline{M}\|\phi\| + b\overline{M}\big(|\eta| + c_1\|\phi\| + 2c_2\big) + \int_0^t M(s)\mu(s)ds \\ + \int_0^t M(s)\psi\big(\mu(s)\big)ds.\end{aligned} \tag{4.18}$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds.

We take the right-hand side of the above inequality as $v(t)$; then we have

$$\begin{gathered}v(0) = \overline{M}\|\phi\| + b\overline{M}\big(|\eta| + c_1\|\phi\| + 2c_2\big), \quad \mu(t) \leq v(t),\ t \in J, \\ v'(t) = M(t)\mu(t) + M(t)\psi\big(\mu(t)\big), \quad t \in J.\end{gathered} \tag{4.19}$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq M(t)\big[v(t) + \psi\big(v(t)\big)\big], \quad t \in J. \tag{4.20}$$

This implies, for each $t \in J$, that

$$\int_{v(0)}^{v(t)} \frac{d\tau}{\tau + \psi(\tau)} \leq \int_0^b M(s)ds < \int_{v(0)}^{\infty} \frac{d\tau}{\tau + \psi(\tau)}. \tag{4.21}$$

This inequality implies that there exists a constant K_2 such that $v(t) \leq K_2$, $t \in J$, and hence $\mu(t) \leq K_2$, $t \in J$. Since for every $t \in J$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_\infty \leq K_2' := \max\{\|\phi\|, K_2\}, \tag{4.22}$$

where K_2' depends only on b, $\overline{M}$, and on the functions p and ψ. This shows that $\mathscr{E}(N_4)$ is bounded.

Set $X := C([-r,b],E)$. As a consequence of Schaefer's theorem (see [20]), we deduce that N_4 has a fixed point y which is a solution to problem (1.2). □

Remark 4.4. The reasoning used above can be applied to obtain the existence results for the following first- and second-order semilinear neutral functional integrodifferential inclusions of Volterra type:

$$\begin{gathered}\frac{d}{dt}\big[y(t) - f(t,y_t)\big] - Ay \in \int_0^t k(t,s)F(s,y_s)ds, \quad \text{a.e. } t \in J, \\ y(t) = \phi(t), \quad t \in [-r,0], \\ \frac{d}{dt}\big[y'(t) - f(t,y_t)\big] - Ay \in \int_0^t k(t,s)F(s,y_s)ds, \quad \text{a.e. } t \in J, \\ y(t) = \phi(t), \quad t \in [-r,0], \\ y'(0) = \eta,\end{gathered} \tag{4.23}$$

where A, F, f, ϕ, and η are as in problems (1.1) and (1.2) and $k : D \to \mathbb{R}$, $D = \{(t,s) \in J \times J : t \geq s\}$.

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