WEAK FORMULATION OF SINGULAR DIFFERENTIAL EXPRESSIONS IN SPACES OF FUNCTIONS WITH MINIMAL DERIVATIVES

M. A. El-GEBEILY

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A weak formulation for singular symmetric differential expressions is presented in spaces of functions which possess minimal differentiability requirements. These spaces are used to characterize the domains of the various operators associated with such expressions. In particular, domains of self-adjoint differential operators are characterized.

1. Introduction

Application of the general theory of self-adjoint operators to the spectral representation of operators associated with the formally self-adjoint differential expression

$$\ell u = \frac{1}{w} \sum_{k=0}^{n} (-1)^k \left(p_{n-k} u^{(k)}\right)^{(k)} \tag{1.1}$$

was carried out to a completion by many researchers in this field. A complete account of this theory can be found in [1, 11]. Account for the parallel theory of partial differential and difference operators can be found in [2, 5]. On the other hand, the differential expression (1.1) gives rise to the formal sesquilinear form

$$a(u,v) = \sum_{k=0}^{n} \int p_{n-k} u^{(k)} \overline{v}^{(k)} \tag{1.2}$$

encountered in the course of studying weak formulations of differential equations. Unlike the differential expressions, the theory behind the sesquilinear forms (1.2) is not yet fully developed. The most general treatment we have so far is for the case when such forms are semibounded or sectorial [10]. The classical Lax-Milgram theorem which is widely used in treatments involving the bilinear forms (1.2) assumes that the underlying form is positive and continuous. While such assumptions suffice to handle regular and some classes of singular differential expressions, they are not sufficient to handle the general singular expressions as they need not be semibounded. The importance of such a theory stems

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from the many important applications it would have in areas such as the calculus of variations and numerical solutions of differential equations. For some of these applications the reader is referred to the papers [3, 4, 7, 9] and the references therein.

In [6] a variational formulation of the second order differential expression

$$\ell u = \frac{1}{w}\{-(pu')' + qu\} \tag{1.3}$$

was presented in regular as well as singular cases. Although no assumptions of semiboundedness were made there, the treatment has two drawbacks. In a general setting, the presentation depended on the existence of a maximal space of definition inferred from Zorn's lemma (see [6, page 43]). The difficulty with this space is the lack of a satisfactory concrete characterization to render it useful for further development. In a more special setting, the treatment relied on more concrete spaces but they require full differentiability assumptions and thus no use is made of the reduced order of differentiation granted by the variational setting ([6, page 48]). This makes the presentation particularly unattractive if we want to devise Galerkin-like numerical methods to solve singular differential equations. These two drawbacks are eliminated in this work. We give here a weak formulation of the more general differential expression (1.1) in spaces which require differentiation properties dictated only by what is necessary for the sesquilinear form (1.2) to be meaningful. We also give full characterizations of various operators associated with the formal operator ℓ in terms of these spaces. These characterizations include the most interesting operators associated with ℓ, namely, self-adjoint operators.

This paper is organized as follows. After this introduction we give a preliminary section in which the notation and the results frequently used in this work are given. The weak formulation of the problem is done in Section 3. In this section the working spaces are defined, the variational form of the problem is set and its equivalence to the original problem is established. In Section 4 some further properties of the defined spaces are explored.

2. Preliminaries

The following notation will be used in this paper. $\mathfrak{D}(a,b)$ denotes the space of test functions on the interval (a,b), $-\infty \le a < b \le \infty$, and $\mathfrak{L}(a,b)$ its dual with respect to the following topology. Denoting by $\langle\cdot,\cdot\rangle$ the pairing between $\mathfrak{D}(a,b)$ and $\mathfrak{L}(a,b)$, a functional $f \in \mathfrak{L}(a,b)$ if and only if for each compact interval $[\alpha,\beta]$ there is a constant C and an integer $r \ge 0$ such that

$$|\langle f,v\rangle| \le C \sup_{0\le k\le r} \|v^{(k)}\|_\infty \tag{2.1}$$

for every function $v \in \mathfrak{D}(a,b)$ with support in $[\alpha,\beta]$ (C and r generally dependent on $[\alpha,\beta]$). $L^2_w(a,b)$ denotes the Hilbert space of complex-valued square integrable functions on the interval (a,b) with respect to the almost everywhere positive weight w. The inner product and norm in this space are denoted by $\langle\cdot,\cdot\rangle_w$ and $\|\cdot\|_w$, respectively. $AC^{(k)}(a,b)$ denotes the space of functions that are absolutely continuous on any compact subinterval of (a,b) together with their derivatives up to order k inclusive. $AC(a,b)$ is used in place of

$AC^{(0)}(a,b)$. $L^1_{\text{loc}}(a,b)$ denotes the space of functions which are integrable on every finite sub-interval $[\alpha,\beta]$ of (a,b). The kth classical derivative of a function u will be denoted as usual by $u^{(k)}$ whereas the notation $u^{[k]}$ will be used to denote the kth pseudo-derivative of u defined by the formulae

$$\begin{aligned} u^{[k]} &= u^{(k)} \quad \text{for } k = 1,2,\dots,(n-1); \qquad u^{[n]} = p_0 u^{(n)}; \\ u^{[n+k]} &= p_k u^{(n-k)} - \left(u^{[n+k-1]}\right)' \quad \text{for } k = 1,2,\dots,n, \end{aligned} \tag{2.2}$$

(see also [11]).

Consider the formally self-adjoint differential expression

$$\ell u = \frac{1}{w}\sum_{k=0}^{n}(-1)^k\left(p_{n-k}u^{(k)}\right)^{(k)} \tag{2.3}$$

defined on the interval (a,b), where $w>0$ almost everywhere on (a,b), the coefficient functions p_0, p_1, $\dots$, p_n are real valued and $1/p_0, p_1,\dots,p_n,w \in L^1_{\text{loc}}(a,b)$. If a, b are finite and the functions $1/p_0$, p_1, $\dots$, p_n, w are integrable on (a,b) then this expression is said to be regular, otherwise it is singular.

The expression ℓ defines the following operators in $L^2_w(a,b)$:

(1) The "maximal" operator L whose domain $\mathcal{D}$ is given by

$$\begin{gathered} \mathcal{D} = \left\{u \in L^2_w(a,b) : u^{[k]} \in AC(a,b),\ k = 1,2,\dots,(2n-1),\ \frac{1}{w}u^{[2n]} \in L^2_w(a,b)\right\}, \\ Lu = \ell u. \end{gathered} \tag{2.4}$$

Note that $\ell u = (1/w)u^{[2n]}$.

(2) The operator L_0' whose domain $\mathcal{D}_0'$ is given by

$$\begin{gathered} \mathcal{D}_0' = \{u \in \mathcal{D} : u \text{ has compact support in } (a,b)\}, \\ L_0' u = \ell u. \end{gathered} \tag{2.5}$$

(3) The "minimal" operator L_0 whose domain $\mathcal{D}_0$ is given by

$$\mathcal{D}_0 = \{u \in \mathcal{D} : [u,v]_b^a = 0\ \forall v \in \mathcal{D}\}, \tag{2.6}$$

where $[u,v]_b^a = [u,v](b) - [u,v](a)$ and $[u,v](x)$ is the Lagrange expression

$$[u,v](x) = \sum_{k=1}^{n}\left(u^{(k-1)}(x)\overline{v}^{[2n-k]}(x) - u^{[2n-k]}(x)\overline{v}^{(k-1)}(x)\right). \tag{2.7}$$

Note that (see [11]) $[u,v](a)$ and $[u,v](b)$ both exist for all $u,v \in \mathcal{D}$.

All three operators are densely defined and the following relationships hold among them

$$L_0' \subset \widetilde{L_0'} = L_0 = L^* \subset L = L_0^*, \tag{2.8}$$

where $\sim$ denotes operator closure. In particular, the operators L_0', L_0 are symmetric and the operators L_0, L are closed. For $\lambda \in \mathbb{C}$, $\operatorname{Im}(\lambda) \neq 0$, put $\mathcal{N}_\lambda = \operatorname{Ker}(L - \lambda I)$. Since the operator ℓ has real coefficients, $u \in D$ if and only if $\overline{u} \in D$ and $Lu = \lambda u$ if and only if $L\overline{u} = \overline{\lambda}\overline{u}$. The common dimension d of the spaces $\mathcal{N}_\lambda$ and $\mathcal{N}_{\overline{\lambda}}$ is called the *deficiency index* of the operator L_0. In fact, $0 \leq d \leq 2n$ and is independent of λ as long as $\operatorname{Im}(\lambda) \neq 0$. Now for a fixed $\lambda \in \mathbb{C}\backslash\mathbb{R}$, the subspaces $\mathcal{D}_0$, $\mathcal{N}_\lambda$ and $\mathcal{N}_{\overline{\lambda}}$ are linearly independent (see [8, 11]) and

$$\mathcal{D} = \mathcal{D}_0 \dotplus \mathcal{N}_\lambda \dotplus \mathcal{N}_{\overline{\lambda}}. \tag{2.9}$$

For any $u \in \mathcal{D}$ write

$$u = u_0 + u_\lambda + u_{\overline{\lambda}}, \tag{2.10}$$

where $u_0 \in \mathcal{D}_0$, $u_\lambda \in \mathcal{N}_\lambda$ and $u_{\overline{\lambda}} \in \mathcal{N}_{\overline{\lambda}}$. Then

$$Lu = L_0 u_0 + \lambda u_\lambda + \overline{\lambda} u_{\overline{\lambda}}. \tag{2.11}$$

Formula (2.9) shows that L_0 is self-adjoint if and only if $d = 0$.

Various characterizations of the domains $\widehat{\mathcal{D}}$ of self-adjoint extensions $\widehat{L}$ of the operator L_0 are given in [11] and elsewhere. We state here two characterizations which will be used in this work.

Theorem 2.1. *Any self-adjoint extension $\widehat{L}$ of the operator L_0 is characterized by a unitary transformation $U : \mathcal{N}_{\overline{\lambda}} \to \mathcal{N}_\lambda$ such that*

$$\begin{aligned} \widehat{\mathcal{D}} &= \mathcal{D}_0 \dotplus (U + I)\mathcal{N}_{\overline{\lambda}}, \\ \widehat{L}u &= L_0 u_0 + (\lambda U + \overline{\lambda} I) u_{\overline{\lambda}}. \end{aligned} \tag{2.12}$$

In other words, there is a one to one correspondence between self-adjoint extensions of L_0 and unitary transformations from $\mathcal{N}_{\overline{\lambda}}$ to $\mathcal{N}_\lambda$.

Theorem 2.2. *Suppose $\widehat{\mathcal{D}}$ is the domain of definition of a self adjoint extension $\widehat{L}$ of L_0. Then there exist functions $w_1, w_2, \ldots, w_d \in \mathcal{D}$ such that*

(1) $w_1, w_2, \ldots, w_d$ are linearly independent modulo $\mathcal{D}_0$,

(2) $[w_i, w_j]_a^b = 0$, $i, j = 1, 2, \ldots, d$,

(3) $\widehat{\mathcal{D}} = \{u \in \mathcal{D} : [u, w_j]_a^b = 0,\ j = 1, 2, \ldots, d\}$.

Conversely, for a set of functions $w_1, w_2, \ldots, w_d$ satisfying the conditions in Part 1 and 2 above, the set $\widehat{\mathcal{D}}$ defined as in Part 3 is the domain of definition of a certain self adjoint extension $\widehat{L}$ of L_0.

In what follows we summarize some results from [6] which will also be needed in this work. From now on, when we state that a complex number exists or is defined we also mean that it is finite. For functions $u, v \in AC^{(n-1)}(a, b)$, we introduce the formal sesquilinear form

$$a(u,v) = \int_a^b \sum_{k=0}^{n} p_{n-k} u^{(k)} \overline{v}^{(k)}, \tag{2.13}$$

if the integral exists. Let us also introduce the brackets

$$\{u,v\}(x) = -\sum_{k=1}^{n} u^{[2n-k]}(x)\overline{v}^{(k-1)}(x) \tag{2.14}$$

and note that

$$[u,v](x) = \{u,v\}(x) - \{\overline{v},\overline{u}\}(x). \tag{2.15}$$

In a similar fashion to the Lagrange expressions we put $\{u,v\}_a^b = \{u,v\}(b) - \{u,v\}(a)$. Suppose the functions $u,v \in L_w^2(a,b)$ possess enough pseudo-derivatives to form the expressions $a(u,v)$, $\langle \ell u,v\rangle_w$ and $\{u,v\}_a^b$, then

$$a(u,v) = \langle \ell u,v\rangle_w - \{u,v\}_a^b. \tag{2.16}$$

Obviously, if all parts of the above equation exist, then

$$a(u,v) = \langle \ell u,v\rangle_w \tag{2.17}$$

if and only if $\{u,v\}_a^b = 0$. For convenience, the following theorem is reproduced from [6].

Theorem 2.3. *For every $u \in \mathscr{D}_0$ and $v \in \mathscr{D}$, $a(u,v)$ exists and*

$$\langle L_0 u,v\rangle_w = a(u,v) = \langle u,Lv\rangle_w. \tag{2.18}$$

Proof. Let $V_1 = \mathscr{D}_0$ equipped with the graph topology of the operator L_0. Then V_1 is a Hilbert space. Let $y \in \mathscr{D}_0'$. Then

$$\begin{aligned} |a(y,v)| &= |\langle L_0 y,v\rangle_w| \leq ||L_0 y||_w \|v\|_w \\ &\leq ||L_0 y||_{V_1} \|v\|_w. \end{aligned} \tag{2.19}$$

Hence, $a(\cdot,v)$ is continuous on $\mathscr{D}_0'$ in the topology of V_1. Since V_1 is the closure of $\mathscr{D}_0'$ in this topology, then $a(\cdot,v)$ is continuous on V_1. On the other hand, since L_0 is the closure of L_0', there exists a sequence $\{u_n\}$ in $\mathscr{D}_0'$ such that $u_n \to u$ and $L_0 u_n \to L_0 u$ in $L_w^2(a,b)$. Therefore, $u_n \to u$ in V_1. Thus $a(u_n,v) \to a(u,v)$. That is, $a(u,v)$ exists. Also

$$a(u,v) = \lim a(u_n,v) = \lim \langle L_0 u_n,v\rangle_w = \langle L_0 u,v\rangle_w = \langle u,Lv\rangle_w. \tag{2.20}$$

□

It immediately follows from (2.18) that

$$\{u,v\}_a^b = \{v,u\}_a^b = 0 \tag{2.21}$$

for all $u \in \mathcal{D}_0$ and $v \in \mathcal{D}$. Hence the description (2.6) of the domain of the minimal operator $\mathcal{D}_0$ may be sharpened to

$$\mathcal{D}_0 = \left\{u \in \mathcal{D} : \{u,v\}_a^b = \{v,u\}_a^b = 0 \ \forall v \in \mathcal{D}\right\}. \tag{2.22}$$

3. Weak formulation

Note that the first and last expressions in (2.18) require $2n$ pseudo derivatives to be formed whereas the middle expression requires only n derivatives. We are thus led to considering the problem of obtaining a weak formulation for the expression ℓ in spaces that require only n derivatives. In this section we give such a formulation within the framework of the space $L_w^2(a,b)$. As stated in the introduction, no assumptions are being made about the semiboundedness of the operators or the forms involved.

Define the following dense subspaces of $L_w^2(a,b)$:

$$\begin{gathered}
\mathcal{V} = \left\{u \in L_w^2(a,b) : \operatorname{supp}(u) \subset (a,b) \text{ compact},\ u \in AC^{(n-1)}(a,b),\ u^{(n)} \in L^1(a,b)\right\}, \\
\mathcal{X} = \left\{u \in L_w^2(a,b) : u \in AC^{(n-1)}(a,b),\ u^{[n]} \in L_{\mathrm{loc}}^\infty(a,b)\right\}, \\
\mathcal{X}_0 = \left\{u \in \mathcal{X} : \{v,u\}_a^b = 0 \ \forall v \in \mathcal{D}\right\}.
\end{gathered} \tag{3.1}$$

Some comments on the choice of the above spaces are now in order. The choice of the space $\mathcal{V}$ was mainly motivated by the requirement that $\mathcal{D}_0' \subset \mathcal{V}$. This requirement, together with the general assumptions we made about the coefficient functions, grant only the local integrability of the derivatives of the functions in $\mathcal{V}$. The space $\mathcal{X}$ is so chosen to include the space $\mathcal{D}$ whose functions have $2n-1$ absolutely continuous pseudo-derivatives on the interval (a,b). Consequently, for a function $u \in \mathcal{D}$, $u^{[n]} = p_0 u^{(n)} \in AC(a,b)$. From this one could infer a local L^p property for any p, $1 \le p \le \infty$. The choice of $L_{\mathrm{loc}}^\infty(a,b)$ is forced by the natural duality with the properties of the space $\mathcal{V}$ in order to insure the existence of the integrals $\int_a^b u^{[n]} \overline{v}^{(n)}$. Finally the space $\mathcal{X}_0$ is chosen to include $\mathcal{D}_0$ and, at the same time not to exceed the differentiability properties granted by functions in the space $\mathcal{X}$. It will be shown below that these spaces are dense in $L_w^2(a,b)$ and give rise to a satisfactory theory for the weak formulation of the singular differentiable operators.

One is interested, in general, in solving variational equations of the form

$$a(u,v) = \langle f, v\rangle_w, \tag{3.2}$$

where $f \in L_w^2(a,b)$ and v varies in some convenient space $\mathcal{W}$. The equality (3.2) means that a continuity requirement with respect to the norm $\|\cdot\|_w$ has to be imposed on the form $a(u,\cdot)$ over $\mathcal{W}$. As we will see, this continuity requirement plays a crucial role in recovering the domains of definition of the operators associated with ℓ. Since this is the only continuity property we are going to need, the phrase "with respect to norm $\|\cdot\|_w$" will be dropped from this point on.

LEMMA 3.1. *$a(\cdot,\cdot)$ is defined on $\mathscr{L}\times\mathscr{V}$.*

Proof. Let $u\in\mathscr{L}$, $v\in\mathscr{V}$ and suppose that $\operatorname{supp}(v)=[\alpha,\beta]\subset(a,b)$.

$$\left|\int_a^b u^{[n]}\overline{v}^{(n)}\right| = \left|\int_\alpha^\beta u^{[n]}\overline{v}^{(n)}\right| \le \|u^{[n]}\|_{L^\infty(\alpha,\beta)}\|v^{(n)}\|_{L^1(a,b)} \tag{3.3}$$

and for $k=0,1,\dots,(n-1)$

$$\left|\int_a^b p_{n-k}u^{(k)}v^{(k)}\right| = \left|\int_\alpha^\beta p_{n-k}u^{(k)}v^{(k)}\right| \le \|u^{(k)}v^{(k)}\|_{L^\infty(\alpha,\beta)}\int_\alpha^\beta |p_{n-k}|. \tag{3.4}$$

Hence, $a(u,v)$ exists. □

LEMMA 3.2. *For $u\in\mathscr{L}_0$ and $v\in\mathscr{D}$*

$$a(u,v)=\langle u,Lv\rangle_w. \tag{3.5}$$

Proof. For $u\in\mathscr{L}_0$ and $v\in\mathscr{D}$, $\langle u,Lv\rangle_w$ exists and, from the definition of $\mathscr{L}_0$, $\{v,u\}_a^b=0$, hence (see the Preliminaries) $a(u,v)$ is defined and the result follows from (2.16). □

THEOREM 3.3. *For $f\in L_w^2(a,b)$, the following are equivalent:*
(I) $u\in\mathscr{D}$, $Lu=f$,
(II) $u\in\mathscr{L}$, $a(u,v)=\langle f,v\rangle_w$ $\forall v\in\mathscr{V}$.
In this case we may write

$$a(u,v)=\langle Lu,v\rangle_w \quad \forall v\in\mathscr{V}. \tag{3.6}$$

Proof. Suppose (I) holds. By the definition of $\mathscr{D}$, $u,u^{[n]}\in AC(a,b)$. Hence, $u^{[n]}$ is bounded on any compact subinterval of (a,b). Therefore, $u^{[n]}\in L^\infty_{\text{loc}}(a,b)$. That is, $u\in\mathscr{L}$. Next let $v\in\mathscr{V}$ and suppose that $\operatorname{supp}(v)=[\alpha,\beta]\subset(a,b)$. Then, with the help of the definitions (2.2) of pseudoderivatives,

$$\begin{aligned}
\langle f,v\rangle_w = \langle Lu,v\rangle_w = \int_\alpha^\beta u^{[2n]}\overline{v} &= \int_\alpha^\beta p_n u\overline{v} - \left(u^{[2n-1]}\right)'\overline{v},\\
&= \int_\alpha^\beta p_n u\overline{v} - \int_\alpha^\beta \left(u^{[2n-1]}\right)'\overline{v} \quad \left(\text{since } \int_\alpha^\beta p_n u\overline{v} \text{ exists}\right)\\
&= \int_\alpha^\beta p_n u\overline{v} + \int_\alpha^\beta u^{[2n-1]}\overline{v}'\\
&= \cdots\\
&= \sum_{k=0}^n\int_\alpha^\beta p_{n-k}u^{(k)}\overline{v}^{(k)} = \int_\alpha^\beta\sum_{k=0}^n p_{n-k}u^{(k)}\overline{v}^{(k)} = a(u,v).
\end{aligned} \tag{3.7}$$

On the other hand, suppose (II) holds. Suppose $v \in \mathfrak{D}(a,b)$. Since $u \in \mathscr{X}$ then $(p_{n-k}u^{(k)})^{(k)} \in \mathfrak{L}(a,b)$, $0 \le k \le n$ (see (2.1)). Hence, $\sum_{k=0}^{n}(-1)^k(p_{n-k}u^{(k)})^{(k)} \in \mathfrak{L}(a,b)$. On the other hand

$$\begin{aligned} \langle f,v\rangle_w = a(u,v) &= \int_a^b \sum_{k=0}^{n} p_{n-k}u^{(k)}\overline{v}^{(k)} \\ &= \sum_{k=0}^{n}\int_\alpha^\beta p_{n-k}u^{(k)}\overline{v}^{(k)} \\ &= \sum_{k=0}^{n}(-1)^{(k)}\Big\langle \big(p_{n-k}u^{(k)}\big)^{(k)},v\Big\rangle \\ &= \Big\langle \sum_{k=0}^{n}(-1)^k\big(p_{n-k}u^{(k)}\big)^{(k)},v\Big\rangle. \end{aligned} \tag{3.8}$$

Since $wf \in L^1_{\text{loc}}(a,b)$, we get

$$u^{[2n]} = wf \quad \text{in } L^1_{\text{loc}}(a,b). \tag{3.9}$$

We proceed to show that $u \in \mathscr{D}$. $u \in L^2_w(a,b)$ by the definition of $\mathscr{X}$. From (3.9) we get

$$\big(u^{[2n-1]}\big)' = p_n u - wf. \tag{3.10}$$

Since the right-hand side of the above equation is integrable over any compact subinterval of (a,b) it follows that $u^{[2n-1]} \in AC(a,b)$. In a similar fashion and with the help of the recursion $(u^{[2n-k-1]})' = p_{n-k}u^{(k)} - u^{[2n-k]}$, $k = 0,2,\ldots,(n-1)$ we get that $u^{[2n-k]} \in AC(a,b)$, $k = 1,2,\ldots,n$. The definition of $\mathscr{X}$ gives $u^{[n-k]} \in AC(a,b)$, $k = 1,2,\ldots,n$. From this and (3.9) again we get that $u \in \mathscr{D}$ and $Lu = f$. □

Corollary 3.4. *For $u \in \mathscr{D}$, the mapping $a(u,\cdot)$ is continuous on $\mathscr{V}$.*

Proof. For $u \in \mathscr{D}$ we have by Theorem 3.3

$$a(u,v) = \langle Lu,v\rangle_w \quad \forall v \in \mathscr{V}. \tag{3.11}$$

Hence, $a(u,\cdot)$ is continuous on $\mathscr{V}$. □

Next we will show that $\mathscr{D}$ is precisely the subspace of $\mathscr{X}$ for which the continuity property of the previous corollary holds. Before establishing this we need the following property.

Lemma 3.5. $\mathscr{D}_0' \subset \mathscr{X}_0 \cap \mathscr{V}$.

Proof. Let $u \in \mathscr{D}_0'$. Clearly u satisfies the two properties defining the space $\mathscr{X}$. On the other hand, let

$$p_0u^{(n)} = g. \tag{3.12}$$

Then g is absolutely continuous on the support of u. Furthermore,

$$u^{(n)} = \frac{g}{p_0}, \tag{3.13}$$

therefore the local integrability of $1/p_0$ implies the integrability of $u^{(n)}$. Thus, $u \in \mathcal{V}$. □

We remark here that the above lemma asserts also that the spaces $\mathcal{X}$, $\mathcal{X}_0$, $\mathcal{V}$ are dense in $L^2_w(a,b)$.

Theorem 3.6. *$\mathcal{D} = \{u \in \mathcal{X} : a(u,\cdot)$ is continuous on $\mathcal{V}\}$.*

Proof. Denote the right-hand side of the above equation by $\mathcal{D}_1$. For $u \in \mathcal{D}_1$ define the antilinear functional $G_u(\cdot)$ on $\mathcal{V}$ by

$$G_u(v) = a(u,v). \tag{3.14}$$

Then $G_u(\cdot)$ is continuous on $\mathcal{V}$. Since $\mathcal{V}$ is dense in $L^2_w(a,b)$ we can extend $G_u(\cdot)$ to all of $L^2_w(a,b)$. Hence, by the Riesz representation theorem, there is a unique element $Tu \in L^2_w(a,b)$ such that

$$G_u(v) = \langle Tu, v\rangle_w \quad \forall v \in \mathcal{V}. \tag{3.15}$$

Now notice that $\mathcal{D} \subset \mathcal{D}_1$ and for $u \in \mathcal{D}$ we have

$$\langle Tu, v\rangle_w = a(u,v) = \langle Lu, v\rangle_w \quad \forall v \in \mathcal{V}. \tag{3.16}$$

This means that the operator T is densely defined and agrees with L on $\mathcal{D}$. That is, $L \subset T$. It follows that $T^* \subset L^* = L_0$. Therefore T^* is a symmetric closed operator. For $v \in \mathcal{D}_0'$ with $\operatorname{supp}(v) = [\alpha,\beta]$, $u \in \mathcal{D}_1$ we have

$$\begin{aligned}
\langle Tu, v\rangle_w &= a(u,v) \quad (\text{since } \mathcal{D}_0' \subset \mathcal{V}) \\
&= \int_a^b \sum_{k=0}^n p_{n-k} u^{(k)} \overline{v}^{(k)} \\
&= \int_\alpha^\beta \sum_{k=0}^n p_{n-k} u^{(k)} \overline{v}^{(k)} \\
&= \sum_{k=0}^n \int_\alpha^\beta p_{n-k} u^{(k)} \overline{v}^{(k)} \\
&= \sum_{k=0}^n (-1)^k \int_\alpha^\beta u \left(p_{n-k} \overline{v}^{(k)}\right)^{(k)} \\
&= \langle u, L_0' v\rangle_w.
\end{aligned} \tag{3.17}$$

This means that $v \in \mathcal{D}(T^*)$ and $T^* v = L_0' v$. Thus we have the chain of operators $L_0' \subset T^* \subset L_0$. This yields $T^* = L_0$ and, hence, $T \subset T^{**} = L_0^* = L$. □

In analogy with this result, we have the following theorem.

Theorem 3.7. *Suppose* $\mathrm{Im}(\lambda) \neq 0$, *then*
(1) $\mathcal{N}_\lambda = \{u \in \mathcal{X} : a(u,v) = \lambda\langle u,v\rangle_w \ \forall v \in \mathcal{V}\}$
(2) $\mathcal{D}_0 = \{u \in \mathcal{X}_0 : a(u,\cdot)$ *is continuous on* $\mathcal{V}\}$.

Proof. (1) This part is an immediate consequence of Theorems 3.3 and 3.6, and the density of $\mathcal{V}$ in $L^2_w(a,b)$.

(2) Let

$$\mathcal{E}_0 = \{u \in \mathcal{X}_0 : a(u,\cdot) \text{ is continuous on } \mathcal{V}\}. \tag{3.18}$$

If $u \in \mathcal{D}_0$ then $u \in \mathcal{X}_0$ and

$$\begin{aligned} a(u,v) &= \langle Lu,v\rangle_w \\ &= \langle L_0 u,v\rangle_w \quad \forall v \in \mathcal{V}, \end{aligned} \tag{3.19}$$

that is, $a(u,\cdot)$ is continuous on $\mathcal{V}$. Hence, $u \in \mathcal{E}_0$. On the other hand, if $u \in \mathcal{E}_0$, then $a(u,\cdot)$ is continuous on $\mathcal{V}$ and can be extended by continuity to all of $L^2_w(a,b)$. In particular $a(u,\cdot)$ is continuous on $\mathcal{D}$ and, by Lemma 3.2,

$$a(u,v) = \langle u,Lv\rangle_w \quad \forall v \in \mathcal{D}. \tag{3.20}$$

Hence, the mapping $v \mapsto \langle u,Lv\rangle_w$ is continuous on $\mathcal{D}$. Therefore, $u \in D(L^*) = D(L_0) = \mathcal{D}_0$.

□

As was stated in the preliminaries, the subspaces $\mathcal{D}_0$, $\mathcal{N}_\lambda$, $\mathcal{N}_{\bar\lambda}$ are linearly independent. Since the space $\mathcal{X}_0$ is a superspace of $\mathcal{D}_0$, the question now arises as to whether the same is true about the spaces $\mathcal{X}_0$, $\mathcal{N}_\lambda$, $\mathcal{N}_{\bar\lambda}$. The affirmative answer is a special case of the following lemma.

Lemma 3.8. *A set of functions* $w_1, w_2, \dots, w_k \in \mathcal{D}$ *are linearly independent modulo* $\mathcal{D}_0$ *if and only if they are linearly independent modulo* $\mathcal{X}_0$.

Proof. The sufficiency part of this lemma is obvious since $\mathcal{D}_0$ is a subspace of $\mathcal{X}_0$. To show the necessity part, assume the functions $w_1, w_2, \dots, w_k \in \mathcal{D}$ are linearly independent modulo $\mathcal{D}_0$ and there exist complex numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ such that

$$\varphi \mathrel{\hat=} \sum_{i=1}^{k} \alpha_i w_i \in \mathcal{X}_0. \tag{3.21}$$

Since $\varphi \in \mathcal{D}$ we can write

$$\varphi = \varphi_0 + \varphi_1 \tag{3.22}$$

with $\varphi_0 \in \mathcal{D}_0$ and $\varphi_1 \in \mathcal{N}_\lambda + \mathcal{N}_{\bar\lambda}$. It follows that $\varphi_1 \in \mathcal{X}_0$, and, since we also have $\varphi_1 \in \mathcal{D}$, we have by Theorem 3.3

$$a(\varphi_1,v) = \langle L\varphi_1,v\rangle_w \quad \forall v \in \mathcal{V}. \tag{3.23}$$

Hence, by Part 2 of Theorem 3.7, $\varphi_1 \in \mathfrak{D}_0$. Thus $\varphi_1 = 0$ and $\varphi \in \mathfrak{D}_0$. This necessarily gives $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$. □

We next give a characterization of self-adjoint extensions of L_0 in terms of unitary operators between the spaces $\mathcal{N}_{\overline{\lambda}}$ and $\mathcal{N}_\lambda$ and the space $\mathfrak{X}_0$. The following theorem may be regarded as a counterpart of Theorem 2.1.

Theorem 3.9. *Suppose $\widehat{L}$ is a self-adjoint extension of the operator L_0 with domain of definition $\widehat{\mathfrak{D}}$ and corresponding unitary operator U. Define the space $\widehat{\mathfrak{X}}$ by*

$$\widehat{\mathfrak{X}} = \mathfrak{X}_0 \dot{+} (U+I)\mathcal{N}_{\overline{\lambda}}. \tag{3.24}$$

Then

$$\widehat{\mathfrak{D}} = \{u \in \widehat{\mathfrak{X}} : a(u,\cdot) \text{ is continuous on } \mathcal{V}\}. \tag{3.25}$$

Conversely, if $U : \mathcal{N}_{\overline{\lambda}} \to \mathcal{N}_\lambda$ is a unitary operator and $\widehat{\mathfrak{X}}$ is defined by (3.24), then the set $\widehat{\mathfrak{D}}$ defined by (3.25) is the domain of definition of a certain self-adjoint extension $\widehat{L}$ of L_0.

Proof. Denote the right-hand side of (3.25) by $\widehat{\mathfrak{D}}_1$. It is straightforward to check that $\widehat{\mathfrak{D}} \subset \widehat{\mathfrak{D}}_1$. On the other hand, for $u \in \widehat{\mathfrak{D}}_1$, write $u = u_0 + (U+I)u_{\overline{\lambda}}$. For $v \in \mathcal{V}$ we get

$$a(u,v) = a(u_0,v) + \langle (\lambda U + \overline{\lambda} I) u_{\overline{\lambda}}, v \rangle_w. \tag{3.26}$$

The continuity of $a(u,\cdot)$ and $\langle (\lambda U + \overline{\lambda} I)u_{\overline{\lambda}}, \cdot\rangle_w$ on $\mathcal{V}$ imply the continuity of $a(u_0,\cdot)$ on $\mathcal{V}$. Since $u_0 \in \mathfrak{X}_0$, we get, by the second part of Theorem 3.7, that $u_0 \in \mathfrak{D}_0$. Hence, $u \in \widehat{\mathfrak{D}}$. The converse statement follows from the characterization in Theorem 2.1 and the first part of this theorem since the definition of $\widehat{\mathfrak{D}}$ implies that

$$\widehat{\mathfrak{D}} = \mathfrak{D}_0 \dot{+} (U+I)\mathcal{N}_{\overline{\lambda}}. \tag{3.27}$$

□

4. Further properties and characterizations

In this section, we give further properties and alternative characterizations of the weak spaces $\mathfrak{X}$, $\mathfrak{X}_0$ and the domains of self-adjoint extensions of L_0 in terms of the so called "boundary condition functions."

It was shown in the previous section that $a(\cdot,\cdot)$ is defined on $\mathfrak{X} \times \mathcal{V}$. Since $\mathfrak{D}_0' \subset \mathcal{V}$, then $a(\cdot,\cdot)$ is defined on $\mathfrak{X} \times \mathfrak{D}_0'$ and, for a fixed $u \in \mathfrak{D}_0'$, the mapping $v \mapsto a(v,u)$ is continuous on $\mathfrak{X}$. The question is, how far can we push the space $\mathfrak{D}_0'$ and retain continuity on $\mathfrak{X}$? The answer is in the corollary to the following lemma.

Lemma 4.1. *For every $u \in \mathfrak{X}$ and $v \in \mathfrak{D}_0$, $a(u,v)$ exists,*

$$a(u,v) = \langle u, L_0 v\rangle_w \tag{4.1}$$

and, consequently, $\{v,u\}_a^b = 0$.

Proof. The proof is similar to that Theorem 2.3 with $\mathfrak{D}$ replaced by $\mathfrak{X}$. □

Corollary 4.2. *For every $u \in \mathfrak{D}_0$, the mapping $v \mapsto a(u,v)$ is continuous on $\mathfrak{X}$.*

We also have the following weakened definition of the space $\mathfrak{X}_0$.

Lemma 4.3. *$\mathfrak{X}_0$ consists precisely of all functions $u \in \mathfrak{X}$ which for a fixed non-real λ satisfy*

$$\{\varphi, u\}_a^b = 0 \tag{4.2}$$

for all functions $\varphi \in \mathcal{N}_\lambda + \mathcal{N}_{\bar{\lambda}}$.

Proof. Equation (4.2) is necessary since $\mathcal{N}_\lambda + \mathcal{N}_{\bar{\lambda}} \subset \mathfrak{D}$. On the other hand, suppose a function $u \in \mathfrak{X}$ satisfies (4.2) for all $\varphi \in \mathcal{N}_\lambda + \mathcal{N}_{\bar{\lambda}}$. Let $v \in \mathfrak{D}$ and write $v = v_0 + \varphi$ for $v_0 \in \mathfrak{D}_0$ and $\varphi \in \mathcal{N}_\lambda + \mathcal{N}_{\bar{\lambda}}$. Then, using Lemma 4.1, we get $\{v,u\}_a^b = \{v_0,u\}_a^b + \{\varphi,u\}_a^b = 0$. Hence, $u \in \mathfrak{X}_0$. □

Corollary 4.4. $\mathfrak{X} = \mathfrak{X}_0 \dotplus \mathcal{N}_\lambda \dotplus \mathcal{N}_{\bar{\lambda}}$.

Proof. We remark first that, by Lemma 3.8, $\mathfrak{X}_0 \dotplus \mathcal{N}_\lambda \dotplus \mathcal{N}_{\bar{\lambda}}$ is a direct sum.

Clearly $\mathfrak{X}_0 \dotplus \mathcal{N}_\lambda \dotplus \mathcal{N}_{\bar{\lambda}} \subset \mathfrak{X}$. On the other hand, let $u \in \mathfrak{X}$ and assume $\varphi_1, \varphi_2, \dots, \varphi_{2d}$ form a basis for $\mathcal{N}_\lambda \dotplus \mathcal{N}_{\bar{\lambda}}$. We claim that the matrix $(\{\varphi_k, \varphi_i\}_a^b)$ has full rank. To see this, assume the contrary. Then there exist scalars $\theta_1, \theta_2, \dots, \theta_{2d}$, not all zeros, such that

$$\sum_{i=1}^{2d} \theta_i \{\varphi_k, \varphi_i\}_a^b = 0, \quad k = 1,2,\dots,2d. \tag{4.3}$$

Define the function $v = \sum_{i=1}^{2d} \theta_i \varphi_i$. It follows from the above equation that $\{\varphi_k, v\}_a^b = 0$, $k = 1,2,\dots,2d$. Hence, by the Lemma 4.3, $v \in \mathfrak{X}_0$. Since $\varphi_1, \varphi_2, \dots, \varphi_{2d}$ are linearly independent modulo $\mathfrak{X}_0$, we must have $\theta_1 = \theta_2 = \cdots = \theta_{2d} = 0$, which is a contradiction. Now let $\alpha_1, \alpha_2, \dots, \alpha_{2d}$ be the solutions of the linear system

$$\{\varphi_k, u\}_a^b = \sum_{i=1}^{2d} \alpha_i \{\varphi_k, \varphi_i\}_a^b, \quad k = 1,2,\dots,2d, \tag{4.4}$$

and let $\varphi = \sum_{i=1}^{2d} \alpha_i \varphi_i$, $u_0 = u - \varphi$. It is easy to check that $\{\varphi_k, u_0\}_a^b = 0$, $k = 1,2,\dots,2d$. Therefore, $u_0 \in \mathfrak{X}_0$, from which we get that $\mathfrak{X}_0 \dotplus \mathcal{N}_\lambda \dotplus \mathcal{N}_{\bar{\lambda}} \supset \mathfrak{X}$. □

Lemma 4.5. *Suppose $\varphi_1, \varphi_2, \dots, \varphi_{2d}$ are $2d$ functions in $\mathfrak{D}$ which are linearly independent modulo $\mathfrak{X}_0$. Then*

$$\begin{gathered} \mathfrak{X}_0 = \left\{ u \in \mathfrak{X} : \{\varphi_k, u\}_a^b = 0,\ k = 1,2,\dots,2d, \right\}, \\ \mathfrak{X} = \mathfrak{X}_0 \dotplus \operatorname{span}[\varphi_1, \varphi_2, \dots, \varphi_{2d}]. \end{gathered} \tag{4.5}$$

Proof. Choose a $\lambda \in \mathbb{C}$ with $\operatorname{Im}(\lambda) \neq 0$ and let $\psi_1, \psi_2, \dots, \psi_{2d}$ be a basis for $\mathcal{N}_\lambda \dotplus \mathcal{N}_{\bar{\lambda}}$. Then we can write

$$\varphi_k = \theta_k + \sum_{i=1}^{2d} \alpha_{ki} \psi_i, \quad k = 1,2,\dots,2d, \tag{4.6}$$

where $\theta_k \in \mathscr{D}_0$ and α_{ki}'s are scalars. We claim that the $2d \times 2d$ matrix $[\alpha_{ki}]$ has full rank. To show this assume that there exist scalars $\gamma_1, \gamma_2, \dots, \gamma_{2d}$ such that $\sum_{i=1}^{2d} \alpha_{ik}\gamma_i = 0$, $k = 1,2,\dots,2d$. It follows that $\sum_{i=1}^{2d} \gamma_i\varphi_i = \sum_{i=1}^{2d} \gamma_i\theta_i$. That is, $\sum_{i=1}^{2d} \gamma_i\varphi_i \in \mathscr{D}_0$. Since $\varphi_1, \varphi_2, \dots, \varphi_{2d}$ are linearly independent modulo $\mathscr{D}_0$, then $\gamma_1 = \gamma_2 = \cdots = \gamma_{2d} = 0$. Hence, we can write

$$\psi_k = \widetilde{\theta}_k + \sum_{i=1}^{2d} \beta_{ki}\varphi_i, \quad k = 1,2,\dots,2d, \tag{4.7}$$

with $\widetilde{\theta}_k \in \mathscr{D}_0$. The results now follow from (4.6), (4.7), Lemma 4.3 and its corollary. □

We turn now to characterizations of domains of self-adjoint extensions of L_0 that parallel Theorem 2.2. It was shown in [11] that the domain of definition $\widehat{\mathscr{D}}$ of self adjoint extensions $\widehat{L}$ of L_0 are characterized by functions $w_1, w_2, \dots, w_d \in \mathscr{D}$ satisfying conditions 1, 2 of Theorem 2.2 such that

$$\widehat{\mathscr{D}} = \mathscr{D}_0 \dotplus \operatorname{span}[w_1, w_2, \dots, w_d]. \tag{4.8}$$

Define the space

$$\widehat{\mathscr{X}} = \{u \in \mathscr{X} : \{w_i, u\}_a^b = 0,\ i = 1,\dots,d\}. \tag{4.9}$$

Lemma 4.6. *For every $u \in \widehat{\mathscr{X}}$ and $v \in \widehat{\mathscr{D}}$, $a(u,v)$ exists, $\{v,u\}_a^b = 0$ and*

$$a(u,v) = \langle u, \widehat{L}v\rangle_w. \tag{4.10}$$

Proof. Let $u \in \widehat{\mathscr{X}}$ and $v \in \widehat{\mathscr{D}}$ and write

$$v = v_0 + \sum_{i=1}^{d} \alpha_i w_i \tag{4.11}$$

with $v_0 \in \mathscr{D}_0$. Using Lemma 4.1 we get

$$\{v,u\}_a^b = \{v_0,u\}_a^b + \sum_{i=1}^{d} \alpha_i \{w_i,u\}_a^b = 0. \tag{4.12}$$

Furthermore, since $\langle u, \widehat{L}v\rangle_w$ exists, it follows that $a(u,v)$ exists. Equation (4.10) now follows from (2.16). □

The following two theorems give a characterization of a class of self-adjoint extensions of L_0.

Theorem 4.7. *Suppose $\widehat{\mathscr{D}}$ is the domain of definition of a self adjoint extension $\widehat{L}$ of L_0 corresponding to the functions $w_1, w_2, \dots, w_d$. Define space $\widehat{\mathscr{X}}$ by (4.9) and the domain*

$$\widehat{\mathscr{D}}_1 = \{u \in \widehat{\mathscr{X}} : a(u,\cdot) \text{ is continuous on } \mathscr{V}\}, \tag{4.13}$$

then

(1) $\widehat{\mathfrak{D}}_1 \subset \widehat{\mathfrak{D}}$,

(2) $\widehat{\mathfrak{D}}_1 = \widehat{\mathfrak{D}}$ *if and only if* $\{w_i, w_j\}_a^b = 0$, $i, j = 1, 2, \dots, d$.

Proof. (1) The proof of this part follows the same lines of that of the second part of Theorem 3.7.

(2) If $\widehat{\mathfrak{D}}_1 = \widehat{\mathfrak{D}}$ then $w_1, w_2, \dots, w_d \in \widehat{\mathfrak{D}}_1 \subset \widehat{\mathfrak{L}}$. Therefore, $\{w_i, w_j\}_a^b = 0$, $i, j = 1, 2, \dots, d$. On the other hand, if $\{w_i, w_j\}_a^b = 0$, $i, j = 1, 2, \dots, d$, then, for $u \in \widehat{\mathfrak{D}} \subset \mathfrak{L}$ we may write

$$u = u_0 + \sum_{i=1}^{d} \alpha_i w_i \tag{4.14}$$

with $u_0 \in \mathfrak{D}_0$. It is easy to check that $\{w_i, u\}_a^b = 0$, $i = 1, \dots, d$ implying $u \in \widehat{\mathfrak{L}}$. Hence, $\widehat{\mathfrak{D}} \subset \widehat{\mathfrak{D}}_1$. □

The foregoing theorem tells us that domains of the type (4.9) cannot be hoped to characterize all self-adjoint extensions of L_0. They rather characterize extensions for which the boundary condition functions satisfy $\{w_i, w_j\}_a^b = 0$, $i, j = 1, 2, \dots, d$. This class of extensions will be called Class I. The following converse theorem applies to this class.

Theorem 4.8. *Suppose there exist functions* $w_1, w_2, \dots, w_d \in \mathfrak{D}$ *such that*

(1) $w_1, w_2, \dots, w_d$ *are linearly independent modulo* $\mathfrak{L}_0$

(2) $\{w_i, w_j\}_a^b = 0$, $i, j = 1, 2, \dots, d$.

Then the set

$$\widehat{\mathfrak{D}} = \{u \in \mathfrak{L} : \{w_j, u\}_a^b = 0,\ j = 1, 2, \dots, d \textit{ and } a(u, \cdot) \textit{ is continuous on } \mathcal{V}\} \tag{4.15}$$

is the domain of definition of a certain Class I self-adjoint extension $\hat{L}$ *of* L_0.

Proof. Conditions 1, 2 above give that $w_1, w_2, \dots, w_d$ are linearly independent modulo $\mathfrak{D}_0$ and $[w_i, w_j]_a^b = 0$, $i, j = 1, 2, \dots, d$. Then, by Theorem 2.2 and (4.8), the set

$$\widehat{\mathfrak{D}}_1 = \mathfrak{D}_0 \dotplus \operatorname{span}[w_1, w_2, \dots, w_d] \tag{4.16}$$

is the domain of definition of a certain self-adjoint Class I extension $\hat{L}$ of L_0. Hence, by Theorem 4.7,

$$\widehat{\mathfrak{D}}_1 = \left\{u \in \mathfrak{L} : \{w_j, u\}_a^b = 0,\ j = 1, 2, \dots, d \text{ and } a(u, \cdot) \text{ is continuous on } \mathcal{V}\right\}. \tag{4.17}$$

That is, $\widehat{\mathfrak{D}}_1 = \widehat{\mathfrak{D}}$. □

For the more general conditions $[w_i, w_j]_a^b = 0$, $i, j = 1, 2, \dots, d$ we may define $\widehat{\mathfrak{L}}$ by

$$\widehat{\mathfrak{L}} = \mathfrak{L}_0 \dotplus \operatorname{span}[w_1, w_2, \dots, w_d], \tag{4.18}$$

in which case a counterpart of Theorem 2.2 may be stated with $\widehat{\mathfrak{D}}$ defined by

$$\widehat{\mathfrak{D}} = \{u \in \widehat{\mathfrak{X}} : a(u, \cdot) \text{ is continuous on } \mathcal{V}\}. \tag{4.19}$$

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M. A. El-Gebeily: Mathematical Sciences Department, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
E-mail address: mgebeily@kfupm.edu.sa