SEMICLASSICAL FUNDAMENTAL SOLUTIONS

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It is the aim of this paper to show how the classical theory, based on fundamental solutions and explicit representations, via special functions can be combined with the functional analytical approach to partial differential equations, to produce semiclassical representation formulae for the solution of equations in cylinder-like domains.

1. Introduction

Partial differential equations (PDEs) in unbounded domains are often encountered in practical applications. If one considers the special case of cylindrical domains

$$\mathbb{R}^m \times \Omega \tag{1.1}$$

for $\Omega \subset \mathbb{R}^n$ open and bounded, $m, n \in \mathbb{N}$, then there are essentially two ways to analyze boundary value problems (BVPs) in such domains which can be found in the literature. On the one hand, we have the classical approach of looking for a fundamental solution, that is, a distribution

$$G \in \mathcal{D}'(\mathbb{R}^m \times \Omega) \tag{1.2}$$

such that PDO $G = \delta$ for a given partial differential operator PDO. If a fundamental solution can be found, then a solution of PDO, $u = f$, can be produced in the form

$$u = G * f \tag{1.3}$$

whenever the convolution makes sense. If boundary conditions are imposed, one needs to modify the approach in order to be able to take care of them, but the basic principle remains the same. On the other hand, we have a fully abstract approach based on operator sums. Letting PDO_1 and PDO_2 be the differential operators which act on the first m variables and on the second n ones, respectively, one tries to construct the solution operator $(\mathrm{PDO}_1 + \mathrm{PDO}_2)^{-1}$ by using functional calculus formulae. Choosing an integral representation based on the resolvents of the two operators, one can obtain results of

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Da Prato-Grisvard type (see [7]), whereas using an imaginary powers integral representation one can produce Dore-Venni-type results (see [8]).

We choose a mixed approach which combines the classical approach with the functional analytical one. The basic idea is to treat the unbounded directions as in the classical theory and to condense the problem in the bounded part of the domain into an abstract operator. To implement this idea, we will need to introduce the concept of operator-valued fundamental solution which is a natural extension of the classical definition. In doing so, we are able to integrate the explicit formulae provided by the classical theory and the abstract operator formulations for BVPs in bounded domains. The result are semiabstract formulae which give a more suggestive and clear insight into the structure of the solution operators to many initial and BVPs. The proposed approach has the advantage of applying equally well to elliptic, parabolic, and hyperbolic problems.

The main results are general representation formulae for the solution of various PDEs and are presented in Theorems 3.1, 3.2, 3.3, 3.4, and 3.6. The tools used in our analysis are the theory of vector-valued distributions as developed in [13] combined with recent results concerning operator-valued Fourier multiplier theorems obtained by [5] and functional calculus results for sectorial and selfadjoint operators. It should be observed that a variety of generalizations for Fourier multipliers theorems to the vector-valued case have been obtained in recent years, see, for instance, [6, 16, 17]. We choose the one in [5] because it does not rest on the assumption that underlying function spaces are UMD. Function spaces of classical regularity are therefore not ruled out.

The paper is organized as follows. In Section 2, we give a brief review of results about vector-valued distribution theory and introduce the concept of generalized fundamental solution. In Section 3, we analyze prototype elliptic, parabolic, and hyperbolic equations to illustrate the simplicity and the strength of the approach.

Although we only consider the linear case, applications to the nonlinear case are possible; see, for instance, [10], where this approach is implicitly followed to analyze singular elliptic BVPs arising in the analysis of a certain degenerate free boundary problem.

2. Generalized fundamental solutions

In this section, we will give a very brief survey of the main concepts and results concerning vector-valued distributions. We will avoid the core issues encountered when attempting to generalize the concept of distribution to the nonscalar case and refer the interested reader to [2, 13]. Since we are especially interested in working with distributions in connection with the Fourier transform, we will limit our attention to tempered distributions.

As in the scalar case, one defines the space of rapidly decreasing test functions by

$$\left\{\varphi \in \mathrm{C}^{\infty}\left(\mathbb{R}^m, E\right) \mid \forall k, j \in \mathbb{N} \sup_{x \in \mathbb{R}^m, |\alpha| \leq k} \left(1+|x|^2\right)^{j/2} \left|\partial^\alpha \varphi(x)\right| \leq c_{k,j} < \infty\right\}, \tag{2.1}$$

where E is a Banach space. For our purposes, we will choose E to be either a space of functions or of linear and continuous operators between function spaces. The seminorms

$$q_{k,j}(\varphi) = \sup_{x \in \mathbb{R}^m, |\alpha| \leq k} \left|\partial^\alpha \varphi(x)\right| \tag{2.2}$$

generate the standard locally convex topology of $\mathcal{S}(\mathbb{R}^m,E)$. If we denote by $\mathcal{S}'(\mathbb{R}^m) = \mathcal{L}(\mathcal{S}(\mathbb{R}^m),\mathbb{K})$ the space of scalar tempered distributions, then the corresponding space of vector-valued distributions is given by

$$\mathcal{S}'(\mathbb{R}^m,E) = \mathcal{L}(\mathcal{S}(\mathbb{R}^m),E). \tag{2.3}$$

Now, as in the scalar case, we have that

$$\left[\varphi \longmapsto \int_{\mathbb{R}^m} f(x)\varphi(x)dx\right] : \mathcal{S}(\mathbb{R}^m) \longrightarrow E \tag{2.4}$$

defines a tempered distribution for any locally integrable E-valued function. It follows that

$$\mathrm{L}_{1,\mathrm{loc}}(\mathbb{R}^m,E) \hookrightarrow \mathcal{S}'(\mathbb{R}^m,E), \tag{2.5}$$

and any distribution which can be represented by a locally integrable function is called *regular*. The Fourier transform is given by

$$\mathcal{F}\varphi = \hat{\varphi} = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^m} e^{-ix\xi}\varphi(x)dx \tag{2.6}$$

for any test function $\varphi \in \mathcal{S}(\mathbb{R}^m,E)$ and by

$$\langle \mathcal{F}u,\varphi\rangle = \langle \hat{u},\varphi\rangle = \langle u,\hat{\varphi}\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^m,E) \tag{2.7}$$

for any tempered distribution $u \in \mathcal{S}'(\mathbb{R}^m,E)$. It can be proven that

$$\mathcal{F} \in \mathcal{GL}(\mathcal{S}(\mathbb{R}^m,E)) \cap \mathcal{GL}(\mathrm{L}_2(\mathbb{R}^m,E)) \cap \mathcal{GL}(\mathcal{S}'(\mathbb{R}^m,E)). \tag{2.8}$$

In [5], vector-valued Besov spaces

$$\mathcal{S}(\mathbb{R}^m,E) \hookrightarrow \mathrm{B}^s_{p,q}(\mathbb{R}^m,E) \hookrightarrow \mathcal{S}'(\mathbb{R}^m,E) \tag{2.9}$$

for $s \in \mathbb{R}$ and $p,q \in [1,\infty]$ are introduced via diadic resolutions of the identity in Fourier space. We refer the reader to the cited paper for the details. The scala of Besov spaces comprises several special function spaces. For instance, classical spaces of Hölder continuous functions

$$\mathrm{BUC}^{k+\alpha}(\mathbb{R}^m,E) = \left\{u \in \mathrm{BUC}^k(\mathbb{R}^m,E) \mid [\partial^\alpha u]_\alpha = \sup_{x\neq y}\frac{\|\partial^\alpha u(x) - \partial^\alpha u(y)\|_E}{|x-y|^\alpha} < \infty,\ |\alpha| = k\right\}, \tag{2.10}$$

where $k \in \mathbb{N}$ and $\alpha \in (0,1)$, are obtained from the general scala setting $s = k+\alpha$ and $p = q = \infty$. The intrinsic norm

$$\|\cdot\|_{\mathrm{BUC}^{k+\alpha}} = \sup_{|\alpha|\leq k}\|\partial^\alpha\cdot\|_\infty + \sup_{|\alpha|=k}[\partial^\alpha\cdot] \tag{2.11}$$

is equivalent to the Besov norm. Other spaces which fall into the Besov category are for instance Sobolev-Slobodeskii spaces

$$\mathrm{W}_p^s(\mathbb{R}^m,E)$$
$$=\left\{u\in\mathrm{W}_p^{[s]}(\mathbb{R}^m,E)\mid[\partial^\alpha u]_{s-[s]}=\int_{\mathbb{R}^m\times\mathbb{R}^m}\frac{\|\partial^\alpha u(x)-\partial^\alpha u(y)\|_E^p}{|x-y|^{m+(s-[s])p}}d(x,y)<\infty,\ |\alpha|=[s]\right\}. \tag{2.12}$$

They correspond to choosing $s\in\mathbb{R}^+\setminus\mathbb{N}$ and $p=q\in(1,\infty)$ and their intrinsic integral norms

$$\|\cdot\|_{\mathrm{W}_p^s}=\left(\sum_{|\alpha|\le[s]}\|\partial^\alpha\cdot\|_{\mathrm{L}_p}^p+\sum_{|\alpha|=[s]}[\partial^\alpha\cdot]_{s-[s]}^p\right)^{1/p} \tag{2.13}$$

are equivalent to the Besov ones. We recall that $[s]=\max_{k\in\mathbb{N}}\{k\le s\}$ and that $\mathrm{W}_p^{[s]}(\mathbb{R}^m,E)$ are the classical Sobolev spaces. In analyzing the mapping properties of generalized fundamental solutions below, we will at times make use of a Fourier multiplier theorem by Amann [5]. To formulate the theorem, we need to introduce multiplier spaces first. Let E and F be Banach spaces and $\mathcal{L}(E,F)$ the space of linear and continuous operators from E into F. Define

$$S^k(\mathbb{R}^m,\mathcal{L}(E,F))$$
$$=\left\{a\in\mathrm{C}^{m+1}(\mathbb{R}^m\setminus\{0\},\mathcal{L}(E,F))\mid\|\partial^\alpha a(x)\|_{\mathcal{L}(E,F)}\le c(1+|x|)^{k-|\alpha|},\ x\neq0,\ |\alpha|\le m+1\right\} \tag{2.14}$$

for $k\in\mathbb{R}$.

Theorem 2.1. *Let $k\in\mathbb{R}$. Then,*

$$a(D)=\mathcal{F}^{-1}a\mathcal{F}\in\mathcal{L}\left(\mathrm{B}_{p,q}^{s+k}(\mathbb{R}^m,E),\mathrm{B}_{p,q}^{s}(\mathbb{R}^m,F)\right) \tag{2.15}$$

for $s\in\mathbb{R}$ and $p,q\in[1,\infty]$ whenever $a\in S^k(\mathbb{R}^m,\mathcal{L}(E,F))$.

This theorem is a special case of [5, Theorem 6.2]. We are now ready for the definition of generalized fundamental solutions.

Definition 2.2. Let $\mathrm{P}=\sum_{|\alpha|\le k}p_\alpha\partial^\alpha$ be a differential operator on $\mathbb{R}^m$ with coefficients $a_\alpha\in\mathcal{L}(E,F)$. Then, a distribution

$$G\in\mathcal{S}'(\mathbb{R}^m,\mathcal{L}(F,E)) \tag{2.16}$$

is called a *fundamental solution* for P if and only if

$$\mathrm{P}G=\mathbf{1}_{\mathcal{L}(E,E)}\delta_x. \tag{2.17}$$

Now, a convolution can be introduced for vector-valued functions and distributions spaces based on continuous multiplication operators for the underlying image spaces. In our situation, we choose the multiplication given by

$$\mathscr{L}(E,F)\times E \longrightarrow F, \qquad (A,x)\longmapsto Ax. \tag{2.18}$$

Then, it is a consequence of [5, Theorem 3.1] that the convolution operator

$$\mathscr{S}'(\mathbb{R}^m,\mathscr{L}(E,F))\times\mathscr{S}(\mathbb{R}^m,E)\longrightarrow \mathrm{C}^\infty(\mathbb{R}^m,F), \qquad (u,\varphi)\longmapsto u*\varphi \tag{2.19}$$

is bilinear and hypocontinuous (which means continuous in both variables, uniformly with respect to each variable separately if the other is restricted to a bounded subset). If u satisfies integrability properties, then the convolution is realized as an integral

$$(u*\varphi)(x)=\int_{\mathbb{R}^m}u(x-\tilde{x})\varphi(\tilde{x})\,d\tilde{x}. \tag{2.20}$$

As in the scalar case, once the existence of a fundamental solution

$$G\in\mathscr{S}'(\mathbb{R}^m,\mathscr{L}(F,E)) \tag{2.21}$$

has been established, one can produce solutions for

$$\mathrm{P}u=f\in\mathscr{S}'(\mathbb{R}^m,F) \tag{2.22}$$

by convolution $u=G*f$, whenever the convolution makes sense. Furthermore, if the Fourier transform of the fundamental solution $\widehat{G}$ can be computed, Theorem 2.1 allows us to analyze its mapping properties. In the next section, we will consider some particular situations in which the above anisotropic fundamental solution approach can be applied directly or indirectly. We will give examples in the elliptic, parabolic, and hyperbolic cases.

3. Applications to linear problems

Many results in nonlinear functional analysis are obtained through a combination of fine linear results and perturbation lemmas which show how to reduce the nonlinear problem to a linear one, at least locally. Green functions in the elliptic case and the variation of constants formula for parabolic problems are only two but very prominent examples. In this spirit, we present the following formulae based on operator-valued fundamental solutions which have the advantage of applying equally well to elliptic, parabolic, and hyperbolic problems. In the parabolic case, they coincide with those obtained by semigroup theory, whereas for the other two classes of problems they give new insightful representations for their solutions and solution operators. In [11], we investigate concrete situations in which operator-valued fundamental solutions can be successfully employed, such as validity of maximum principles, large space behavior, singular perturbations, and the problem of imposing appropriate boundary conditions on numerically introduced artificial boundaries.

3.1. Elliptic boundary value problems. We first consider the case of elliptic BVPs on cylinder-like domains and illustrate how to use the Fourier transform and Dunford functional calculus techniques to obtain generalized fundamental solutions. We illustrate the idea with the help of a general second-order problem and a special fourth-order one. We choose to work in classical spaces of bounded and uniformly Hölder continuous functions and in Sobolev-Slobodeskii spaces. We therefore set

$$\mathrm{E}_p^s(\Omega) = \begin{cases} \mathrm{W}_p^s(\Omega), & 1 < p < \infty, \\ \mathrm{BUC}^s(\Omega), & p = \infty, \end{cases} \tag{3.1}$$

for $s \in \mathbb{R}^+$ if $p \in (1,\infty)$ and $s \in \mathbb{R}^+ \setminus \mathbb{N}$ if $p = \infty$. Consider the uniformly strongly elliptic BVP

$$\begin{aligned} \mathcal{A}(y,\partial)u &= f \quad \text{in } \Omega, \\ \mathcal{B}(y,\partial)u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{3.2}$$

for differential and boundary operators given by

$$\begin{aligned} \mathcal{A}(y,\partial)u &= -\operatorname{div}(\Lambda\nabla u + bu) + (c|\nabla u) + du, \\ \mathcal{B}(y,\partial)u &= (1-\delta)u + \delta\left(\partial_{\nu_\Lambda} u + \left[(\gamma_\partial b|\nu) + \beta_0\right]\gamma_\partial u\right). \end{aligned} \tag{3.3}$$

Assume for simplicity that all data are smooth and that $\delta \in \mathrm{C}(\partial\Omega,\{0,1\})$. Then, $\partial\Omega$ has disconnected components $\Gamma_j = \delta^{-1}(j)$, $j = 0,1$. The vector ν is the unit outward normal to $\partial\Omega$, $\partial_{\nu_\Lambda} u = (\nabla u|\Lambda\nu)$ is the conormal derivative of u and γ_∂ is the trace operator. The uniform ellipticity assumption means that there exists $\underline{\alpha} > 0$ with

$$(\Lambda(y)\eta|\eta) \geq \underline{\alpha}|\eta|^2 \quad \text{for any } \eta \in \mathbb{R}^m,\ y \in \Omega. \tag{3.4}$$

Classical regularity results imply

$$(\mathcal{A},\mathcal{B}) \in \mathcal{GL}\left(\mathrm{E}_p^{2+s}(\Omega), \mathrm{E}_p^s(\Omega) \times E_p^{2+s-1/p}(\Gamma_0) \times \mathrm{E}_p^{1+s-1/p}(\Gamma_1)\right), \tag{3.5}$$

where we have set $1/\infty = 0$ and defined the boundary spaces in the natural way. These facts can be found in [3, 12]. An abstract operator A on $\mathrm{L}_p(\Omega)$ for $p \in (1,\infty)$ can be associated to $(\mathcal{A},\mathcal{B})$ with

$$\operatorname{dom}(A) = \mathrm{E}_{p,\mathcal{B}}^{2+s}(\Omega) = \left\{u \in \mathrm{E}_p^{2+s}(\Omega) \,\middle|\, \mathcal{B}u = 0\right\} \tag{3.6}$$

and $Au = \mathcal{A}u$ for $u \in \operatorname{dom}(A)$, where we choose $s = 0$ if $p \in (1,\infty)$ and $s \in (0,1)$ if $p = \infty$. It can be proved that the operator A is sectorial, see [3, 12], where a definition of sectorial operator can also be found. If $\delta \neq 1$ or, if $\delta = 1$ but $\beta_0 \neq 0$, we have that $0 \in \rho(A)$. We assume that this is the case from now on. The case of inhomogeneous boundary conditions can be reduced to the homogeneous case by trace theorems (extension theorems), as it is sketched in [11] or in [3].

We are now ready to analyze elliptic BVPs in cylinder-like domains. Consider

$$\begin{aligned} -\triangle_x u + \mathcal{A}(y,\partial)u &= f \quad \text{in } \mathbb{R}^m \times \Omega, \\ \mathcal{B}(y,\partial)u &= 0 \quad \text{on } \mathbb{R}^m \times \partial\Omega, \end{aligned} \tag{3.7}$$

which in view of the above sketch can be reformulated as an abstract elliptic problem

$$-\triangle u + Au = f \quad \text{in } \mathbb{R}^m \tag{3.8}$$

in $\mathrm{E}_p^s(\Omega)$ which fits into the framework of generalized fundamental solutions. Either by taking a Fourier transform or, directly by analogy to the scalar case, a fundamental solution can be found. The following theorem deals with dimensions 1, 2, and 3. Analogous formulae are valid in higher dimensions.

Theorem 3.1. *A generalized fundamental solution can be found for $-\triangle + A$ and is given by*

$$G_m(x,A) = \begin{cases} \dfrac{1}{2}A^{-1/2}e^{-|x|A^{1/2}}, & m=1, \\ K_0\left(|x|A^{1/2}\right), & m=2, \\ \dfrac{1}{|x|}e^{-|x|A^{1/2}}, & m=3, \end{cases} \quad x \in \mathbb{R}^m. \tag{3.9}$$

A proof of this result which is based on the Dunford functional calculus for sectorial operators can be found in [11]. It is interesting to observe that the fundamental solution can be understood in terms of an analytic function of the pseudodifferential operator $\sqrt{A}$. In odd dimensions, even in terms of the semigroup generated by $\sqrt{A}$. Based on Theorem 2.1, it is also easy to obtain associated regularity results in anisotropic spaces.

Theorem 3.2.

$$\begin{aligned} &\left[f \longmapsto G_m(\cdot,A) * f\right] \\ &\quad \in \mathcal{L}_{is}\left(\mathrm{B}_{q,r}^t\left(\mathbb{R}^m, E_p^s(\Omega)\right), \mathrm{B}_{q,r}^{t+2}\left(\mathbb{R}^m, E_p^s(\Omega)\right) \cap \mathrm{B}_{q,r}^t\left(\mathbb{R}^m, E_{p,\mathcal{B}}^{s+2}(\Omega)\right)\right). \end{aligned} \tag{3.10}$$

Proof. We know that $\widehat{G}_m(\cdot,A) = (\xi^2 + A)^{-1}$. Our assumptions for BVP (3.2) entail that the operator A is sectorial. In particular, we have

$$\begin{aligned} \left|\left|\left(\xi^2 + A\right)^{-1}\right|\right|_{\mathcal{L}(E_p^s(\Omega), E_p^{s+2}(\Omega))} &\le c, \\ \left|\left|\left(1+\xi^2\right)\left(\xi^2 + A\right)^{-1}\right|\right|_{\mathcal{L}(E_p^s(\Omega))} &\le c. \end{aligned} \tag{3.11}$$

Moreover,

$$\partial_\xi^\alpha\left(\xi^2 + A\right)^{-1} = p_\alpha(\xi)\left(\xi^2 + A\right)^{-|\alpha|-1} \tag{3.12}$$

for any $\alpha \in \mathbb{N}^m$ and some polynomial p_α of order at most $|\alpha|$. Theorem 2.1 therefore applies and we obtain the desired regularity results. □

To further illustrate the applicability of ideas, we consider a fourth-order BVP in an infinite strip. The BVP reads

$$\begin{aligned} \triangle^2 u = f(x,y) &\quad \text{in } \mathbb{R} \times (0,1) \ni (x,y), \\ u = \partial_y u = 0 &\quad \text{on } \mathbb{R} \times \{0\}, \\ u = \partial_y u = 0 &\quad \text{on } \mathbb{R} \times \{1\}. \end{aligned} \tag{3.13}$$

Again, we can "pack" the part of the operator acting on the bounded direction into an abstract operator

$$A = \partial_{yyyy} : \operatorname{dom}(A) \subset E_p^s(0,1) \longrightarrow E_p^s(0,1) \tag{3.14}$$

defined through

$$\operatorname{dom}(A) = \{u \in E_{p,\mathcal{B}}^{4+s}(0,1) \mid u(j) = u_y(j) = 0,\ j = 0,1\} \tag{3.15}$$

and look for a solution satisfying the abstract (infinite-dimensional) elliptic problem

$$(\partial_{xxxx} - 2\sqrt{A}\partial_{xx} + A)u = f(x) \tag{3.16}$$

for the vector-valued function $u : \mathbb{R} \to E_p^s(0,1)$. As in the second-order case, we choose $s = 0$ if $p \in (0,\infty)$ and $s \in (0,1)$ otherwise. Dunford-Schwartz operator calculus and Fourier transform allow us to explicitly compute the fundamental solution in this case too.

Theorem 3.3. *The abstract elliptic problem (3.16) possesses an operator-valued fundamental solution $G \in \mathrm{L}_1(\mathbb{R}, \mathcal{L}(\mathrm{L}_p(0,1)))$ given by*

$$G(x,A) = \frac{1}{2}A^{-1/2}(|x| + A^{-1/4})e^{-|x|A^{1/4}}, \quad x \in \mathbb{R}. \tag{3.17}$$

Proof. The fourth root of the sectorial operator A can be defined by functional calculus and G can be interpreted by means of the semigroup generated by $-A^{1/4}$. A direct calculation then shows that it is indeed a fundamental solution. Alternatively, the use of the Fourier transform allows us to relate the operator-valued case to the underlying scalar one. In fact,

$$\begin{aligned} 2G(x,A) &= A^{-1/2}(|x| + A^{-1/4})e^{-|x|A^{1/4}} \\ &= \int_\Gamma \lambda^{-1/2}(|x| + \lambda^{-1/4})e^{-|x|\lambda^{1/4}}(\lambda - A)^{-1}d\lambda \\ &= \int_\Gamma \mathcal{F}^{-1}(\xi^2 + \lambda)^{-2}(\lambda - A)^{-1}\mathcal{F}\,d\lambda = \mathcal{F}^{-1}(\xi^2 + A)^{-2}\mathcal{F}, \end{aligned} \tag{3.18}$$

where the first integral converges since A is an invertible sectorial operator and the scalar factor is exponentially decaying along any path of integration avoiding (enclosing) the spectrum of A. The second equality is valid since the Fourier transform is continuous, for instance on $\mathrm{L}_1(\mathbb{R}, \mathrm{L}_p(0,1))$, and because of the smooth dependence of the functional calculus on the variable $x \in \mathbb{R}$ (cf. [4, Lemma 4.1.1]). In the last equality, we made use of the fact that the Fourier transform of the scalar factor in the integrand is known, see, for instance, [9, Chapter 17]. □

The solution can therefore be understood in terms of semigroups generated by first-order pseudodifferential operators and their resolvents. This shows how "the different directions" contribute to the full solution operator. Using Theorem 2.1, it is possible to characterize the anisotropic mapping properties of the associated convolution operator.

THEOREM 3.4. *The following holds:*

$$\begin{aligned}&\big[f \longmapsto G(\cdot,A) * f\big]\\&\quad\in \mathcal{L}is\big(\mathrm{B}^{t}_{q,r}\big(\mathbb{R},E^{s}_{p}(0,1)\big),\mathrm{B}^{t+4}_{q,r}\big(\mathbb{R},E^{s}_{p}(0,1)\big)\cap \mathrm{B}^{t}_{q,r}\big(\mathbb{R},E^{4+s}_{p,\mathcal{B}}(0,1)\big)\big).\end{aligned}\tag{3.19}$$

Proof. The proof follows from an easy estimate of the polynomial symbol $(\xi^2+A)^{-1}$ since, again, A is a sectorial operator on $\mathrm{L}_p(0,1)$. □

Remark 3.5. It should be observed that the availability of a representation formula for the fundamental solution allows for another, direct way to analyze its mapping properties. Thus, on the one hand, one does not really need to resort to Fourier multiplier theorems to obtain regularity results and on the other, one also gains insight into the qualitative behavior of the solution operator.

3.2. Parabolic equations. Based on the properties of the underlying elliptic operators, it is possible to obtain existence, uniqueness, and regularity results in anisotropic spaces for parabolic evolution equations as well. As the example will show, this approach allows one to work in space of mixed classical and generalized regularity, which can be advantageous in dealing with nonlinear problems.

THEOREM 3.6. *Let $\rho \in (0,1)$ and assume that*

$$\begin{aligned}&f \in \mathrm{C}^{\rho}\big(J,buc^{s}(\mathbb{R}^n,\mathrm{L}_p(\Omega))\big),\\&g \in \mathrm{C}^{1+\rho}\big(J,buc^{s}(\mathbb{R}^n,W^{2-1/p}_{p}(\Gamma_0)\times W^{1-1/p}_{p}(\Gamma_1))\big)\\&\qquad\cap \mathrm{C}^{\rho}\big(J,buc^{s+2}(\mathbb{R}^n,W^{2-1/p}_{p}(\Gamma_0)\times W^{1-1/p}_{p}(\Gamma_1))\big),\\&u_0 \in buc^{s}(\mathbb{R}^n,\mathrm{L}_p(\Omega)).\end{aligned}\tag{3.20}$$

Then, there exists a unique solution u of

$$\begin{gathered}u_t-\triangle u+\mathcal{A}u=f,\quad (t,x,y)\in(0,\infty)\times\mathbb{R}^n\times\Omega,\\\mathcal{B}u=g,\quad (t,x,y)\in(0,\infty)\times\mathbb{R}^n\times\partial\Omega,\\u(0)=u_0\end{gathered}\tag{3.21}$$

with

$$\begin{aligned}&u\in \mathrm{C}^{\rho}\big(J\setminus\{0\},buc^{s+2}(\mathbb{R}^n,\mathrm{L}_p(\Omega))\cap buc^{s}(\mathbb{R}^n,W^2_p(\Omega))\big)\\&\qquad\cap \mathrm{C}^{1+\rho}\big(J\setminus\{0\},buc^{s}(\mathbb{R}^n,\mathrm{L}_p(\Omega))\big).\end{aligned}\tag{3.22}$$

Proof. The proof is an immediate consequence of [11, Main Theorem 3] and of [4, Theorem 1.2.1]. □

The solution of the above initial BVP can be represented in terms of a generalized fundamental solution which can be easily computed since the spatial operators commute. It is given by

$$G(t,x)=H(t)e^{-|x|^2/4t}e^{-tA},\tag{3.23}$$

where H is the Heaviside function and e^{-tA} is the semigroup generated by the operator $-A$. In this case, apart from the fact that we can consider anisotropic spaces, the presented approach does not lead to any insights which cannot already be obtained by semigroup theory.

3.3. Hyperbolic equations. We now consider a specific instance of a hyperbolic equation which shows that our approach is capable of producing solution formulae revealing more of the solution structure than a purely abstract (semigroup) approach. It also retains some of the transparency of the abstract approach as opposed to the fully concrete classical approach based on special functions. Consider the wave equation in a tube-like domain

$$\begin{aligned} u_{tt} - \partial_{xx}u - \triangle_y u &= f \quad \text{in } \mathbb{R}\times\mathbb{R}\times\Omega \ni (t,x,y), \\ u &= 0 \quad \text{in } \mathbb{R}\times\mathbb{R}\times\partial\Omega, \\ u(0,\cdot) &= u_0 \quad \text{in } \mathbb{R}\times\Omega, \\ u_t(0,\cdot) &= u_1 \quad \text{in } \mathbb{R}\times\Omega. \end{aligned} \tag{3.24}$$

In the classical approach, the role of a fundamental solution is played by the Riemann function (see, e.g.,[15, page 221]) which solves the system for the data $(f,u_0,u_1) = (0,0,\delta_{(x,y)})$.

In this case, we take a Hilbert space approach working in L_2 and taking advantage of the functional calculus for selfadjoint operators (see, e.g., [14, Chapter 5]). We need to resort to this Hilbert space only functional calculus, because of the well-known lack of regularization of the wave equation and the possibility of exactly characterizing Fourier multipliers in a Hilbert space setting. We go through the same steps as before. Firstly, we reformulate the equation as an abstract equation in $L_2(\Omega)$, which allows us to incorporate the boundary condition into the function space. Let $A : \mathrm{dom}(A) \subset L_2(\Omega) \to L_2(\Omega)$ be defined by

$$\mathrm{dom}(A) = \{u \in H^2(\Omega) \mid \gamma_{\partial\Omega}u = 0\}, \quad Au = -\triangle_y u \quad \text{for } u \in \mathrm{dom}(A) \tag{3.25}$$

and rewrite the above system as the following $L_2(\Omega)$-valued wave equation on $\mathbb{R}$ for $u : \mathbb{R}\times\mathbb{R} \to L_2(\Omega)$:

$$\begin{aligned} u_{tt} - \partial_{xx}u + Au &= f \quad \text{in } \mathbb{R}\times\mathbb{R} \ni (t,x), \\ u(0,\cdot) &= u_0 \quad \text{in } \mathbb{R}, \\ u_t(0,\cdot) &= u_1 \quad \text{in } \mathbb{R}. \end{aligned} \tag{3.26}$$

A generalized Riemann function R_t would now be a distribution satisfying this equation for $(f,u_0,u_1) = (0,0,\mathbf{1}_{\mathcal{L}(L_2(\Omega))}\delta_x)$. At this point, we take a vector-valued Fourier transform in the x-direction to obtain the equivalent system

$$\begin{aligned} \hat{u}_{tt} + (\xi^2 + A)\hat{u} &= \hat{f} \quad \text{in } \mathbb{R}\times\mathbb{R} \ni (t,x), \\ \hat{u}(0,\cdot) &= \hat{u}_0 \quad \text{in } \mathbb{R}, \\ \hat{u}_t(0,\cdot) &= \hat{u}_1 \quad \text{in } \mathbb{R}, \end{aligned} \tag{3.27}$$

from which the solution $\widehat{R}_t$ can be easily computed by semigroup theory,

$$\widehat{R}_t = (\xi^2 + A)^{-1/2} \sin\left(t\sqrt{\xi^2 + A}\right). \tag{3.28}$$

Here, we of course use the calculus for selfadjoint operators to make sense of the formula and to lift the formula for the inverse Fourier transform for $\widehat{R}_t$ from the scalar to the vector-valued situation. In fact, we have

$$\widehat{R}_t = \int_{\sigma(A)} (\xi^2 + \lambda)^{-1/2} \sin\left(t\sqrt{\xi^2 + \lambda}\right) dP(\lambda) \tag{3.29}$$

for the spectral measure associated to A (see, e.g., [14]).

Remark 3.7. Since we are considering the case of Ω bounded here, the spectral representation can actually be rewritten as

$$\widehat{R}_t = \sum_{k=1}^{\infty} (\xi^2 + \lambda_k)^{-1/2} \sin\left(t\sqrt{\xi^2 + \lambda_k}\right) (\,\cdot\,|\varphi_k)_{L_2(\Omega)} \tag{3.30}$$

in terms of the eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ and associated eigenfunctions $(\varphi_k)_{k\in\mathbb{N}}$ of A. Formula (3.29), however, shows that the result remains valid on more general assumptions about Ω.

By continuity of the functional calculus for selfadjoint operators, the Fourier transform can be taken inside the integral in (3.29). This gives the following theorem.

Theorem 3.8. *The Riemann function for (3.26) is given by*

$$R_t(x, A) = \begin{cases} J_0\left(\sqrt{A}\,(t^2 - |x|^2)^{1/2}\right), & |x| < t, \\ 0, & \text{otherwise.} \end{cases} \tag{3.31}$$

The general solution can consequently be written as

$$u(t,x) = \frac{d}{dt}\left(R_t(\cdot, A)\right) *_x u_0 + R_t(\cdot, A) *_x u_1 + \int_0^t R_{t-\tau}(\cdot, A) *_x f(\tau)\,d\tau, \tag{3.32}$$

where J_0 is the Bessel function of first kind of order 0 *(see* [1]*). Since $J_0' = -J_1$, for the Bessel function J_1 of order* 1, *$(d/dt)(R_t(\cdot, A))$ can also be represented by classical functions.*

It should be pointed out that a pure semigroup approach would have led to the introduction of the cosine and sine families generated by the full spatial operator. In the corresponding representation formula for the solution, though equivalent, the light cone would not appear. The above formula retains both the concision and the transparency of the semigroup representation without giving up all the details of an explicit representation via classical functions. This is very important since a direct approach gives us quantitative insight into the solution, whereas the functional analytical one leads to a better understanding of the mapping properties of the solution operator.

In this particular case, we can use the representation formula (3.31) to obtain the following physically interesting solution.

Remark 3.9. Define the light cone L by

$$L := \{(t,x) \in \mathbb{R} \times \mathbb{R} \mid |x| \leq t\} \tag{3.33}$$

and consider the solution u of (3.26) with initial datum

$$(u_0, u_1) = (0, \delta_x \otimes f) \tag{3.34}$$

for $f \in \mathrm{L}_2(\Omega)$ and where δ_x is the Dirac distribution supported at $x = 0$. Then, by (3.31) and (3.32), u is given by

$$u(t,x,\cdot) = \begin{cases} J_0\left(\sqrt{A}\left(t^2 - |x|^2\right)^{1/2}\right) f, & (t,x) \in L, \\ 0, & \text{otherwise.} \end{cases} \tag{3.35}$$

Since $J_0(0) = 1$, it can be seen from (3.29) or (3.30) that

$$[s \longmapsto J_0(s\sqrt{A})] \tag{3.36}$$

is strongly continuous, that is, for every $f \in \mathrm{L}_2(\Omega)$,

$$[s \longmapsto J_0(s\sqrt{A}) f] \in \mathrm{C}([0,\infty), \mathrm{L}_2(\Omega)). \tag{3.37}$$

Notice that convergence in (3.30) is ensured by

$$(f \mid \varphi_k)_{k \in \mathbb{N}} \in l_2(\mathbb{N}). \tag{3.38}$$

Moreover, $J_0(0\sqrt{A}) = \mathrm{id}_{\mathrm{L}_2(\Omega)}$. We therefore obtain

$$[(t,x) \longmapsto u(t,x)] \in \mathrm{C}(L, \mathrm{L}_2(\Omega)), \quad u(t, \pm t) = f. \tag{3.39}$$

Thus, the initial impulse f produces a wave which appears of shape f propagating down the wave guide.

Taking into account the decay properties of the Bessel function

$$|J_0(s)| \leq \frac{c}{s^{1/2}}, \quad s \geq 0, \tag{3.40}$$

we also obtain some regularization in the interior of the light cone, that is,

$$[(t,x) \longmapsto u(t,x)] \in \mathrm{C}\left(\mathring{L}, \mathrm{H}^{1/2-\varepsilon}(\Omega)\right) \tag{3.41}$$

for any $0 < \varepsilon \leq 1/2$. This follows from (3.30) and the fact that

$$\operatorname{dom}(A^\alpha) \subset \mathrm{H}^{2\alpha}(\Omega), \quad \alpha \in (0,1). \tag{3.42}$$

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