ON THE RATE OF THE VOLUME GROWTH FOR SYMMETRIC VISCOUS HEAT-CONDUCTING GAS FLOWS WITH A FREE BOUNDARY

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The system of quasilinear equations for symmetric flows of a viscous heat-conducting gas with a free external boundary is considered. For global in time weak solutions having nonstrictly positive density, the linear in time two-sided bounds for the gas volume growth are established.

1. Introduction

We consider the system of quasilinear equations describing symmetric flows of a viscous heat-conducting perfect polytropic gas [1]

$$\eta_t = (r^k v)_x, \tag{1.1}$$

$$v_t = r^k \sigma_x, \tag{1.2}$$

$$c_V \theta_t = (r^k \pi)_x + \sigma (r^k v)_x - 2k\mu (r^{k-1} v^2)_x, \tag{1.3}$$

$$r_t = v, \tag{1.4}$$

$$\sigma = \nu\rho (r^k v)_x - R\rho\theta, \quad \pi = \varkappa \rho r^k \theta_x, \quad \rho = \frac{1}{\eta}, \tag{1.5}$$

in the domain $Q := \Omega \times \mathbb{R}^+ = (0,M) \times (0,\infty)$. The system is supplemented with the boundary and initial conditions

$$v|_{x=0} = 0, \qquad \left(\sigma - 2k\mu \frac{v}{r}\right)\Big|_{x=M} = 0, \qquad (r^k \pi)|_{x=0,M} = 0 \quad \text{for } t > 0, \tag{1.6}$$

$$\{\eta, v, \theta, r\}|_{t=0} = \{\eta^0(x), v^0(x), \theta^0(x), r^0(x)\} \quad \text{for } x \in \Omega. \tag{1.7}$$

The parameter k takes the values 1 or 2 accordingly to the cylindrical or spherical symmetry.

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The unknown functions $\eta > 0$, v, $\theta \geq 0$ and $r \geq r_0$ depend on the Lagrangian mass coordinates (x,t) and denote the specific volume, the velocity, the absolute temperature and the Eulerian coordinate that is the radius of a gas particle. The functions ρ, σ and $-\pi$ are the density, the stress and the heat flux. We consider flows around a hard core so that $r_0 > 0$ is its radius, and the internal boundary $(x = 0)$ is one with the core. The external boundary $(x = M)$ is free; both boundaries are thermally isolated, see (1.6).

The quantities $\nu > 0$, $\mu, R > 0$, $c_V > 0$ and $\varkappa > 0$ are physical constants; $M > 0$ is the total mass of the gas. We impose the standard condition on the viscosity coefficients ν and μ

$$\nu_1 := \nu - \frac{2k}{k+1}\mu > 0. \tag{1.8}$$

The initial function r^0 is not arbitrary but rather connected to η^0 by the physical relation

$$\left(r^0\right)^{k+1}(x) = r_0^{k+1} + (k+1)\int_0^x \eta^0(\xi)\,d\xi \quad \text{for } x \in \overline{\Omega}. \tag{1.9}$$

In the simpler case of the planar symmetry $(k = 0)$, the asymptotic behavior of solutions was studied in detail in [5] and more recently in [6, 7] for other boundary conditions. In the case of the spherical symmetry, some results on the growth of the (scaled) gas volume $V(t) := \int_\Omega \eta(x,t)\,dx$ as $t \to \infty$ are available in [2].

We prove the sharp result establishing the linear growth of V both in the cases of the spherical and cylindrical symmetries like that for the planar one. In contrast to [2, 5–7], we treat essentially more general global in time *weak* solutions to the problem whose density is non-strictly positive only.

2. Results

We introduce the integration operators

$$Iz(x) := \int_0^x z(\xi)\,d\xi, \quad I^*z(x) := \int_x^M z(\xi)\,d\xi \quad \text{for } z \in L^1(\Omega). \tag{2.1}$$

They are connected by the identity

$$\int_\Omega (Iz_1)z_2\,dx = \int_\Omega z_1 I^* z_2\,dx \quad \text{for any } z_1, z_2 \in L^1(\Omega). \tag{2.2}$$

Let $V_q(Q_T)$ be the space of functions $w \in L^{q,\infty}(Q_T)$ having the derivative $w_x \in L^q(Q_T)$, for $q = 1, 2$ and $Q_T := \Omega \times (0,T)$; recall that $\|w\|_{L^{q,s}(Q_T)} = \|\|w\|_{L^q(\Omega)}\|_{L^s(0,T)}$, for $q, s \in [1,\infty]$.

We study *global in time weak solution* to the problem (1.1)–(1.7) such that:

(1) the properties

$$\begin{gathered} \eta, \eta_t \in L^{1,2}(Q_T), \qquad \frac{1}{\eta} \in L^\infty(Q_T), \qquad v \in V_2(Q_T), \\ \theta \in V_1(Q_T), \qquad r, r_x, r_t \in L^{1,\infty}(Q_T) \end{gathered} \tag{2.3}$$

together with $\eta > 0$, $\theta \geq 0$, $r \geq r_0$ (almost everywhere in Q_T) and $v|_{x=0} = 0$ are valid;

(2) equations (1.1) and (1.4) together with the initial conditions $\eta|_{t=0} = \eta^0$ and $r|_{t=0} = r^0$ are satisfied;

(3) the integral identities

$$\int_{Q_T} \Big\{ -\nu\varphi_t + \sigma(r^k\varphi)_x \Big\} dx\,dt = \int_\Omega \nu^0\varphi|_{t=0}dx + 2k\mu\int_0^T (\nu r^{k-1})|_{x=M}\varphi|_{x=M}dt, \tag{2.4}$$

for any $\varphi \in H^1(Q_T)$ with $\varphi|_{x=0} = 0$ and $\varphi|_{t=T} = 0$, as well as

$$\int_{Q_T} \Big\{ -c_V\theta\psi_t + r^k\pi\psi_x - \Big[\sigma(r^k\nu)_x - 2k\mu(r^{k-1}\nu^2)_x\Big]\psi \Big\} dx\,dt = \int_\Omega c_V\theta^0\psi|_{t=0}dx, \tag{2.5}$$

for any $\psi \in C^1(\overline{Q}_T)$ with $\psi|_{t=T} = 0$, are valid, where relations (1.5) are assumed to hold.

Hereafter $T > 0$ is arbitrary and it is assumed that $\eta^0 \in L^1(\Omega)$, $\nu^0 \in L^2(\Omega)$, $\theta^0 \in L^1(\Omega)$ as well as $\eta^0 > 0$ and $\theta^0 \geq 0$ (almost everywhere in Ω).

We have to justify correctness of the definition of the weak solution. First notice that actually $\eta \in L^{1,\infty}(Q_T)$ and $r \in L^\infty(Q_T)$ according to properties (2.3). Next, we recall that (1.1) and (1.4) together with relation (1.9) imply the following relation between r and η

$$r^{k+1} = r_0^{k+1} + (k+1)I\eta. \tag{2.6}$$

In particular, actually $r \geq r_0$ and $\rho r_x = r^{-k}$. Consequently

$$\sigma = \rho(\nu r^k \nu_x - R\theta) + \nu k r^{-1}\nu \in L^2(Q_T), \tag{2.7}$$

where the embedding $V_1(Q_T) \subset L^2(Q_T)$ is taken into account. Moreover, for any $\varphi \in V_2(Q_T)$, we have

$$\sigma(r^k\varphi)_x = \sigma r^k\varphi_x + kr^{-1}(\nu r^k\nu_x - R\theta)\varphi + \nu k^2 r^{k-2} r_x \nu\varphi, \tag{2.8}$$

and since $V_2(Q_T) \subset L^{\infty,4}(Q_T)$ [4], we obtain

$$\sigma(r^k\varphi)_x \in L^1(Q_T). \tag{2.9}$$

If in addition $\varphi_x \in L^{2,\infty}(Q_T)$, then

$$\sigma(r^k\varphi)_x \in L^{1,2}(Q_T). \tag{2.10}$$

Furthermore

$$(r^{k-1}\nu^2)_x = 2r^{k-1}\nu\nu_x + (k-1)r^{k-2}r_x\nu^2 \in L^{1,2}(Q_T). \tag{2.11}$$

Consequently identities (2.4) and (2.5) are well-defined.

Notice also that

$$\sigma\eta = \nu r^k\nu_x - R\theta + \nu k r^{-1}\nu\eta \in L^{1,2}(Q_T). \tag{2.12}$$

Concerning the existence of strong and weak solutions, see in particular [1, 3, 8].

We will need the energy conservation law. Let us set $\sigma_\Gamma := 2k\mu(v/r)|_{x=M}$; notice that $\sigma_\Gamma \in L^4(0,T)$.

Lemma 2.1. *The total kinetic energy* $(1/2)\int_\Omega v^2 dx$ *and the total internal energy* $\int_\Omega c_V\theta\, dx$ *are absolutely continuous functions on* $[0,T]$ *for any* $T>0$ *having the derivatives*

$$\frac{d}{dt}\frac{1}{2}\int_\Omega v^2 dx = -\int_\Omega (\sigma-\sigma_\Gamma)(r^k v)_x dx, \qquad \frac{d}{dt}\int_\Omega c_V\theta\, dx = \int_\Omega (\sigma-\sigma_\Gamma)(r^k v)_x dx. \tag{2.13}$$

Consequently the total energy conservation law holds

$$\mathscr{E} := \int_\Omega \left(\frac{1}{2}v^2 + c_V\theta\right)dx \equiv \mathscr{E}^0 \quad on\ \overline{\mathbb{R}}^+, \tag{2.14}$$

where $\mathscr{E}^0 := \int_\Omega((1/2)(v^0)^2 + c_V\theta^0)dx$ *is the total initial energy.*

Proof. Though results of the stated type are known, we prefer to present an independent proof.

(1) We first notice that if a function $w \in L^2(Q_T)$ has the derivatives $w_x, (I^*w)_t \in L^2(Q_T)$ and $w|_{x=0} = 0$, then the function $\int_\Omega w^2 dx$ is absolutely continuous on $[0,T]$ and has the derivative

$$\frac{d}{dt}\int_\Omega w^2 dx = 2\int_\Omega (I^*w)_t w_x dx. \tag{2.15}$$

Actually, under the additional condition $w_t \in L^2(Q_T)$, by exploiting identity (2.2) we have

$$2\int_{t_1}^{t_2}\int_\Omega (I^*w)_t w_x dx\, dt = \int_\Omega w^2 dx \bigg|_{t_1}^{t_2} \tag{2.16}$$

for all $0 \le t_1 \le t_2 \le T$. In the general case, by applying (2.16) for w mollified with respect to t and passing to the limit there, we establish (2.16) for almost all t_1 and t_2 such that $0 \le t_1 \le t_2 \le T$. This leads to (2.15).

(2) We rewrite identity (2.4) in the form

$$\int_{Q_T}\left\{-v\varphi_t + (\sigma-\sigma_\Gamma)(r^k\varphi)_x\right\}dx\, dt = \int_\Omega v^0\varphi|_{t=0} dx. \tag{2.17}$$

Since $(r^k\varphi)_x = r^k\varphi_x + (r^k)_x\varphi$, by choosing $\varphi := I\zeta$ with $\zeta \in C^1(\overline{Q}_T)$ having $\zeta|_{t=0,T} = 0$ and applying (2.2), we get

$$\int_{Q_T}\left\{-(I^*v)\zeta_t + (\sigma-\sigma_\Gamma)r^k\zeta + \{I^*[(\sigma-\sigma_\Gamma)(r^k)_x]\}\zeta\right\}dx\, dt = 0. \tag{2.18}$$

Thus by definition there exists the weak derivative

$$(I^*v)_t = -(\sigma - \sigma_\Gamma)r^k - I^*\left[(\sigma - \sigma_\Gamma)(r^k)_x\right] \in L^2(Q_T), \tag{2.19}$$

see properties (2.7) and (2.10) for $\varphi \equiv 1$. By integrating over Ω this equality multiplied by v_x we have

$$\int_\Omega (I^*v_t)v_x dx = -\int_\Omega \left\{(\sigma - \sigma_\Gamma)r^k v_x + (\sigma - \sigma_\Gamma)(r^k)_x v\right\} dx = -\int_\Omega (\sigma - \sigma_\Gamma)(r^k v)_x dx, \tag{2.20}$$

where property (2.9) for $\varphi = v$ is also taken into account. This together with formula (2.15) imply the first formula (2.13).

The second formula (2.13) arises simpler after choosing $\psi \in C^1[0,T]$ with $\psi|_{t=0,T} = 0$ in identity (2.5). □

Let us establish the key equality in the paper. We set $V_0 := r_0^{k+1}/(k+1)$.

LEMMA 2.2. *The following equality holds*

$$\frac{dW}{dt} = \int_\Omega \left\{\frac{1}{k+1}\left[1 + k\left(\frac{r_0}{r}\right)^{k+1}\right]v^2 + R\theta\right\} dx, \tag{2.21}$$

where the function

$$W := \nu_1 V + \frac{2k}{k+1}\mu V_0 \log(V_0 + V) + \int_\Omega \frac{\nu}{r^k} I\eta\, dx \tag{2.22}$$

is absolutely continuous on $[0,T]$ *for any* $T > 0$.

Proof. Equation (1.1) and the definition of σ imply

$$\nu\eta_t = \sigma\eta + R\theta = \sigma_\Gamma\eta + (\sigma - \sigma_\Gamma)\eta + R\theta. \tag{2.23}$$

By integrating this equality over Ω we get

$$\nu\frac{dV}{dt} = \sigma_\Gamma V + \int_\Omega (\sigma - \sigma_\Gamma)\eta\, dx + \int_\Omega R\theta\, dx. \tag{2.24}$$

Let us transform the first and second summands in the right-hand side. By integrating (1.1) over Ω we get

$$\frac{dV}{dt} = (r^k v)|_{x=M}. \tag{2.25}$$

Using this equality together with (2.6) for $x = M$, we obtain

$$\sigma_\Gamma V = 2k\mu\frac{(r^k v)|_{x=M}}{r^{k+1}|_{x=M}}V = \frac{2k}{k+1}\mu\frac{V}{V_0 + V}\frac{dV}{dt} = \frac{2k}{k+1}\mu\frac{d}{dt}\left[V - V_0\log(V_0 + V)\right]. \tag{2.26}$$

Let $\zeta \in C^1(\overline{Q}_T)$ and $\zeta|_{t=0,T} = 0$. By choosing $\varphi := I\zeta/r^k$ in identity (2.17), using the formula

$$\left(\frac{I\zeta}{r^k}\right)_t = \frac{I\zeta_t}{r^k} - k\frac{I\zeta}{r^{k+1}}v \tag{2.27}$$

(see (1.4)) and applying identity (2.2), we find

$$\int_{Q_T}\left\{-\left(I^*\frac{v}{r^k}\right)\zeta_t + k\left(I^*\frac{v^2}{r^{k+1}}\right)\zeta + (\sigma-\sigma_\Gamma)\zeta\right\}dx\,dt = 0. \tag{2.28}$$

This means that there exists the derivative

$$\left(I^*\frac{v}{r^k}\right)_t = -kI^*\left(\frac{v^2}{r^{k+1}}\right) - (\sigma-\sigma_\Gamma) \in L^2(Q_T). \tag{2.29}$$

Moreover, $(I^*(v/r^k))_t\eta \in L^{1,2}(Q_T)$ according to property (2.12). By integrating over Ω the last equality multiplied by η we have

$$\int_\Omega(\sigma-\sigma_\Gamma)\eta\,dx = -\frac{d}{dt}\int_\Omega\left(I^*\frac{v}{r^k}\right)\eta\,dx + \int_\Omega\left(I^*\frac{v}{r^k}\right)\eta_t dx - \int_\Omega kI^*\left(\frac{v^2}{r^{k+1}}\right)\eta\,dx. \tag{2.30}$$

Therefore by applying identity (2.2), equalities $I\eta_t = r^k v$ and $I\eta = (r^{k+1}/(k+1)) - (r_0^{k+1}/(k+1))$, see (1.1) and (2.6), we obtain

$$\int_\Omega(\sigma-\sigma_\Gamma)\eta\,dx = -\frac{d}{dt}\int_\Omega\frac{v}{r^k}I\eta\,dx + \int_\Omega\left\{\frac{1}{k+1}v^2 + \frac{k}{k+1}v^2\left(\frac{r_0}{r}\right)^{k+1}\right\}dx. \tag{2.31}$$

Inserting equality (2.26) together with the last one into (2.24), we complete the proof. □

Now we are in a position to prove the main result. Let $V^0 := \int_\Omega \eta^0 dx$ be the initial volume.

Proposition 2.3. *The following two-sided bounds for the gas volume hold*

$$\alpha_{1\varepsilon}\mathscr{E}^0 t + \beta_{1\varepsilon} \le V(t) \le \alpha_{2\varepsilon}\mathscr{E}^0 t + \beta_{2\varepsilon} \quad \text{for any } t \ge 0, \tag{2.32}$$

with any $0 < \varepsilon < \nu_1$ *and*

$$\alpha_{1\varepsilon} := \frac{\min\{2/(k+1), R/c_V\}}{\nu_1+\varepsilon}, \qquad \alpha_{2\varepsilon} := \frac{\max\{2, R/c_V\}}{\nu_1-\varepsilon}, \\ \beta_{i\varepsilon} = \beta_{i\varepsilon}(V^0,\mathscr{E}^0,\nu,\mu,M,V_0), \quad i=1,2. \tag{2.33}$$

Proof. By virtue of the energy conservation law we have

$$\min\left\{\frac{2}{k+1},\frac{R}{c_V}\right\}\mathscr{E}^0 \le \int_\Omega \left\{\frac{1}{k+1}\left[1+k\left(\frac{r_0}{r}\right)^{k+1}\right]v^2+R\theta\right\}dx \le \max\left\{2,\frac{R}{c_V}\right\}\mathscr{E}^0, \tag{2.34}$$

$$\|v\|_\Omega \le \sqrt{2\mathscr{E}^0}. \tag{2.35}$$

The latter bound and equality (2.6) together with the Young inequality imply

$$\begin{aligned}\left|\int_\Omega \frac{v}{r^k}I\eta\,dx\right| &\le \|v\|_{L^1(\Omega)}\left\|\left(\frac{I\eta}{r^{k+1}}\right)^{k/(k+1)}\right\|_{C(\overline{\Omega})}V^{1/(k+1)}\\ &\le \sqrt{2M\mathscr{E}^0}\frac{1}{(k+1)^{k/(k+1)}}V^{1/(k+1)}\\ &\le \frac{1}{k+1}\left(\varepsilon_0 V + c^0\varepsilon_0^{-1/k}\right),\end{aligned} \tag{2.36}$$

with $c^0 := c_{0k}(M\mathscr{E}^0)^{(k+1)/(2k)}$ and $c_{0k} > 0$ depending on k only, for any $\varepsilon_0 > 0$. Therefore

$$|W - \nu_1 V| \le \frac{1}{k+1}\left[(2k|\mu|\varepsilon_1+\varepsilon_0)V + 2k|\mu|V_0(|\log V_0| + c_{\varepsilon_1}) + c^0\varepsilon_0^{-1/k}\right], \tag{2.37}$$

with $c_{\varepsilon_1} := \log(\varepsilon_1^{-1}) + \varepsilon_1 - 1$, for any $\varepsilon_1 > 0$. This inequality remains valid for W and V replaced by $W(0)$ and V^0.

By integrating the key equality (2.21) and applying inequalities (2.34) and (2.37) with suitable ε_0 and ε_1 together with condition (1.8), we obtain the two-sided bounds (2.32). □

Notice that the assumption $r_0 > 0$ has been not so crucial, the quantities $\beta_{i\varepsilon}$ in (2.32) are bounded as $r_0 \to 0$ and thus the case without core, that is, $r_0 = 0$, could be also covered (at least for classical solutions) but we would not like to come into these details here.

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