# ON THE DISCRETENESS OF THE SPECTRA OF THE DIRICHLET AND NEUMANN $p$-BIHARMONIC PROBLEMS

JIŘÍ BENEDIKT

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We are interested in a nonlinear boundary value problem for $(|u''|^{p-2}u'')'' = \lambda|u|^{p-2}u$ in $[0,1]$, $p > 1$, with Dirichlet and Neumann boundary conditions. We prove that eigenvalues of the Dirichlet problem are positive, simple, and isolated, and form an increasing unbounded sequence. An eigenfunction, corresponding to the $n$th eigenvalue, has precisely $n-1$ zero points in $(0,1)$. Eigenvalues of the Neumann problem are nonnegative and isolated, 0 is an eigenvalue which is not simple, and the positive eigenvalues are simple and they form an increasing unbounded sequence. An eigenfunction, corresponding to the $n$th positive eigenvalue, has precisely $n+1$ zero points in $(0,1)$.

## 1. Main results

We are concerned with structure of eigenvalues and eigenfunctions of the nonlinear Dirichlet boundary value problem for the $p$-biharmonic operator

$$\begin{gathered}\left(|u''(t)|^{p-2}u''(t)\right)'' = \lambda|u(t)|^{p-2}u(t), \quad t \in [0,1], \\ u(0) = u'(0) = u(1) = u'(1) = 0,\end{gathered} \tag{1.1}$$

and the Neumann boundary value problem

$$\begin{gathered}\left(|u''(t)|^{p-2}u''(t)\right)'' = \lambda|u(t)|^{p-2}u(t), \quad t \in [0,1], \\ u''(0) = \left(|u''(t)|^{p-2}u''(t)\right)'\Big|_{t=0} = u''(1) = \left(|u''(t)|^{p-2}u''(t)\right)'\Big|_{t=1} = 0,\end{gathered} \tag{1.2}$$

where $\lambda \in \mathbb{R}$ and $p > 1$.

Drábek and Ôtani proved in [4, Theorem 1.3] that the Navier boundary value problem ($u(0) = u''(0) = u(1) = u''(1) = 0$) for the $p$-biharmonic operator possesses infinitely many eigenvalues, all simple, forming a sequence $0 < \lambda_1(p) < \lambda_2(p) < \cdots \to +\infty$. An eigenfunction, corresponding to $\lambda_n(p)$, has precisely $n-1$ zero points in $(0,1)$. We prove a similar result for the Dirichlet and the Neumann problem. Note that the method used


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in [4] is based on transferring the Navier problem to a Dirichlet problem for a system of two second-order equations (for $u$ and $|u''|^{p-2}u''$). Hence this method cannot be adopted for the problem (1.1) or (1.2).

The Dirichlet problem (1.1) (with nonconstant coefficients) was studied by Kratochvíl and Nečas in [7]. They proved that eigenvalues of this problem form a sequence $0 < \lambda_1(p) < \lambda_2(p) < \cdots \to +\infty$, and the set of the corresponding eigenfunctions is discrete. Moreover, it is shown in [7] that for every eigenvalue, there are only finitely many linearly independent corresponding eigenfunctions. This result was proved in [7] only for $p \ge 2$, not for $p \in (1,2)$.

Boundary value problems with $p$-biharmonic operator and general Robin-type boundary conditions were studied in [1, Corollary 4]. It is proved there that (1.1) has only positive simple eigenvalues (see [1, Example 8]). Problem (1.2) has only nonnegative eigenvalues, the positive ones are simple, and, clearly, (1.2) has also the eigenvalue $\lambda = 0$ which is not simple since any linear function $u$ is a solution of (1.2) with $\lambda := 0$ (see [1, Example 9]).

Our main results follow (see Section 2 for related definitions).

Theorem 1.1 (Dirichlet problem). *The set of all eigenvalues of (1.1) forms a sequence $0 < \lambda_1^D(p) < \lambda_2^D(p) < \cdots \to +\infty$. Every $\lambda_n^D(p)$, $n \in \mathbb{N}$, is a simple eigenvalue and any corresponding eigenfunction has precisely $n-1$ zero points in $(0,1)$. The set of all eigenfunctions is discrete in the sense that in some $C^2[0,1]$-neighborhood of every eigenfunction, the only other eigenfunctions are its multiples.*

Theorem 1.2 (Neumann problem). *The set of all eigenvalues of (1.2) forms a sequence $0 = \lambda_0^N(p) < \lambda_1^N(p) < \cdots \to +\infty$. Every $\lambda_n^N(p)$, $n > 0$, is a simple eigenvalue while $\lambda_0^N(p) = 0$ is not. An eigenfunction, corresponding to $\lambda_n^N(p)$, $n > 0$, has precisely $n+1$ zero points in $(0,1)$. The set of all eigenfunctions, corresponding to the positive eigenvalues, is discrete in the above sense. Moreover, there is a relation between the positive eigenvalues of (1.2) and (1.1):*

$$\lambda_n^N(p) = \left(\lambda_n^D\left(\frac{p}{p-1}\right)\right)^{p-1}, \quad n \in \mathbb{N}. \tag{1.3}$$

*For $n > 0$, any eigenfunction $u$ of (1.2), corresponding to $\lambda_n^N(p)$, and any eigenfunction $v$ of (1.1) for $p$ replaced by $p/(p-1)$, corresponding to $\lambda_n^D(p/(p-1))$, there exists a $\kappa \in \mathbb{R} \setminus \{0\}$ such that*

$$u = \kappa \left| v'' \right|^{(2-p)/(p-1)} v''. \tag{1.4}$$

Taking $p = 2$ in (1.1), we obtain the one-dimensional linear clamped plate equation. It is known (see [3, 6]) that the first eigenvalue of the clamped plate equation on a ball in $\mathbb{R}^N$ is simple, and the corresponding eigenfunction has a fixed sign. On the other hand, there are numerous counterexamples showing that on some domains in $\mathbb{R}^N$, the first eigenvalue of the clamped plate equation can be negative and the corresponding eigenfunction can change its sign. Theorem 1.1 states that on $[0,1]$ (a ball in $\mathbb{R}$), the first eigenvalue of (1.1) is positive and the corresponding eigenfunction is of fixed sign even for $p > 1$ arbitrary.

Nevertheless, the proof for $p = 2$ relies on the positivity of Green's function, and so it is useless for the nonlinear $p$-biharmonic operator.

The organization of this paper is as follows. In Section 2, we define the solution, the spectrum, the eigenfunctions, and the simplicity of the eigenvalues of (1.1) and (1.2). In Section 3, we prove Theorem 1.1 and in Section 4, we give a proof of Theorem 1.2. In Section 5, we introduce some open problems.

## 2. Preliminaries

We define the solution of (1.1) and (1.2) in accordance with [1]. We adopt the notation $\psi_p(s) = |s|^{p-2}s$, $s \in \mathbb{R} \setminus \{0\}$, $\psi_p(0) = 0$, $p > 1$. We denote $p' = p/(p-1)$ ($\psi_p$ and $\psi_{p'}$ are then inverse functions).

We put $u_1 := u$ and $u_3 := \psi_p(u'')$. Then (1.1) is equivalent to the boundary value problem for a system of four first-order equations

$$\begin{aligned} u_1'(t) &= u_2(t), \\ u_2'(t) &= \psi_{p'}(u_3(t)), \\ u_3'(t) &= u_4(t), \\ u_4'(t) &= \lambda\psi_p(u_1(t)), \quad t \in [0,1], \\ u_1(0) &= u_2(0) = u_1(1) = u_2(1) = 0. \end{aligned} \tag{2.1}$$

Similarly, the Neumann problem (1.2) is equivalent to

$$\begin{aligned} u_1'(t) &= u_2(t), \\ u_2'(t) &= \psi_{p'}(u_3(t)), \\ u_3'(t) &= u_4(t), \\ u_4'(t) &= \lambda\psi_p(u_1(t)), \quad t \in [0,1], \\ u_3(0) &= u_4(0) = u_3(1) = u_4(1) = 0. \end{aligned} \tag{2.2}$$

*Definition 2.1.* A vector function $\mathbf{u} = [u_1, u_2, u_3, u_4]^{\mathrm{T}} \in (C^1[0,1])^4$ is called a *solution* of (2.1) or (2.2) if it satisfies the equations in (2.1) or (2.2), respectively, for all $t \in [0,1]$, and fulfills the boundary conditions.

By a *solution* of (1.1) or (1.2), we understand a function $u \in C^2[0,1]$ such that $[u, u', \psi_p(u''), (\psi_p(u''))']^{\mathrm{T}}$ is a solution of the corresponding problem (2.1) or (2.2), respectively.

*Definition 2.2.* By an *eigenvalue* of (1.1) or (1.2), we mean $\lambda \in \mathbb{R}$ for which (1.1) or (1.2), respectively, has a nontrivial solution, called an *eigenfunction*, corresponding to the eigenvalue $\lambda$.

We say that an eigenvalue $\lambda$ is *simple* if all corresponding eigenfunctions are multiples of one of them.

The spectrum (i.e., the set of all eigenvalues) of (1.1) is sketched in Figure 2.1.

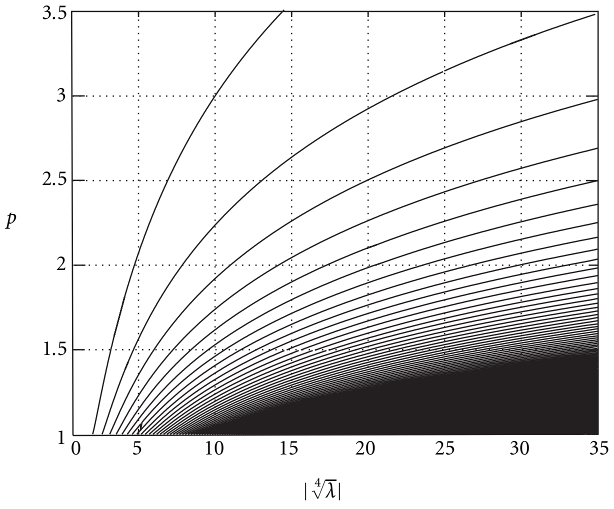


Figure 2.1. Spectrum of (1.1).

In order to prove Theorems 1.1 and 1.2, we use the fact that the corresponding initial value problem

$$\begin{aligned} u_1'(t) &= u_2(t), & u_1(t_0) &= \alpha, \\ u_2'(t) &= \psi_{p'}(u_3(t)), & u_2(t_0) &= \beta, \\ u_3'(t) &= u_4(t), & u_3(t_0) &= \gamma, \\ u_4'(t) &= \lambda\psi_p(u_1(t)), & u_4(t_0) &= \delta, \end{aligned} \tag{2.3}$$

$t \in [t_0, t_1]$, has a unique solution (see [2, Corollaries 1.4 and 1.8]). The solution of (2.3) is defined in [2] in accordance with Definition 2.1 as a vector function $\mathbf{u} = [u_1, u_2, u_3, u_4]^{\mathrm{T}} \in (C^1[t_0, t_1])^4$ satisfying the equations in (2.3) at every $t \in [t_0, t_1]$ and the initial conditions.

In the sequel, we often use the following lemma concerning the integration of a differential inequality. Notice that by $\mathbf{u} \le \mathbf{v}$ and $\mathbf{u} < \mathbf{v}$, we mean $u_i \le v_i$ and $u_i < v_i$, respectively, for all $i \in \{1,2,3,4\}$. By $\mathbf{u} \ne \mathbf{v}$, we mean $u_i \ne v_i$ for at least one $i \in \{1,2,3,4\}$.

Lemma 2.3. *Let $\mathbf{u}$ and $\mathbf{v}$ be solutions of (2.3), where $\lambda > 0$. If $\mathbf{u}(t_0) \le \mathbf{v}(t_0)$ and $\mathbf{u}(t_0) \ne \mathbf{v}(t_0)$, then*

$$\mathbf{u}(t) < \mathbf{v}(t) \quad \forall t \in (t_0, t_1]. \tag{2.4}$$

*Proof.* See, for example, [5, Chapter III, Section 4] and compare to [1, Lemma 20]. □

The next lemma is important for investigation of the number of zero points of the eigenfunctions.

Lemma 2.4. *Let $t_0 < t_m < t_1 < t_r$. Let $\mathbf{u}$ be a solution of (2.3) on the interval $[t_0, t_1]$ and let $\mathbf{v}$ be a solution of (2.3) on $[t_0, t_r]$. Let $\lambda > 0$.*

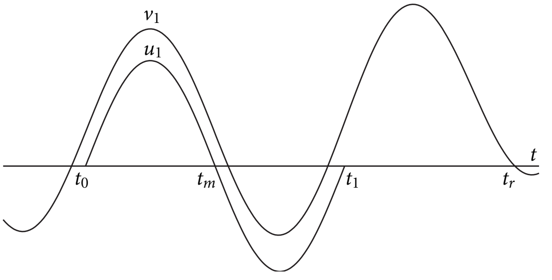


Figure 2.2. Lemma 2.4.

*Assume* $u_1(t_0) = u_1(t_m) = u_1(t_1) = 0$, $u_1(t) > 0$ *for* $t \in (t_0,t_m)$, *and* $u_1(t) < 0$ *for* $t \in (t_m,t_1)$. *Let further* $v_1(t_r) = 0$ *and let* $v_1$ *have finitely many zero points in* $(t_m,t_1)$, *all being simple. Moreover, let* $\mathbf{u}(t) < \mathbf{v}(t)$ *for all* $t \in (t_0,t_1]$. *(See Figure 2.2.)*

*Then* $v_1$ *has exactly two zero points in* $(t_0,t_1]$.

*Proof.* Since $v_1(t) > u_1(t) \geq 0$ for all $t \in (t_0,t_m] \cup \{t_1\}$, all zero points of $u_1$ from $(t_0,t_1]$ are in $(t_m,t_1)$. The number of them is finite and even, because they are simple.

By contradiction, we eliminate the possibility that $v_1$ has no zero point in $(t_m,t_1)$, that is, $v_1 > 0$ on $[t_0,t_1]$. We prove that it would mean $\mathbf{v}(t_1) > \mathbf{0}$. Obviously, $v_1(t_1) > u_1(t_1) = 0$ and $v_2(t_1) > u_2(t_1) \geq 0$. The maximum principle for linear second-order equations yields $u_1''(\tilde{t}_1) < 0$ for some $\tilde{t}_1 \in (t_0,t_m)$, and $u_1''(\tilde{t}_2) > 0$ for some $\tilde{t}_2 \in (t_m,t_1)$. Thus $u_3(\tilde{t}_1) = \psi_p(u_1''(\tilde{t}_1)) < 0$ and $u_3(\tilde{t}_2) > 0$. The mean value theorem implies the existence of a point $\tilde{t}_m \in (\tilde{t}_1,\tilde{t}_2)$, such that $u_4(\tilde{t}_m) = u_3'(\tilde{t}_m) > 0$, and so $v_4(\tilde{t}_m) > u_4(\tilde{t}_m) > 0$. Since we suppose that $v_1 > 0$ on $[t_0,t_1]$ and $\lambda > 0$, we have

$$v_4(t) = v_4(\tilde{t}_m) + \lambda \int_{\tilde{t}_m}^{t} \psi_p(v_1(\tau))\,d\tau > 0 \tag{2.5}$$

for any $t \in [\tilde{t}_m,t_1]$. It remains to show that $v_3(t_1) > 0$. Since $\tilde{t}_2 > \tilde{t}_m$, (2.5) yields $v_4 > 0$ on $[\tilde{t}_2,t_1]$. Moreover, $v_3(\tilde{t}_2) > u_3(\tilde{t}_2) > 0$, and so

$$v_3(t_1) = v_3(\tilde{t}_2) + \int_{\tilde{t}_2}^{t_1} v_4(t)\,dt > 0. \tag{2.6}$$

Now that we proved $\mathbf{v}(t_1) > \mathbf{0}$, we apply Lemma 2.3 (we take the zero solution as $\mathbf{u}$) to conclude that $\mathbf{v}(t_r) > \mathbf{0}$, and so $v_1(t_r) > 0$, a contradiction.

It remains to show that $v_1$ cannot have more than two zero points in $(t_m,t_1)$. We suppose that it has at least three. Since the number of them is finite, we can denote the first three by $\bar{t}_1 < \bar{t}_2 < \bar{t}_3$. Consequently, $v_1 < 0$ on $(\bar{t}_1,\bar{t}_2)$ and $v_1 > 0$ on $(\bar{t}_2,\bar{t}_3)$. At the same time, $u_1 < 0$ on $[\bar{t}_1,\bar{t}_3]$. Hence $\hat{\mathbf{u}} := -\mathbf{v}$, $\hat{\mathbf{v}} := -\mathbf{u}$, $\hat{t}_0 := \bar{t}_1$, $\hat{t}_m := \bar{t}_2$, $\hat{t}_1 := \bar{t}_3$, and $\hat{t}_r := t_1$ satisfy the assumptions of this lemma. We already proved that $\hat{v}_1$ has at least two zero points in $(\hat{t}_m,\hat{t}_1)$, that is, $u_1$ has at least two zero points in $[\bar{t}_1,\bar{t}_3]$, a contradiction. This completes the proof. □

## 3. Dirichlet problem

We already know (see [1, Example 8]) that all eigenvalues of (1.1) are simple and positive. First, we prove some basic properties of the eigenfunctions of (1.1).

Lemma 3.1. *Let $\mathbf{u}$ be a solution of (2.1) with $\lambda > 0$. If $u_1(t_0) = u_2(t_0) = u_3(t_0) = 0$ for some $t_0 \in [0,1]$, then $\mathbf{u}(t) = \mathbf{0}$ for all $t \in [0,1]$.*

*Proof.* We prove by contradiction that $u_4(t_0) = 0$. If $u_4(t_0) \neq 0$, we can assume $u_4(t_0) > 0$. Let $\mathbf{v}$ be the zero vector function on $[0,1]$. Thus $\mathbf{u}(t_0) \geq \mathbf{v}(t_0)$ and $\mathbf{u}(t_0) \neq \mathbf{v}(t_0)$. If $t_0 < 1$, then by Lemma 2.3, $u_1(1) > 0$, a contradiction. If $t_0 = 1$, then we consider $\tilde{\mathbf{u}} := [u_1(1-t), -u_2(1-t), u_3(1-t), -u_4(1-t)]^{\mathrm{T}}$, which is clearly also a solution of (2.1). Hence $\tilde{\mathbf{u}}(0) \leq \mathbf{v}(0)$ and $\tilde{\mathbf{u}}(0) \neq \mathbf{v}(0)$ similarly yield $u_1(0) = \tilde{u}_1(1) < 0$. It contradicts again the Dirichlet boundary conditions.

We have proved $u_4(t_0) = 0$. Since the zero function is a solution of (2.3) on $[t_0,1]$, the uniqueness of the solution of (2.3) implies $\mathbf{u}(t) = \mathbf{0}$ for $t \in [t_0,1]$. Similarly $\mathbf{u}(t) = \mathbf{0}$ for $t \in [0,t_0]$. □

Lemma 3.2. *Let $\lambda$ be an eigenvalue of (1.1) and $u$ a corresponding eigenfunction. Then*

(i) *$u$ has finitely many zero points in $[0,1]$;*

(ii) *if $u(t_0) = 0$ for some $t_0 \in (0,1)$, $u'(t_0) \neq 0$;*

(iii) $u''(0) \neq 0$.

*Proof.* We denote by $\mathbf{u} := [u, u', \psi_p(u''), (\psi_p(u''))']^{\mathrm{T}}$ the corresponding solution of (2.1). We have $\lambda > 0$ by [1, Corollary 4(iii)].

(i) Assume by contrary that there is a sequence $\{t_n\}_{n=1}^{\infty} \subset [0,1]$, $u(t_n) = 0$. We can suppose that $t_n \to t_0$ for some $t_0 \in [0,1]$, and $t_n \neq t_0$ for all $n \in \mathbb{N}$. Clearly,

$$\begin{gathered} u_1(t_0) = \lim_{n\to\infty} u(t_n) = 0, \qquad u_2(t_0) = \lim_{n\to\infty} \frac{u(t_n)}{t_n - t_0} = 0, \\ u_2'(t_0) = \lim_{n\to\infty} \frac{2u(t_n)}{(t_n - t_0)^2} = 0, \end{gathered} \tag{3.1}$$

and so $u_3(t_0) = \psi_p(u_2'(t_0)) = 0$. Since $\lambda > 0$, Lemma 3.1 yields $\mathbf{u} \equiv \mathbf{0}$, a contradiction to the nontriviality of $u = u_1$.

(ii) We proceed again by contradiction. Let $u_1(t_0) = u_2(t_0) = 0$, $t_0 \in (0,1)$. Lemma 3.1 implies $u_3(t_0) \neq 0$, and we can assume $u_3(t_0) > 0$.

Let, first, $u_4(t_0) \geq 0$. Hence $\mathbf{u}(t_0) \geq \mathbf{0}$, $\mathbf{u}(t_0) \neq \mathbf{0}$, and Lemma 2.3 then implies $u(1) = u_1(1) > 0$, a contradiction.

It remains now to investigate the opposite case $u_4(t_0) < 0$. We denote by $\tilde{\mathbf{u}}(t) := [u_1(1-t), -u_2(1-t), u_3(1-t), -u_4(1-t)]^{\mathrm{T}}$ a solution of (2.1). Then, $\tilde{\mathbf{u}}(1-t_0) = [0, 0, u_3(t_0), -u_4(t_0)]^{\mathrm{T}} \geq \mathbf{0}, \neq \mathbf{0}$. Hence, by Lemma 2.3, $u(0) = u_1(0) = \tilde{u}_1(1) > 0$, a contradiction again.

(iii) It is a direct consequence of Lemma 3.1. □

We use the results [4, Theorems 1.1 and 1.3] by Drábek and Ôtani that the Navier problem

$$\left(|u''(t)|^{p-2}u''(t)\right)'' = \lambda|u(t)|^{p-2}u(t), \quad t\in[0,1],$$
$$u(0) = u''(0) = u(1) = u''(1) = 0 \tag{3.2}$$

has the least positive eigenvalue, which we denote by $\tilde{\lambda}_1(p)$. There is a corresponding eigenfunction $\tilde{u}_1(p)$ that satisfies $\tilde{u}_1(p) > 0$ and $\tilde{u}_1(p)'' < 0$ in $(0,1)$, and $\tilde{u}_1(p)'(0) = 1$. Moreover, $\tilde{u}_1(p)$ is even with respect to $1/2$, and so

$$\tilde{u}_1(p)'(1) = -\tilde{u}_1(p)'(0), \qquad \left(\psi_p(\tilde{u}_1(p)'')\right)'\big|_{t=1} = -\left(\psi_p(\tilde{u}_1(p)'')\right)'\big|_{t=0}. \tag{3.3}$$

The eigenfunctions, corresponding to higher eigenvalues, are all constructed from $\tilde{u}_1(p)$ in [4]. We will construct the eigenfunctions of the Dirichlet problem (1.1) using the function $\tilde{u}_1(p)$ too.

For fixed $p > 1$, we define a function $\tilde{u} : [0,+\infty) \to \mathbb{R}$ by

$$\tilde{u}(t) = (-1)^n\tilde{u}_1(p)(t-n) \quad \text{for } t\in[n,n+1),\ n\in\{0,1,2,\dots\}. \tag{3.4}$$

We denote $\tilde{\mathbf{u}} := [\tilde{u},\tilde{u}',\psi_p(\tilde{u}''),(\psi_p(\tilde{u}''))']^{\mathrm{T}}$. The properties of $\tilde{u}_1(p)$ guarantee that $\tilde{\mathbf{u}}$ is a solution of (2.3) on $[0,+\infty)$ for $\lambda := \tilde{\lambda}_1(p) > 0$, and with the initial condition $\tilde{\mathbf{u}}(0) = [0,1,0,\tilde{u}_4(0)]^{\mathrm{T}}$. Obviously, $\tilde{u}_4(0) < 0$ since otherwise, $\tilde{\mathbf{u}}(0) \geq \mathbf{0}$, $\tilde{\mathbf{u}}(0) \neq \mathbf{0}$, and Lemma 2.3 would imply $\tilde{u}_1 > 0$ on $(0,+\infty)$.

Lemma 3.3. *Let $n\in\mathbb{N}$ be arbitrary. Then there exists an eigenfunction $u_{4n-2}$ of (1.1), $u''_{4n-2}(0) = 1$, having precisely $4n-3$ zero points in $(0,1)$.*

*Proof.* Let a mapping $T : \mathbb{R} \to (C[0,2n])^4$ assign to a $\xi\in\mathbb{R}$ the solution of (2.3), where $t_0 := 0$, $t_1 := 2n$, $\lambda := \tilde{\lambda}_1(p)$, and $[\alpha,\beta,\gamma,\delta]^{\mathrm{T}} := [0,1,0,\xi]^{\mathrm{T}}$. Clearly, $T(\tilde{u}_4(0)) = \tilde{\mathbf{u}}$. Let $\mathbf{w} := T(0)$. Then $\mathbf{w}(0) \geq \mathbf{0}$, $\mathbf{w}(0) \neq \mathbf{0}$, and Lemma 2.3 implies $\mathbf{w} > \mathbf{0}$ on $(0,2n]$. The continuous dependence of the solution of the initial value problem (2.3) on the initial conditions (see [2, Corollary 1.10]) means that $T$ is continuous.

We define a mapping $f : (C[0,2n])^4 \to \mathbb{R}$ by

$$f(\mathbf{u}) := \min_{t\in[2n-1,2n]} u_1(t), \quad \mathbf{u}\in\left(C[0,2n]\right)^4. \tag{3.5}$$

Then $f$ and also $g : \mathbb{R}\to\mathbb{R}$, $g := f\circ T$, are continuous. Now clearly $g(\tilde{u}_4(0)) < 0$ and $g(0) > 0$. Consequently, there exists a constant $K\in(\tilde{u}_4(0),0)$ (we recall that $\tilde{u}_4(0) < 0$) such that $g(K) = 0$. We denote $\mathbf{v} := T(K)$.

Since $\mathbf{v}(0) \geq \tilde{\mathbf{u}}(0)$ and $\mathbf{v}(0) \neq \tilde{\mathbf{u}}(0)$, Lemma 2.3 yields $\mathbf{v} > \tilde{\mathbf{u}}$ on $(0,2n]$. We have

$$\min_{t\in[2n-1,2n]} v_1(t) = 0 \tag{3.6}$$

(see Figure 3.1). Due to the continuity of $v_1$, we can take the first point in $[2n-1,2n]$, where the minimum (3.6) is achieved, and denote it by $\bar{t}$. Since $v_1(2n-1) > \tilde{u}_1(2n-1) = 0$ and $v_1(2n) > \tilde{u}_1(2n) = 0$, it must be $\bar{t}\in(2n-1,2n)$. Hence $v_1(\bar{t}) = v_2(\bar{t}) = 0$.

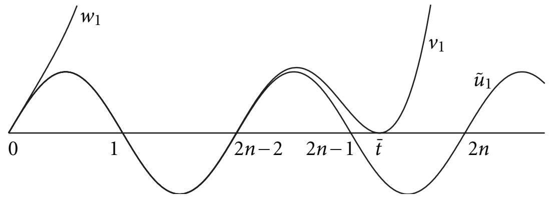


Figure 3.1. Proof of Lemma 3.3 for $n = 2$.

We now define a function $\hat{u}_{4n-2} : [0,1] \to \mathbb{R}$ by

$$\hat{u}_{4n-2}(t) := \begin{cases} v_1\big(\bar{t}(1-2t)\big) & \text{for } t \in \left[0, \frac{1}{2}\right], \\ -v_1\big(\bar{t}(2t-1)\big) & \text{for } t \in \left(\frac{1}{2}, 1\right]. \end{cases} \tag{3.7}$$

We immediately get $\hat{u}_{4n-2}(0) = \hat{u}'_{4n-2}(0) = \hat{u}_{4n-2}(1) = \hat{u}'_{4n-2}(1) = 0$. Since $v_1(0) = v_3(0) = 0$, the vector function

$$\hat{\mathbf{u}}_{4n-2} := \left[\hat{u}_{4n-2}, \hat{u}'_{4n-2}, \psi_p\big(\hat{u}''_{4n-2}\big), \big(\psi_p\big(\hat{u}''_{4n-2}\big)\big)'\right]^{\mathrm{T}} \tag{3.8}$$

is of class $C^1[0,1]$, and one can easily check that $\hat{\mathbf{u}}_{4n-2}$ is a solution of (2.1) with $\lambda := (2\bar{t})^{2p}\tilde{\lambda}_1(p)$. Hence, $\hat{u}_{4n-2}$ is an eigenfunction of (1.1), corresponding to the eigenvalue $\lambda^{\mathrm{D}}_{4n-2}(p) := (2\bar{t})^{2p}\tilde{\lambda}_1(p)$. Since $\bar{t} \in (2n-1, 2n)$, we obtain the estimate

$$\lambda^{\mathrm{D}}_{4n-2}(p) \in \big((4n-2)^{2p}\tilde{\lambda}_1(p), (4n)^{2p}\tilde{\lambda}_1(p)\big), \quad n \in \mathbb{N}. \tag{3.9}$$

Lemma 3.2(iii) implies $\hat{u}''_{4n-2}(0) \neq 0$, and so

$$u_{4n-2} := \frac{1}{\hat{u}''_{4n-2}(0)} \hat{u}_{4n-2} \tag{3.10}$$

is an eigenfunction of (1.1), corresponding to $\lambda^{\mathrm{D}}_{4n-2}(p)$ and satisfying, moreover, $u''_{4n-2}(0) = 1$.

We now show that $u_{4n-2}$ has precisely $4n-3$ zeros in $(0,1)$. Lemma 3.2(i) and (ii) state that $u_{4n-2}$ has finitely many zero points in $(0,1)$, which all are simple.

Let $k \in \{0, 1, \dots, n-2\}$ be arbitrary. Then we have already verified all assumptions of Lemma 2.4, where $\mathbf{u} := \tilde{\mathbf{u}}$, $\mathbf{v} := \mathbf{v}$, $t_0 := 2k$, $t_m := 2k+1$, $t_1 := 2k+2$, and $t_r := \bar{t}$. Hence Lemma 2.4 yields that $v_1$ has exactly two zeros in $(2k, 2k+2]$, that is, $v_1$ has exactly $2n-2$ zeros in $(0, 2n-2]$. By the choice of $\bar{t}$, $v_1 > 0$ on $(2n-2, \bar{t})$, and so $v_1$ has precisely $2n-2$ zeros even in $(0, \bar{t})$. The definitions (3.10) of $u_{4n-2}$ and (3.7) immediately yield that $u_{4n-2}$ has precisely $2(2n-2)+1 = 4n-3$ zeros in $(0,1)$, and the proof is complete. □

Lemma 3.4. *Let $n \in \mathbb{N}$ be arbitrary. Then there exists an eigenfunction $u_{4n}$ of (1.1), $u''_{4n}(0) = 1$, having precisely $4n-1$ zero points in $(0,1)$.*

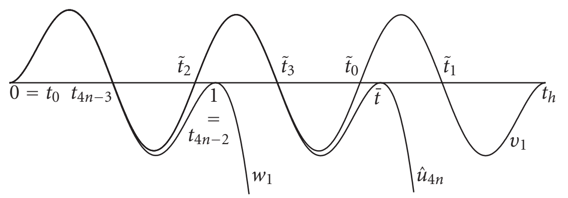


Figure 3.2. Proof of Lemma 3.4 for $n = 1$.

*Proof.* We denote

$$t_h := \left(\frac{\lambda^{\mathrm{D}}_{4n+2}(p)}{\lambda^{\mathrm{D}}_{4n-2}(p)}\right)^{1/(2p)} > \left(\frac{(4n+2)^{2p}\tilde{\lambda}_1(p)}{(4n)^{2p}\tilde{\lambda}_1(p)}\right)^{1/(2p)} > 1. \tag{3.11}$$

(We used estimate (3.9).) We define a function $v_1 : [0, t_h] \to \mathbb{R}$ by

$$v_1(t) = t_h^2 u_{4n+2}\left(\frac{t}{t_h}\right), \quad t \in [0, t_h] \tag{3.12}$$

(see Figure 3.2), where the functions $u_{4n+2}$, $n \in \mathbb{N}$, were defined in the previous proof. We denote $\mathbf{v} := [v_1, v_1', \psi_p(v_1''), (\psi_p(v_1''))']^{\mathrm{T}}$. Clearly, $v_1(0) = v_2(0) = v_1(t_h) = v_2(t_h) = 0$, $v_3(0) = 1$, and $v_1$ has precisely $4n+1$ zero points in $(0, t_h)$. Since $u_{4n+2}$ is a solution of (1.1) with $\lambda := \lambda^{\mathrm{D}}_{4n+2}(p)$, we get by substituting $\mathbf{v}$ into (2.3) that $\mathbf{v}$ is a solution of (2.3) on $[0, t_h]$, with $\lambda := \lambda^{\mathrm{D}}_{4n+2}(p)t_h^{-2p} = \lambda^{\mathrm{D}}_{4n-2}(p)$ and the initial condition $[\alpha, \beta, \gamma, \delta]^{\mathrm{T}} := [0, 0, 1, v_4(0)]^{\mathrm{T}}$.

Similarly, as in the proof of Lemma 3.3, we define a mapping $T : \mathbb{R} \to (C[0, t_h])^4$ that assigns to $\xi \in \mathbb{R}$ the solution of (2.3), with $t_0 := 0$, $t_1 := t_h$, $\lambda := \lambda^{\mathrm{D}}_{4n-2}(p)$, and $[\alpha, \beta, \gamma, \delta]^{\mathrm{T}} := [0, 0, 1, \xi]^{\mathrm{T}}$. $T$ is again continuous by [2, Corollary 1.10]. Obviously, $\mathbf{v} = T(v_4(0))$.

We denote $K_1 := (\psi_p(u''_{4n-2}(t)))'|_{t=0}$ and $\mathbf{w} := T(K_1)$. The uniqueness of the solution of the initial value problem implies $w_1 = u_{4n-2}$ on $[0, 1]$. It must be $K_1 < 0$ since otherwise, $\mathbf{w} \geq \mathbf{0}$, $\mathbf{w} \neq \mathbf{0}$, and so $\mathbf{w} > \mathbf{0}$ on $(0, t_h]$ by virtue of Lemma 2.3. But $w_1(1) = u_{4n-2}(1) = 0$. From the definition of $u_{4n-2}$, we see that $u_{4n-2}$ is odd with respect to 1/2. Thus $w_1(t) = -w_1(1-t)$ and $w_3(t) = -w_3(1-t)$ for all $t \in [0, 1]$. Hence $w_3(1) = -w_3(0) = -1$ and $w_4(1) = w_4(0) = K_1 < 0$. Now, $\mathbf{w}(1) \leq \mathbf{0}$, $\mathbf{w}(1) \neq \mathbf{0}$, and Lemma 2.3 gives $\mathbf{w} < \mathbf{0}$ on $(1, t_h]$.

Now we prove that $K_1 < v_4(0)$. If $K_1 > v_4(0)$, then $\mathbf{w} > \mathbf{v}$ on $(0, t_h]$ by Lemma 2.3. But $w_1(t_h) < 0 = v_1(t_h)$. If $K_1 = v_4(0)$, then $\mathbf{w} = \mathbf{v}$ by the uniqueness of the solution of (2.3). This is not possible for the same reason. Hence $K_1 < v_4(0)$, and Lemma 2.3 implies $\mathbf{w} < \mathbf{v}$ on $(0, t_h]$.

Since $w_1$ coincides with $u_{4n-2}$ on $[0, 1]$, it has precisely $4n-1$ zero points in $[0, 1]$, which we denote by $0 = t_0 < t_1 < \cdots < t_{4n-3} < t_{4n-2} = 1$. We take $k \in \{0, 1, \ldots, 2n-2\}$ arbitrary. Now all assumptions of Lemma 2.4, where $\mathbf{u} := \mathbf{w}$, $\mathbf{v} := \mathbf{v}$, $t_0 := t_{2k}$, $t_m := t_{2k+1}$, $t_1 := t_{2k+2}$, and $t_r := t_h$, are satisfied. Hence $v_1$ has exactly two zeros in each $(t_{2k}, t_{2k+2}]$, and so exactly $4n-2$ zeros in $(0, 1]$. We already know that $v_1$ has $4n+1$ zeros in $(0, t_h)$. We denote by $\tilde{t}_0 < \tilde{t}_1$ the last two. Obviously, $\tilde{t}_0, \tilde{t}_1 > 1$. Since $v_1''(0) = 1$, Lemma 3.2(ii) yields $v_1 > 0$ on $(\tilde{t}_0, \tilde{t}_1)$.

Similarly, as in the previous proof, we define a mapping $f : (C[0,t_h])^4 \to \mathbb{R}$ by

$$f(\mathbf{u}) := \max_{t\in[\tilde{t}_0,\tilde{t}_1]} u_1(t), \quad \mathbf{u} \in \left(C[0,t_h]\right)^4. \tag{3.13}$$

Again $f$ and $g : \mathbb{R} \to \mathbb{R}$, $g = f \circ T$, are continuous. We have $g(K_1) = f(\mathbf{w}) < 0$ and $g(v_4(0)) = f(\mathbf{v}) > 0$. Thus there exists $K \in (K_1, v_4(0))$ such that $g(K) = 0$. We denote $\hat{\mathbf{u}}_{4n} := T(K)$ and $\hat{u}_{4n}$, the first component of $\hat{\mathbf{u}}_{4n}$. Since $K < v_4(0)$, we have $\hat{\mathbf{u}}_{4n}(0) \le \mathbf{v}(0)$ and $\hat{\mathbf{u}}_{4n}(0) \ne \mathbf{v}(0)$, and Lemma 2.3 yields $\hat{\mathbf{u}}_{4n} < \mathbf{v}$ on $(0,t_h]$. In particular, $\hat{u}_{4n}(\tilde{t}_0) < v_1(\tilde{t}_0) = 0$ and $\hat{u}_{4n}(\tilde{t}_1) < v_1(\tilde{t}_1) = 0$. Thus the maximum

$$\max_{t\in[\tilde{t}_0,\tilde{t}_1]} \hat{u}_{4n}(t) = 0 \tag{3.14}$$

must be achieved in $(\tilde{t}_0,\tilde{t}_1)$. We again denote by $\bar{t}$ the least zero point of $\hat{u}_{4n}$ in $(\tilde{t}_0,\tilde{t}_1)$. Obviously, $\hat{u}_{4n}(0) = \hat{u}'_{4n}(0) = \hat{u}_{4n}(\bar{t}) = \hat{u}'_{4n}(\bar{t}) = 0$, and so the function $u_{4n}$, defined by

$$u_{4n}(t) = \frac{1}{\bar{t}^2}\hat{u}_{4n}(\bar{t}t), \quad t \in [0,1], \tag{3.15}$$

is an eigenfunction of (1.1), satisfying $u''_{4n}(0) = 1$. Lemma 3.2(i) and (ii) yield that $u_{4n}$ in $[0,1]$, and also $\hat{u}_{4n}$ in $[0,\bar{t}]$, have finitely many zero points, all being simple.

Since $K_1 < K$, we can show (similarly as we did for $\mathbf{v}$) that $\hat{\mathbf{u}}_{4n}$ satisfies $\mathbf{w} < \hat{\mathbf{u}}_{4n}$ on $(0,t_h]$, and $\hat{u}_{4n}$ has exactly $4n-2$ zeros in $(0,1]$. We denote by $\tilde{t}_2$ and $\tilde{t}_3$ the $(4n-2)$th and the $(4n-1)$th, respectively, zero point of $v_1$ in $(0,t_h)$. Hence $\tilde{t}_2 \le 1 < \tilde{t}_3$ and $v_1 > 0$ on $(\tilde{t}_2,\tilde{t}_3)$. We showed that $\hat{u}_{4n}(\tilde{t}_2) < v_1(\tilde{t}_2) = 0$, $\hat{u}_{4n}(1) > w_1(1) = 0$, and $\hat{u}_{4n}(\tilde{t}_3) < v_1(\tilde{t}_3) = 0$. Consequently, $\tilde{t}_2 < 1$ and $\hat{u}_{4n}$ has at least one zero point in each of the intervals $(\tilde{t}_2, 1)$ and $(1,\tilde{t}_3)$.

If $\hat{u}_{4n}$ had at least two zeros in $(1,\tilde{t}_3)$, then all assumptions of Lemma 2.4, where $\mathbf{u} := \hat{\mathbf{u}}_{4n}$, $\mathbf{v} := \mathbf{v}$, $t_r := \tilde{t}_3$, and $t_0, t_m, t_1$ are the first three zero points of $\hat{u}_{4n}$ in $(\tilde{t}_2,\tilde{t}_3)$, would be verified, and Lemma 2.4 would imply that $v_1$ had at least two zeros in $(\tilde{t}_2,\tilde{t}_3)$. But $v_1 > 0$ there. Hence $\hat{u}_{4n}$ has exactly $4n-1$ zeros in $(0,\tilde{t}_3)$. Since $\hat{u}_{4n} < v_1 \le 0$ on $[\tilde{t}_3,\tilde{t}_0]$, and due to the choice of $\bar{t}$, $\hat{u}_{4n}$ has precisely $4n-1$ zero points even in $(0,\bar{t})$, so as $u_{4n}$ in $(0,1)$. This finishes the proof. □

Lemma 3.5. *Let $n \in \mathbb{N}$ be arbitrary. Then there exists an eigenfunction $u_n$ of (1.1), $u''_n(0) = 1$, having precisely $n-1$ zero points in* $(0,1)$.

*Proof.* It remains to prove the existence of an eigenfunction with $2m-2$ zeros in $(0,1)$, $m \in \mathbb{N}$. The proof is very similar to that of Lemma 3.3.

First, we define a mapping $T : \mathbb{R} \to (C[0,1])^4$ assigning to a $\xi \in \mathbb{R}$ the solution of (2.3) with $t_0 := 0$, $t_1 := 1$, $\lambda := \lambda^{\mathrm{D}}_{2m}(p)$, and $[\alpha,\beta,\gamma,\delta]^{\mathrm{T}} := [0,0,1,\xi]^{\mathrm{T}}$. Again, [2, Corollary 1.10] guarantees the continuity of $T$. The constants $\lambda^{\mathrm{D}}_{2m}(p)$ were defined in the previous two proofs.

We denote $\mathbf{u}_{2m} := [u_{2m}, u'_{2m}, \psi_p(u''_{2m}), (\psi_p(u''_{2m}))']^{\mathrm{T}}$, where the eigenfunctions $u_{2m}$ of (1.1) were constructed in the previous two proofs too. We denote $K_1 := (\psi_p(u''_{2m}(t)))'|_{t=0}$. The uniqueness of the solution of (2.3) implies that $T(K_1) = \mathbf{u}_{2m}$. Let $\mathbf{w} := T(0)$. As in the proof of Lemma 3.3, $\mathbf{w} > \mathbf{0}$ on $(0,1]$ (see Figure 3.3).

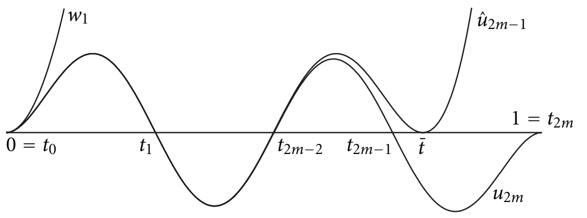


Figure 3.3. Proof of Lemma 3.5 for $m = 2$.

We denote by $0 = t_0 < t_1 < \cdots < t_{2m-1} < t_{2m} = 1$ all the zero points of $u_{2m}$. Then $f : (C[0,1])^4 \to \mathbb{R}$ defined by

$$f(\mathbf{u}) := \min_{t\in[t_{2m-1},t_{2m}]} u_1(t), \quad \mathbf{u} \in (C[0,1])^4, \tag{3.16}$$

and $g : \mathbb{R} \to \mathbb{R}$, $g = f \circ T$, are both continuous. Since $K_1 < 0$, $g(K_1) < 0$, and $g(0) > 0$, there is a $K \in (K_1, 0)$ such that $g(K) = 0$. We denote $\hat{\mathbf{u}}_{2m-1} := T(K)$, and denote by $\hat{u}_{2m-1}$ the first component of $\hat{\mathbf{u}}_{2m-1}$. Since $K_1 < K$, we have $\mathbf{u}_{2m} < \hat{\mathbf{u}}_{2m-1}$ on $(0,1]$ by Lemma 2.3. Let $\bar{t}$ be the first $t \in (t_{2m-1}, t_{2m})$, where $\hat{u}_{2m-1}(t) = 0$. Hence $\hat{u}_{2m-1}(\bar{t}) = \hat{u}'_{2m-1}(\bar{t}) = 0$, and the function

$$u_{2m-1}(t) := \frac{1}{\bar{t}^2}\hat{u}_{2m-1}(\bar{t}t), \quad t \in [0,1], \tag{3.17}$$

is an eigenfunction of (1.1) with $u''_{2m-1}(0) = 1$.

Due to Lemma 3.2(i) and (ii), $u_{2m-1}$ and even $\hat{u}_{2m-1}$ have in $[0,1]$ and $[0,\bar{t}]$, respectively, finitely many zeros, all being simple. Thus, similarly as in the proof of Lemma 3.3, we can use Lemma 2.4 to show that $\hat{u}_{2m-1}$ has exactly two zeros in each of the intervals $(0,t_2],(t_2,t_4],\ldots,(t_{2m-4},t_{2m-2}]$. Since $u''_{2m}(0) = 1$, we have $\hat{u}_{2m-1} > u_{2m} \geq 0$ on $(t_{2m-2}, t_{2m-1}]$ by virtue of Lemma 3.2(ii), and obviously $\hat{u}_{2m-1} > 0$ on $(t_{2m-1}, \bar{t})$. Consequently, $\hat{u}_{2m-1}$, as well as $u_{2m-1}$, have precisely $2m-2$ zero points in $(0,\bar{t})$ and $(0,1)$, respectively. □

Lemma 3.6. *Let $u_1$ and $u_2$ be eigenfunctions of (1.1), corresponding to eigenvalues $\lambda_1$ and $\lambda_2$, respectively, and having precisely $n_1 - 1$ and $n_2 - 1$, respectively, zero points in $[0,1]$, $n_1, n_2 \in \mathbb{N}$. Assume $u''_1(0) = u''_2(0) = 1$. Then*

(i) $n_1 < n_2 \Leftrightarrow \lambda_1 < \lambda_2$,
(ii) $n_1 = n_2 \Leftrightarrow \lambda_1 = \lambda_2 \Leftrightarrow u_1 = u_2$ *on* $[0,1]$.

*Proof.* It suffices to show that

(1) $\lambda_1 < \lambda_2 \Rightarrow n_1 < n_2$,
(2) $\lambda_1 = \lambda_2 \Rightarrow u_1 = u_2$ on $[0,1]$.

Since $u''_1(0) = u''_2(0) = 1$, the second implication is a consequence of the positivity and the simplicity of all eigenvalues of (1.1) (see [1, Example 8]). So, it remains to prove the first one.

Let $\lambda_1 < \lambda_2$. Substituting into (1.1), we realize that $\hat{u}_1(t) := u_1(1-t)$, $t \in [0,1]$, is an eigenfunction of (1.1), corresponding to $\lambda_1$. Hence the simplicity of $\lambda_1$ implies existence of $\kappa \in \mathbb{R}$ such that $\hat{u}_1 = \kappa u_1$. Taking any $t_0 \in [0,1]$, where $u_1(t_0) \neq 0$, we get

$$u_1(t_0) = \hat{u}_1(1-t_0) = \kappa u_1(1-t_0) = \kappa \hat{u}_1(t_0) = \kappa^2 u_1(t_0) \neq 0. \tag{3.18}$$

Thus $\kappa^2 = 1$, that is, the function $u_1$ is either even or odd with respect to 1/2. We discuss the former case only; for the latter one, the proof is analogous.

Now $u_1$ is even, and so $u_1'$ is odd with respect to 1/2. Thus $u_1'(1/2) = 0$ and, due to Lemma 3.2(ii), $u_1(1/2) \neq 0$. Consequently, $u_1$ has an even number of zero points in $(0,1)$, and $n_1$ is odd.

Let

$$\tilde{t} := \left(\frac{\lambda_2}{\lambda_1}\right)^{1/(2p)} > 1. \tag{3.19}$$

Let $\mathbf{v}$ be the solution of (2.3), where $t_0 := 0$, $t_1 := \tilde{t}$, $\lambda := \lambda_1$, and $[\alpha,\beta,\gamma,\delta]^{\mathrm{T}} := [0,0,1,K]^{\mathrm{T}}$, $K = (\psi_p(u_1''(t)))'|_{t=0}$. The uniqueness of the solution of (2.3) implies $v_1 = u_1$ on $[0,1]$. Since $u_1$ is even with respect to 1/2, we have $v_3(1) = \psi_p(u_1''(1)) = 1$ and $v_4(1) = -K$. Clearly, $K < 0$ because if $K \geq 0$, then $\mathbf{v} \geq \mathbf{0}$, $\mathbf{v} \neq \mathbf{0}$, and thus $\mathbf{v} > \mathbf{0}$ on $(0,1]$ by Lemma 2.3. But $v_1(1) = 0$. Hence $v_4(1) > 0$, $\mathbf{v}(1) \geq \mathbf{0}$, $\mathbf{v}(1) \neq \mathbf{0}$, and Lemma 2.3 yields $\mathbf{v} > \mathbf{0}$ on $(1,\tilde{t}]$.

We define a function $w_1 : [0,\tilde{t}] \to \mathbb{R}$ by

$$w_1(t) := \tilde{t}^2 u_2\left(\frac{t}{\tilde{t}}\right), \quad t \in [0,\tilde{t}]. \tag{3.20}$$

We denote $\mathbf{w} := [w_1, w_1', \psi_p(w_1''), (\psi_p(w_1''))']^{\mathrm{T}}$. Then $\mathbf{w}$ is a solution of (2.3), with $t_0 := 0$, $t_1 := \tilde{t}$, $\lambda := \lambda_2 \tilde{t}^{-2p} = \lambda_1$, and $[\alpha,\beta,\gamma,\delta]^{\mathrm{T}} := [0,0,1,w_4(0)]^{\mathrm{T}}$. We prove that $K > w_4(0)$. If $K < w_4(0)$, then $\mathbf{v}(0) \leq \mathbf{w}(0)$, $\mathbf{v}(0) \neq \mathbf{w}(0)$, and so $\mathbf{v} < \mathbf{w}$ on $(0,\tilde{t}]$ by Lemma 2.3, which is not possible since $v_1(\tilde{t}) > 0 = w_1(\tilde{t})$. If $K = w_4(0)$, then the uniqueness of the solution of (2.3) would imply $\mathbf{v} = \mathbf{w}$ on $[0,\tilde{t}]$, which cannot be true for the same reason.

We denote by $0 = t_0 < t_1 < \cdots < t_{n_1-1} < t_{n_1} = 1$ all zero points of $v_1 = u_1$ in $[0,1]$. We have $\mathbf{v}(0) \geq \mathbf{w}(0)$, $\mathbf{v}(0) \neq \mathbf{w}(0)$, and so Lemma 2.3 yields $\mathbf{v} > \mathbf{w}$ on $(0,\tilde{t}]$. Since $w_1''(0) = 1$, it must be $w_1 > 0$ on $(0,\varepsilon)$ for some $\varepsilon > 0$. We have $w_1(t_1) < v_1(t_1) = 0$, and so $w_1$ has at least one zero in $(0,t_1)$. We prove that it has only one. Assume by contrary that it has at least two. Then all assumptions of Lemma 2.4, where $t_0 := 0$, $t_m < t_1$ are the first two zeros of $w_1$ in $(0,t_1)$, $t_r := t_1$, $\mathbf{u} := \mathbf{w}$, and $\mathbf{v} := \mathbf{v}$, are satisfied, and thus $v_1$ has at least two zero points in $(0,t_1)$. This is a contradiction.

We prove that $w_1$ has exactly $n_1$ zeros in $(0,1]$. For $n_1 = 1$, we have proved it already. For $n_1 > 1$, take arbitrary $k \in \{0,1,\dots,(n_1-3)/2\}$. We have verified all assumptions of Lemma 2.4 with $\mathbf{u} := -\mathbf{v}$, $\mathbf{v} := -\mathbf{w}$, $t_0 := t_{2k+1}$, $t_m := t_{2k+2}$, $t_1 := t_{2k+3}$, and $t_r := \tilde{t}$. Consequently, both $-w_1$ and $w_1$ have exactly two zeros in $(t_{2k+1}, t_{2k+3}]$, and altogether $1 + 2((n_1-3)/2+1) = n_1$ zeros in $(0,1]$. Thus the number of zeros of $w_1$ in $(0,\tilde{t})$, which is equal to the number of zeros of $u_2$ in $(0,1)$, is at least $n_1$. Hence $n_2 > n_1$, and the proof is finished. □

*Proof of Theorem 1.1.* Lemma 3.5 gives us the existence of the sequence $\{u_n\}_{n=1}^{\infty}$, $u_n''(0) = 1$, of eigenfunctions of (1.1) having precisely $n-1$ zero points in $(0,1)$. We denote the corresponding eigenvalues by $\lambda_n^{\mathrm{D}}(p) > 0$ in accordance with the proof of Lemma 3.3. Then

Lemma 3.6(i) yields

$$\lambda_1^{\mathrm{D}}(p) < \lambda_2^{\mathrm{D}}(p) < \cdots, \tag{3.21}$$

and estimate (3.9) implies $\lambda_n^{\mathrm{D}}(p) \to +\infty$ as $n \to \infty$.

On the other hand, if we take any eigenfunction $u$ of (1.1), then $u''(0) \neq 0$ according to Lemma 3.2(iii), and $u$ has a finite number of zero points in $(0,1)$ (denote it by $n_0$) by Lemma 3.2(i). Lemma 3.6(ii) then yields

$$\frac{u}{u''(0)} = u_{n_0}. \tag{3.22}$$

Consequently, the sequences $\{\lambda_n^{\mathrm{D}}(p)\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ contain all eigenvalues and eigenfunctions (up to normalization) of (1.1).

Simplicity of the eigenvalues $\lambda_n^{\mathrm{D}}(p)$ of (1.1) is a consequence of [1, Corollary 4(i)].

It now remains to prove the discreteness of the set of eigenfunctions of (1.1), which is a standard consequence of the above facts. We take an eigenfunction $u_n$ of (1.1). We denote by $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$ all the zero points of $u_n$. We know that $u_n''(0) \neq 0$ (Lemma 3.2(iii)) and $u_n'(t_i) \neq 0$, $i \in \{1,2,\ldots,n-1\}$ (Lemma 3.2(ii)). Since $u_n$ is even or odd with respect to $1/2$ (see the proof of Lemma 3.6), we have $|u_n''(1)| = |u_n''(0)| \neq 0$.

Consequently, there exist neighborhoods $\mathcal{U}_i \subset [0,1]$ of $t_i$, where $i \in \{0,1,\ldots,n-1,n\}$, and a constant $K > 0$ such that

$$\begin{aligned} |u_n''(t)| &\geq K \quad \text{for } t \in \mathcal{U}_0 \cup \mathcal{U}_n, \\ |u_n'(t)| &\geq K \quad \text{for } t \in \bigcup_{i\in\{1,2,\ldots,n-1\}} \mathcal{U}_i, \\ |u_n(t)| &\geq K \quad \text{for } t \in [0,1] \setminus \bigcup_{i\in\{0,1,\ldots,n\}} \mathcal{U}_i. \end{aligned} \tag{3.23}$$

Now, if we take $0 < \varepsilon < K$, then any eigenfunction $u$ of (1.1) such that $\|u - u_n\|_{C^2[0,1]} < \varepsilon$ has $n-1$ zero points in $(0,1)$, so as $u_n$. Thus

$$\frac{u}{u''(0)} = u_n \tag{3.24}$$

by Lemma 3.6(ii). This completes the proof of Theorem 1.1. □

## 4. Neumann problem

We describe the set of positive eigenvalues and the corresponding eigenfunctions of (1.2) by means of the (positive) eigenvalues and the corresponding eigenfunctions of (1.1), showing that they are in one-to-one correspondence. The zero eigenvalue must be treated separately, and by [1, Corollary 4(ii)], neither (1.1) nor (1.2) has a negative eigenvalue.

We divide the proof of Theorem 1.2 into the following three assertions.

Proposition 4.1. *The set of all eigenvalues of (1.2) forms a sequence* $0 = \lambda_0^N(p) < \lambda_1^N(p) < \cdots \to +\infty$. *Every* $\lambda_n^N(p)$, $n > 0$, *is a simple eigenvalue while* $\lambda_0 = 0$ *is not. Moreover, (1.3) holds true, and for* $n > 0$, *for any eigenfunction* $u$ *of (1.2), corresponding to* $\lambda_n^N(p)$, *and for* *any eigenfunction $v$ of (1.1), corresponding to $\lambda_n^{\mathrm{D}}(p/(p-1))$, (1.4) holds true for some $\kappa \in \mathbb{R} \setminus \{0\}$.*

*Proof.* We take any positive eigenvalue $\lambda$ of (1.2) and a corresponding eigenfunction $u_1$. It means that $\mathbf{u} = [u_1, u_1', \psi_p(u_1''), (\psi_p(u_1''))']^{\mathrm{T}}$ is a solution of (2.2). Since $\lambda > 0$, we can multiply the first two equations in (2.2) by $\psi_{p'}(\lambda)$ to obtain the equivalent problem

$$
\begin{gathered}
u_3'(t) = u_4(t),\\
u_4'(t) = \psi_{(p')'}\big(\psi_{p'}(\lambda)u_1(t)\big),\\
\big(\psi_{p'}(\lambda)u_1(t)\big)' = \big(\psi_{p'}(\lambda)u_2(t)\big),\\
\big(\psi_{p'}(\lambda)u_2(t)\big)' = \big(\psi_{p'}(\lambda)\big)\psi_{(p')}\big(u_3(t)\big), \quad t \in [0,1],\\
u_3(0) = u_4(0) = u_3(1) = u_4(1) = 0.
\end{gathered}
\tag{4.1}
$$

We immediately see that $[u_3, u_4, \psi_{p'}(\lambda)u_1, \psi_{p'}(\lambda)u_2]^{\mathrm{T}}$ is a solution of the Dirichlet problem (2.1), with $p := p'$ and $\lambda := \psi_{p'}(\lambda)$. The function $u_3$ cannot be the zero function since otherwise, $u_1 = \psi_{p'}(u_3''/\lambda) = 0$ on $[0,1]$. But $u_1$ is an eigenfunction of (1.2). Consequently, $u_3$ is an eigenfunction of the Dirichlet problem (1.1) with $p := p'$ and $\lambda := \psi_{p'}(\lambda)$, and so $\psi_{p'}(\lambda) = \lambda_n^{\mathrm{D}}(p')$ for some $n \in \mathbb{N}$. Hence $\lambda = \psi_p(\lambda_n^{\mathrm{D}}(p'))$. Thus we proved that any positive eigenvalue $\lambda$ of (1.2) equals $\lambda_n^{\mathrm{N}}(p)$ for some $n \in \mathbb{N}$. The sequence $\{\lambda_n^{\mathrm{N}}(p)\}_{n=1}^{\infty}$ is defined by (1.3).

To show that $\lambda_n^{\mathrm{N}}(p)$ is an eigenvalue of (1.2) for any $n \in \mathbb{N}$, we take the eigenvalue $\lambda_n^{\mathrm{D}}(p')$ of (1.1), with $p := p'$, and a corresponding eigenfunction, denoted by $v_1$ here. Then $\mathbf{v} := [v_1, v_1', \psi_{p'}(v_1''), (\psi_{p'}(v_1''))']^{\mathrm{T}}$ is a solution of the Dirichlet problem (2.1), where $p := p'$ and $\lambda := \lambda_n^{\mathrm{D}}(p')$. Substituting into (4.1), one can check that it is equivalent to the claim that

$$
\mathbf{u} = \left[\frac{v_3}{\lambda_n^{\mathrm{D}}(p')}, \frac{v_4}{\lambda_n^{\mathrm{D}}(p')}, v_1, v_2\right]^{\mathrm{T}}
\tag{4.2}
$$

is a solution of (4.1), with $\lambda := \psi_p(\lambda_n^{\mathrm{D}}(p')) = \lambda_n^{\mathrm{N}}(p) > 0$. But for $\lambda > 0$, (4.1) is equivalent to (2.2), and so $\mathbf{u}$ is also a solution of the corresponding problem (2.2). Again, $v_3$ is not the zero function since if it was, then we would conclude from (2.1) that $v_1 = \psi_{(p')'}(v_3''/\lambda_n^{\mathrm{D}}(p')) = 0$ on $[0,1]$, which is not true. Hence $u_1$ is an eigenfunction of (1.2), corresponding to the eigenvalue $\lambda_n^{\mathrm{N}}(p)$. This proves that the positive eigenvalues of (1.2) form the sequence $\lambda_n^{\mathrm{N}}(p)$, $n \in \mathbb{N}$, defined by (1.3). Their simplicity is a consequence of [1, Corollary 4(i)].

We showed (see (4.2)) that any eigenfunction $\zeta u_1$, $\zeta \in \mathbb{R} \setminus \{0\}$, of (1.2), corresponding to $\lambda_n^{\mathrm{N}}(p) > 0$, can be written as $\kappa v_3 = \kappa \psi_{p'}(v_1'')$, where $\kappa = \zeta/\lambda_n^{\mathrm{D}}(p')$, and $v_1$ is an eigenfunction of the Dirichlet problem (1.1) for $p := p'$, corresponding to $\lambda_n^{\mathrm{D}}(p')$. This proves (1.4).

Obviously, $\lambda_0^{\mathrm{N}}(p) = 0$ is an eigenvalue of (1.2) since any linear function is a solution of (1.2) with $\lambda := 0$. Consequently, $\lambda_0^{\mathrm{N}}(p) = 0$ is not a simple eigenvalue. Thus we proved that all eigenvalues of (1.2) form the sequence $0 = \lambda_0^{\mathrm{N}}(p) < \lambda_1^{\mathrm{N}}(p) < \cdots$. Since $\lambda_n^{\mathrm{D}}(p') \to +\infty$ as $n \to \infty$, the relation (1.3) immediately yields that even $\lambda_n^{\mathrm{N}}(p) \to +\infty$ as $n \to \infty$. ∎

Now we prove that an eigenfunction, corresponding to an eigenvalue $\lambda_n^{\mathrm{N}}(p)$ of (1.2), $n > 0$, has precisely $n+1$ zero points in $(0,1)$. Due to (1.4), the zero points of an eigenfunction of (1.2), corresponding to $\lambda_n^{\mathrm{N}}(p)$, $n \in \mathbb{N}$, coincide with zero points of the second derivative of an (arbitrary) eigenfunction of (1.1) for $p := p'$, corresponding to $\lambda_n^{\mathrm{D}}(p')$. Hence we can restate the assertion as follows.

Lemma 4.2. *Let $u$ be an eigenfunction of (1.1), corresponding to $\lambda_n^D(p')$ and satisfying $u''(0) = 1$. Then $u''$ has precisely $n+1$ zero points in $(0,1)$.*

*Proof.* We denote by $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ all the zero points of $u$. In each interval $(t_i, t_{i+1})$, $i \in \{0,1,\dots,n-1\}$, $u$ has at least one local maximum for $i$ even, and minimum for $i$ odd. We choose one in each interval and denote it by $\tilde{t}_i \in (t_i, t_{i+1})$. Hence we have $\operatorname{sgn}(u(\tilde{t}_i)) = (-1)^i$ and $u'(\tilde{t}_i) = 0$. We denote $\mathbf{u} := [u, u', \psi_{p'}(u''), (\psi_{p'}(u''))']^{\mathrm{T}}$.

First, we prove that $u''(\tilde{t}_i) \neq 0$ for all $i \in \{0,1,\dots,n-1\}$. We proceed by contradiction—assume $u'(\tilde{t}_i) = u''(\tilde{t}_i) = 0$. We suppose that $u(\tilde{t}_i) > 0$; for $u(\tilde{t}_i) < 0$, the proof is similar. Similarly as in the proof of Lemma 3.2(ii), we distinguish two cases. For $u_4(\tilde{t}_i) \geq 0$, Lemma 2.3 yields $\mathbf{u} > \mathbf{0}$ on $(\tilde{t}_i, 1]$, and for $u_4(\tilde{t}_i) < 0$, the same lemma implies $\mathbf{u} > \mathbf{0}$ on $[0, \tilde{t}_i)$, a contradiction in both cases. This proves that $\operatorname{sgn}(u''(\tilde{t}_i)) = (-1)^{i+1}$.

Clearly, $u > 0$ on $(0, \tilde{t}_0)$. Now we show that $u''$ has exactly one zero point in $(0, \tilde{t}_0)$. Since $u''(0) = 1$ and $u''(\tilde{t}_0) < 0$, it has at least one. Assume by contrary that $u''(a_1) = u''(a_2) = 0$, $0 < a_1 < a_2 < \tilde{t}_0$. Then $u_3(a_1) = u_3(a_2) = 0$ and $u_3(\tilde{t}_0) < 0$. The mean value theorem implies the existence of $b_1 \in (a_1, a_2)$ and $b_2 \in (a_2, \tilde{t}_0)$ such that $u_4(b_1) = 0$ and $u_4(b_2) < 0$. Hence $u_4'(c) < 0$ for some $c \in (b_1, b_2)$. This is a contradiction since

$$u(c) = u_1(c) = \psi_p\left(\frac{u_4'(c)}{\lambda_n^{\mathrm{D}}(p')}\right) < 0, \tag{4.3}$$

but $c \in (0, \tilde{t}_0)$. Similarly, one can prove that $u''$ has exactly one zero point in $(\tilde{t}_{n-1}, 1)$.

We now consider an interval $(\tilde{t}_i, \tilde{t}_{i+1})$ for arbitrary $i \in \{0,1,\dots,n-2\}$. We assume that $i$ is even; for $i$ odd, the proof is analogous. Thus $u_1 > 0$ on $[\tilde{t}_i, t_{i+1})$, $u_1 < 0$ on $(t_{i+1}, \tilde{t}_{i+1}]$, $u_3(\tilde{t}_i) < 0$, and $u_3(\tilde{t}_{i+1}) > 0$. Again, $u''$ has at least one zero in $(\tilde{t}_i, \tilde{t}_{i+1})$, and we prove by contradiction that it has exactly one. So, let $u_3(a_1) = u_3(a_2) = 0$, $\tilde{t}_i < a_1 < a_2 < \tilde{t}_{i+1}$. Then the mean value theorem yields $u_4(b_1) > 0$, $u_4(b_2) = 0$, and $u_4(b_3) > 0$ for some $b_1 \in (\tilde{t}_i, a_1)$, $b_2 \in (a_1, a_2)$, and $b_3 \in (a_2, \tilde{t}_{i+1})$. Hence $u_4'(c_1) < 0$ and $u_4'(c_2) > 0$ for some $c_1 \in (b_1, b_2)$ and $c_2 \in (b_2, b_3)$. Since $u_1 = \psi_p(u_4'/\lambda_n^{\mathrm{D}}(p'))$, we have $u_1(c_1) < 0$ and $u_1(c_2) > 0$. Consequently, $c_1 > t_{i+1}$ and $c_2 < t_{i+1}$, a contradiction.

We now see that $u''$ has precisely $n+1$ zero points in $(0,1)$. □

Proposition 4.3. *For any eigenfunction $u$ of (1.2), corresponding to a positive eigenvalue, there exists an $\varepsilon > 0$ such that if $v$ is an eigenfunction of (1.2) and $\|u - v\|_{C^2[0,1]} < \varepsilon$, then $v = \kappa u$ for some $\kappa \in \mathbb{R}$.*

*Proof.* The reader is invited to verify that Lemma 3.2(ii) holds true even for the eigenfunctions of (1.2), corresponding to positive eigenvalues. Then, similarly as in the proof of Theorem 1.1, we can take $\varepsilon > 0$ so small that $v$ has the same number of zeros as $u$. The assertion is then a consequence of Lemma 4.2, (1.4), and the simplicity of positive eigenvalues of (1.2) (see [1, Example 9]). □

Proving Proposition 4.3, we finished the proof of Theorem 1.2.

## 5. Open problems

There are many open questions concerning the functions $p \mapsto \lambda_n^{\mathrm{D}}(p)$ and $p \mapsto \lambda_n^{\mathrm{N}}(p)$, $p \in (1,\infty)$, $n \in \mathbb{N}$. We know only that they are positive.

(1) Are $\lambda_n^{\mathrm{D}}(p)$ and $\lambda_n^{\mathrm{N}}(p)$, $n \in \mathbb{N}$, continuous as in the case of the Navier problem (3.2) (see [4])? Or even of the class $C^\infty$ as in the second-order case? Are they monotone?

(2) Is it possible to investigate $\lim_{p\to 1_+}\lambda_n^{\mathrm{D}}(p)$, $\lim_{p\to\infty}\lambda_n^{\mathrm{D}}(p)$, $\lim_{p\to 1_+}\lambda_n^{\mathrm{N}}(p)$, and $\lim_{p\to\infty}\lambda_n^{\mathrm{N}}(p)$?

(3) Is there a relation between $\lambda_n^{\mathrm{D}}(p)$ and $\lambda_n^{\mathrm{N}}(p)$, $n \in \mathbb{N}$? (Note that (1.3) is a relation between $\lambda_n^{\mathrm{D}}(p)$ and $\lambda_n^{\mathrm{N}}(p')$.) Can they be compared with the eigenvalues of the Navier problem? We already know that $\lambda_{4n-2}^{\mathrm{D}}(p)$, $n \in \mathbb{N}$, lies between the $(4n-2)$th and the $(4n)$th eigenvalues of the Navier problem (see (3.9) and [4]). When do we have $\lambda_n^{\mathrm{D}}(p) = \lambda_n^{\mathrm{N}}(p)$, $\lambda_n^{\mathrm{D}}(p) < \lambda_n^{\mathrm{N}}(p)$, and $\lambda_n^{\mathrm{D}}(p) > \lambda_n^{\mathrm{N}}(p)$? We only know that for $p = 2$ and any $n \in \mathbb{N}$, the equality holds true.

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## References

[1] J. Benedikt, *On simplicity of spectra of p-biharmonic equations*, to appear in Nonlinear Anal.

[2] ———, *Uniqueness theorem for p-biharmonic equations*, Electron. J. Differential Equations **2002** (2002), no. 53, 1–17.

[3] T. Boggio, *Sulle funzioni di Green d'ordine m* [*On the mth-order Green function*], Rend. Circ. Mat. Palermo **20** (1905), 97–135 (Italian).

[4] P. Drábek and M. Ôtani, *Global bifurcation result for the p-biharmonic operator*, Electron. J. Differential Equations **2001** (2001), no. 48, 1–19.

[5] P. Hartman, *Ordinary Differential Equations*, S. M. Hartman, Maryland, 1973.

[6] P. Jentzsch, *Über Integralgleichungen mit positivem Kern* [*On integral equations with positive kernel*], J. reine angew. Math. **141** (1912), 235–244 (German).

[7] A. Kratochvíl and J. Nečas, *The discreteness of the spectrum of a nonlinear Sturm-Liouville equation of fourth order*, Comment. Math. Univ. Carolinae **12** (1971), 639–653 (Russian).

Jiří Benedikt: Centre of Applied Mathematics, University of West Bohemia, Univerzitní 22, 306 14 Plzeň, Czech Republic

*E-mail address*: benedikt@kma.zcu.cz