SINGLE BLOW-UP SOLUTIONS FOR A SLIGHTLY SUBCRITICAL BIHARMONIC EQUATION

KHALIL EL MEHDI

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We consider a biharmonic equation under the Navier boundary condition and with a nearly critical exponent (P_ε): $\Delta^2 u = u^{9-\varepsilon}$, $u > 0$ in Ω and $u = \Delta u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^5$, $\varepsilon > 0$. We study the asymptotic behavior of solutions of (P_ε) which are minimizing for the Sobolev quotient as ε goes to zero. We show that such solutions concentrate around a point $x_0 \in \Omega$ as $\varepsilon \to 0$, moreover x_0 is a critical point of the Robin's function. Conversely, we show that for any nondegenerate critical point x_0 of the Robin's function, there exist solutions of (P_ε) concentrating around x_0 as $\varepsilon \to 0$.

1. Introduction and results

Let us consider the following biharmonic equation under the Navier boundary condition

$$\begin{aligned} \Delta^2 u = u^{p-\varepsilon}, &\quad u > 0 \text{ in } \Omega \\ \Delta u = u = 0 &\quad \text{on } \partial\Omega, \end{aligned} \tag{Q_ε}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 5$, ε is a small positive parameter, and $p+1 = 2n/(n-4)$ is the critical Sobolev exponent of the embedding $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^{2n/(n-4)}(\Omega)$.

It is known that (Q_ε) is related to the limiting problem (Q_0) (when $\varepsilon = 0$) which exhibits a lack of compactness and gives rise to solutions of (Q_ε) which blow up as $\varepsilon \to 0$. The interest of the limiting problem (Q_0) grew from its resemblance to some geometric equations involving Paneitz operator and which have widely been studied in these last years (for details one can see [4, 6, 10, 12–14, 17] and references therein).

Several authors have studied the existence and behavior of blowing up solutions for the corresponding second order elliptic problem (see, e.g., [1, 3, 9, 18, 21, 22, 24–26] and references therein). In sharp contrast to this, very little is known for fourth order elliptic equations. In this paper we are mainly interested in the asymptotic behavior and

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the existence of solutions of (Q_ε) which blow up around one point, and the location of this blow up point as $\varepsilon \to 0$.

The existence of solutions of (Q_ε) for all $\varepsilon \in (0, p-1)$ is well known for any domain Ω (see, e.g., [16]). For $\varepsilon = 0$, the situation is more complex, Van Der Vorst showed in [28] that if Ω is starshaped (Q_0) has no solution whereas Ebobisse and Ould Ahmedou proved in [15] that (Q_0) has a solution provided that some homology group of Ω is nontrivial. This topological condition is sufficient, but not necessary, as examples of contractible domains Ω on which a solution exists show [19].

In view of this qualitative change in the situation when $\varepsilon = 0$, it is interesting to study the asymptotic behavior of the subcritical solution u_ε of (Q_ε) as $\varepsilon \to 0$. Chou and Geng [11], and Geng [20] made a first study, when Ω is strictly convex. The convexity assumption was needed in their proof in order to apply the method of moving planes (MMP for short) in proving a priori estimate near the boundary. Notice that in the Laplacian case (see [21]), the MMP has been used to show that blow up points are away from the boundary of the domain. The process is standard if domains are convex. For nonconvex regions, the MMP still works in the Laplacian case through the applications of Kelvin transformations [21]. For (Q_ε), the MMP also works for convex domains [11]. However, for nonconvex domains, a Kelvin transformation does not work for (Q_ε) because the Navier boundary condition is not invariant under the Kelvin transformation of biha rmonic operator. In [5], Ben Ayed and El Mehdi removed the convexity assumption of Chou and Geng for higher dimensions, that is $n \geq 6$. The aim of this paper is to prove that the results of [5] are true in dimension 5. In order to state precisely our results, we need to introduce some notations.

We consider the following problem

$$\begin{aligned} \Delta^2 u = u^{9-\varepsilon}, \quad & u > 0 \text{ in } \Omega \\ \Delta u = u = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{P_ε}$$

where Ω is a smooth bounded domain in $\mathbb{R}^5$ and ε is a small positive parameter.

Let us define on Ω the following Robin's function

$$\varphi(x) = H(x,x), \quad \text{with } H(x,y) = |x-y|^{-1} - G(x,y), \text{ for } (x,y) \in \Omega \times \Omega, \tag{1.1}$$

where G is the Green's function of Δ^2, that is,

$$\begin{aligned} \forall x \in \Omega \quad \Delta^2 G(x,\cdot) = c\delta_x \quad & \text{in } \Omega \\ \Delta G(x,\cdot) = G(x,\cdot) = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where δ_x denotes the Dirac mass at x and $c = 3\omega_5$, with ω_5 is the area of the unit sphere of $\mathbb{R}^5$. For $\lambda > 0$ and $a \in \mathbb{R}^5$, let

$$\delta_{a,\lambda}(x) = \frac{c_0 \lambda^{1/2}}{\left(1 + \lambda^2 |x-a|^2\right)^{1/2}}, \quad c_0 = (105)^{1/8}. \tag{1.3}$$

It is well known (see [23]) that $\delta_{a,\lambda}$ are the only solutions of

$$\Delta^2 u = u^9, \quad u > 0 \text{ in } \mathbb{R}^5 \tag{1.4}$$

and are also the only minimizers of the Sobolev inequality on the whole space, that is

$$S = \inf\left\{|\Delta u|^2_{L^2(\mathbb{R}^5)}|u|^{-2}_{L^{10}(\mathbb{R}^5)},\ \text{s.t. } \Delta u \in L^2,\ u \in L^{10},\ u \neq 0\right\}. \tag{1.5}$$

We denote by $P\delta_{a,\lambda}$ the projection of $\delta_{a,\lambda}$ on $\mathcal{H}(\Omega) := H^2(\Omega) \cap H^1_0(\Omega)$, defined by

$$\Delta^2 P\delta_{a,\lambda} = \Delta^2 \delta_{a,\lambda} \quad \text{in } \Omega, \qquad \Delta P\delta_{a,\lambda} = P\delta_{a,\lambda} = 0 \quad \text{on } \partial\Omega. \tag{1.6}$$

Let

$$\begin{gathered}\theta_{a,\lambda} = \delta_{a,\lambda} - P\delta_{a,\lambda}, \\ \|u\| = \left(\int_\Omega |\Delta u|^2\right)^{1/2}, \qquad \langle u,v\rangle = \int_\Omega \Delta u \Delta v, \quad u,v \in H^2(\Omega) \cap H^1_0(\Omega) \\ \|u\|_q = |u|_{L^q(\Omega)}.\end{gathered} \tag{1.7}$$

Thus we have the following result.

Theorem 1.1. *Let (u_ε) be a solution of (P_ε), and assume that*

$$||u_\varepsilon||^2 ||u_\varepsilon||^{-2}_{10-\varepsilon} \longrightarrow S \quad \text{as } \varepsilon \longrightarrow 0, \tag{H}$$

where S is the best Sobolev constant in $\mathbb{R}^5$ defined by (1.5). Then (up to a subsequence) there exist $a_\varepsilon \in \Omega$, $\lambda_\varepsilon > 0$, $\alpha_\varepsilon > 0$ and v_ε such that u_ε can be written as

$$u_\varepsilon = \alpha_\varepsilon P\delta_{a_\varepsilon,\lambda_\varepsilon} + v_\varepsilon \tag{1.8}$$

with $\alpha_\varepsilon \to 1$, $\|v_\varepsilon\| \to 0$, $a_\varepsilon \in \Omega$ and $\lambda_\varepsilon d(a_\varepsilon,\partial\Omega) \to +\infty$ as $\varepsilon \to 0$.

In addition, a_ε converges to a critical point $x_0 \in \Omega$ of φ and we have

$$\lim_{\varepsilon\to 0} \varepsilon ||u_\varepsilon||^2_{L^\infty(\Omega)} = (c_1 c_0^2/c_2)\varphi(x_0), \tag{1.9}$$

where $c_1 = c_0^{10}\int_{\mathbb{R}^5}(dx/(1+|x|^2)^{9/2})$, $c_2 = c_0^{10}\int_{\mathbb{R}^5}(\log(1+|x|^2)(1-|x|^2)/(1+|x|^2)^6)dx$ and $c_0 = (105)^{1/8}$.

Our next result provides a kind of converse to Theorem 1.1.

Theorem 1.2. *Assume that $x_0 \in \Omega$ is a nondegenerate critical point of φ. Then there exists an $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0,\varepsilon_0]$, (P_ε) has a solution of the form*

$$u_\varepsilon = \alpha_\varepsilon P\delta_{a_\varepsilon,\lambda_\varepsilon} + v_\varepsilon \tag{1.10}$$

with $\alpha_\varepsilon \to 1$, $\|v_\varepsilon\| \to 0$, $a_\varepsilon \to x_0$ and $\lambda_\varepsilon d(a_\varepsilon,\partial\Omega) \to +\infty$ as $\varepsilon \to 0$.

Our strategy to prove the above results is the same as in higher dimensions. However, as usual in elliptic equations involving critical Sobolev exponent, we need more refined estimates of the asymptotic profiles of solutions when $\varepsilon \to 0$ to treat the lower dimensional case. Such refined estimates, which are of self interest, are highly nontrivial and use in a crucial way careful expansions of the Euler-Lagrange functional associated to (P_ε), and its gradient near a small neighborhood of highly concentrated functions. To perform such expansions we make use of the techniques developed by Bahri [2] and Rey [25, 27] in the framework of the *Theory of critical points at infinity*.

The outline of the paper is the following: in Section 2 we perform some crucial estimates needed in our proofs and Section 3 is devoted to the proof of our results.

2. Some crucial estimates

In this section, we prove some crucial estimates which will play an important role in proving our results. We first recall some results.

Proposition 2.1 [8]. *Let $a \in \Omega$ and $\lambda > 0$ such that $\lambda d(a,\partial\Omega)$ is large enough. For $\theta_{(a,\lambda)} = \delta_{(a,\lambda)} - P\delta_{(a,\lambda)}$, we have the following estimates*

$$0 \le \theta_{(a,\lambda)} \le \delta_{(a,\lambda)}, \quad \theta_{(a,\lambda)} = c_0\lambda^{-1/2}H(a,\cdot) + f_{(a,\lambda)}, \tag{2.1}$$

where $f_{(a,\lambda)}$ satisfies

$$f_{(a,\lambda)} = O\left(\frac{1}{\lambda^{5/2}d^3}\right), \qquad \lambda\frac{\partial f_{(a,\lambda)}}{\partial\lambda} = O\left(\frac{1}{\lambda^{5/2}d^3}\right), \qquad \frac{1}{\lambda}\frac{\partial f_{(a,\lambda)}}{\partial a} = O\left(\frac{1}{\lambda^{7/2}d^4}\right), \tag{2.2}$$

where d is the distance $d(a,\partial\Omega)$,

$$\begin{gathered} |\theta_{(a,\lambda)}|_{L^{10}} = O((\lambda d)^{-1/2}), \qquad ||\theta_{(a,\lambda)}|| = O((\lambda d)^{-1/2}), \\ \left|\lambda\frac{\partial\theta_{(a,\lambda)}}{\partial\lambda}\right|_{L^{10}} = O\left(\frac{1}{(\lambda d)^{1/2}}\right), \qquad \left|\frac{1}{\lambda}\frac{\partial\theta_{(a,\lambda)}}{\partial a}\right|_{L^{10}} = O\left(\frac{1}{(\lambda d)^{3/2}}\right). \end{gathered} \tag{2.3}$$

Proposition 2.2 [5]. *Let u_ε be a solution of (P_ε) which satisfies (H). Then, there exist $a_\varepsilon \in \Omega$, $\alpha_\varepsilon > 0$, $\lambda_\varepsilon > 0$ and v_ε such that*

$$u_\varepsilon = \alpha_\varepsilon P\delta_{a_\varepsilon,\lambda_\varepsilon} + v_\varepsilon \tag{2.4}$$

with $\alpha_\varepsilon \to 1$, $\lambda_\varepsilon d(a_\varepsilon,\partial\Omega) \to \infty$, $c_0^{-2}\|u_\varepsilon\|_\infty^2/\lambda_\varepsilon \to 1$, $\|u_\varepsilon\|_\infty^\varepsilon \to 1$ and $\|v_\varepsilon\| \to 0$.

Furthermore, $v_\varepsilon \in E_{(a_\varepsilon,\lambda_\varepsilon)}$ which is the set of $v \in \mathcal{H}(\Omega)$ such that

$$\langle v, P\delta_{a_\varepsilon,\lambda_\varepsilon}\rangle = \langle v, \partial P\delta_{a_\varepsilon,\lambda_\varepsilon}/\partial\lambda_\varepsilon\rangle = 0, \qquad \langle v_\varepsilon, \partial P\delta_{a_\varepsilon,\lambda_\varepsilon}/\partial a\rangle = 0. \tag{V_0}$$

Lemma 2.3 [5]. *$\lambda_\varepsilon^\varepsilon = 1 + o(1)$ as ε goes to zero implies that*

$$\delta_\varepsilon^{-\varepsilon} - c_0^{-\varepsilon}\lambda_\varepsilon^{\varepsilon(4-n)/2} = O(\varepsilon\log(1+\lambda_\varepsilon^2|x-a_\varepsilon|^2)) \quad \text{in } \Omega, \tag{2.5}$$

where $\delta_\varepsilon = \delta_{a_\varepsilon,\lambda_\varepsilon}$ and $d_\varepsilon = d(a_\varepsilon,\partial\Omega)$.

Proposition 2.4 [5]. *Let (u_ε) be a solution of (P_ε) which satisfies (H). Then v_ε occurring in Proposition 2.2 satisfies*

$$||v_\varepsilon|| \le C(\varepsilon + (\lambda_\varepsilon d_\varepsilon)^{-1}), \tag{2.6}$$

where C is a positive constant independent of ε.

Now, we are going to state and prove the crucial estimates needed in the proof of our theorems. In order to simplify the notations, we set $\delta_\varepsilon = \delta_{a_\varepsilon,\lambda_\varepsilon}$, $P\delta_\varepsilon = P\delta_{a_\varepsilon,\lambda_\varepsilon}$, $\theta_\varepsilon = \theta_{a_\varepsilon,\lambda_\varepsilon}$ and $d_\varepsilon = d(a_\varepsilon,\partial\Omega)$.

Lemma 2.5. *For ε small, we have the following estimates*

(i) $\int_\Omega \delta_\varepsilon^9(1/\lambda_\varepsilon)(\partial P\delta_\varepsilon/\partial a) = -(c_1/2\lambda_\varepsilon^2)(\partial H/\partial a)(a_\varepsilon,a_\varepsilon) + O(1/(\lambda_\varepsilon d_\varepsilon)^3)$,

(ii) $\int_\Omega P\delta_\varepsilon^{9-\varepsilon}(1/\lambda_\varepsilon)(\partial P\delta_\varepsilon/\partial a) = -(c_1/\lambda_\varepsilon^{2+\varepsilon/2})(\partial H/\partial a)(a_\varepsilon,a_\varepsilon) + O(1/(\lambda_\varepsilon d_\varepsilon)^3 + \varepsilon/(\lambda_\varepsilon d_\varepsilon)^2)$,

where c_1 is the constant defined in Theorem 1.1.

Proof. Notice that

$$\int_{\Omega\setminus B_\varepsilon} \delta_\varepsilon^{10} = O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^5}\right). \tag{2.7}$$

Thus, we have, for $1 \le k \le 5$

$$\int_\Omega \delta_\varepsilon^9 \frac{1}{\lambda_\varepsilon}\frac{\partial P\delta_\varepsilon}{\partial a_k} = \int_\Omega \delta_\varepsilon^9 \frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k} - \int_\Omega \delta_\varepsilon^9 \frac{1}{\lambda_\varepsilon}\frac{\partial \theta_\varepsilon}{\partial a_k} = -\int_{B_\varepsilon} \delta_\varepsilon^9 \frac{1}{\lambda_\varepsilon}\frac{\partial \theta_\varepsilon}{\partial a_k} + O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^5}\right), \tag{2.8}$$

where $B_\varepsilon = B(a_\varepsilon,d_\varepsilon)$. Expanding $\partial\theta_\varepsilon/\partial a_k$ around a_ε and using Proposition 2.1, we obtain

$$\int_{B_\varepsilon} \delta_\varepsilon^9 \frac{1}{\lambda_\varepsilon}\frac{\partial \theta_\varepsilon}{\partial a_k} = \frac{c_0}{2\lambda_\varepsilon^{3/2}}\frac{\partial H(a_\varepsilon,a_\varepsilon)}{\partial a}\int_{B_\varepsilon}\delta_\varepsilon^9 + O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^3}\right). \tag{2.9}$$

Estimating the integral on the right-hand side in (2.9) and using (2.8), we easily derive claim (i). To prove claim (ii), we write

$$\begin{aligned}\int_\Omega P\delta_\varepsilon^{9-\varepsilon}\frac{1}{\lambda_\varepsilon}\frac{\partial P\delta_\varepsilon}{\partial a_k} &= \int_\Omega \delta_\varepsilon^{9-\varepsilon}\frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k} - \int_\Omega \delta_\varepsilon^{9-\varepsilon}\frac{1}{\lambda_\varepsilon}\frac{\partial \theta_\varepsilon}{\partial a_k} - (9-\varepsilon)\int_\Omega \delta_\varepsilon^{8-\varepsilon}\theta_\varepsilon\frac{1}{\lambda}\frac{\partial \delta_\varepsilon}{\partial a_k}\\ &\quad + \frac{(9-\varepsilon)(8-\varepsilon)}{2}\int_\Omega \delta_\varepsilon^{7-\varepsilon}\theta_\varepsilon^2\frac{1}{\lambda}\frac{\partial \delta_\varepsilon}{\partial a_k} + O\left(\int_\Omega \delta_\varepsilon^{8-\varepsilon}\theta_\varepsilon\left|\frac{1}{\lambda_\varepsilon}\frac{\partial\theta_\varepsilon}{\partial a_k}\right| + \int \delta_\varepsilon^{7-\varepsilon}\theta_\varepsilon^3\right)\end{aligned} \tag{2.10}$$

and we have to estimate each term on the right-hand side of (2.10).

Using Proposition 2.1 and Lemma 2.3, we have

$$\begin{gathered}\int_\Omega \delta_\varepsilon^{7-\varepsilon}\theta_\varepsilon^3 \le c\|\theta_\varepsilon\|_\infty^3\int\delta_\varepsilon^7 = O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^3}\right),\\ \int_\Omega \delta_\varepsilon^{8-\varepsilon}\theta_\varepsilon\left|\frac{1}{\lambda_\varepsilon}\frac{\partial\theta_\varepsilon}{\partial a_k}\right| \le c\|\theta_\varepsilon\|_\infty\left|\left|\frac{1}{\lambda_\varepsilon}\frac{\partial\theta_\varepsilon}{\partial a_k}\right|\right|_\infty\int_\Omega\delta_\varepsilon^8 = O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^3}\right).\end{gathered} \tag{2.11}$$

We also have

$$\int_\Omega \delta_\varepsilon^{9-\varepsilon}\frac{1}{\lambda_\varepsilon}\frac{\partial\delta_\varepsilon}{\partial a_k} = \int_{\Omega\setminus B_\varepsilon}\delta_\varepsilon^{9-\varepsilon}\frac{1}{\lambda_\varepsilon}\frac{\partial\delta_\varepsilon}{\partial a_k} = O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^5}\right). \tag{2.12}$$

Expanding θ_ε around a_ε and using Proposition 2.1 and Lemma 2.3, we obtain

$$9\int_{B_\varepsilon}\delta_\varepsilon^{8-\varepsilon}\theta_\varepsilon\frac{1}{\lambda_\varepsilon}\frac{\partial\delta_\varepsilon}{\partial a_k}=\frac{c_1}{2\lambda_\varepsilon^{2+\varepsilon/2}}\frac{\partial H(a_\varepsilon,a_\varepsilon)}{\partial a}+O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^3}+\frac{\varepsilon}{(\lambda_\varepsilon d_\varepsilon)^2}\right),$$
$$\int_{B_\varepsilon}\delta_\varepsilon^{7-\varepsilon}\theta_\varepsilon^2\frac{1}{\lambda_\varepsilon}\frac{\partial\delta_\varepsilon}{\partial a_k}=O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^3}\right). \tag{2.13}$$

In the same way, we find

$$\int_\Omega\delta_\varepsilon^{9-\varepsilon}\frac{1}{\lambda_\varepsilon}\frac{\partial\theta_\varepsilon}{\partial a_k}=\frac{c_1}{2\lambda_\varepsilon^{2+\varepsilon/2}}\frac{\partial H(a_\varepsilon,a_\varepsilon)}{\partial a}+O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^3}+\frac{\varepsilon}{(\lambda_\varepsilon d_\varepsilon)^2}\right). \tag{2.14}$$

Combining (2.10)–(2.14), we obtain claim (ii). □

To improve the estimates of the integrals involving v_ε, we use an idea of Rey [27], namely we write

$$v_\varepsilon=\Pi v_\varepsilon+w_\varepsilon, \tag{2.15}$$

where Πv_ε denotes the projection of v_ε onto $H^2\cap H_0^1(B_\varepsilon)$, that is

$$\Delta^2\Pi v_\varepsilon=\Delta^2 v_\varepsilon\quad\text{in } B_\varepsilon;\qquad \Delta\Pi v_\varepsilon=\Pi v_\varepsilon=0\quad\text{on }\partial B_\varepsilon, \tag{2.16}$$

where $B_\varepsilon=B(a_\varepsilon,d_\varepsilon)$. We split Πv_ε in an even part Πv_ε^e and an odd part Πv_ε^o with respect to $(x-a_\varepsilon)_k$, thus we have

$$v_\varepsilon=\Pi v_\varepsilon^e+\Pi v_\varepsilon^o+w_\varepsilon\quad\text{in } B_\varepsilon \text{ with } \Delta^2 w_\varepsilon=0 \text{ in } B_\varepsilon. \tag{2.17}$$

Notice that it is difficult to improve the estimate (2.6) of the v_ε-part of solutions. However, it is sufficient to improve the integrals involving the odd part of v_ε with respect to $(x-a_\varepsilon)_k$, for $1\le k\le 5$ and to know the exact contribution of the integrals containing the w_ε-part of v_ε. Let us start by the terms involving w_ε.

Lemma 2.6. *For ε small, we have that*

$$\int_{B_\varepsilon}\delta_\varepsilon^8\left(\delta_\varepsilon^{-\varepsilon}-\frac{1}{c_0^\varepsilon\lambda_\varepsilon^{\varepsilon/2}}\right)\frac{1}{\lambda_\varepsilon}\frac{\partial\delta_\varepsilon}{\partial a_k}w_\varepsilon=O\left(\frac{\varepsilon\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^{1/2}}\right). \tag{2.18}$$

Proof. Let ψ be the solution of

$$\Delta^2\psi=\delta_\varepsilon^8\left(\delta_\varepsilon^{-\varepsilon}-\frac{1}{c_0^\varepsilon\lambda_\varepsilon^{\varepsilon/2}}\right)\frac{1}{\lambda_\varepsilon}\frac{\partial\delta_\varepsilon}{\partial a_k}\quad\text{in } B_\varepsilon;\qquad \Delta\psi=\psi=0\quad\text{on }\partial B_\varepsilon. \tag{2.19}$$

Thus we have

$$I_\varepsilon:=\int_{B_\varepsilon}\Delta^2\psi w_\varepsilon=\int_{\partial B_\varepsilon}\frac{\partial\psi}{\partial\nu}\Delta w_\varepsilon+\int_{\partial B_\varepsilon}\frac{\partial\Delta\psi}{\partial\nu}w_\varepsilon. \tag{2.20}$$

Let G_ε be the Green's function for the biharmonic operator on B_ε with the Navier boundary conditions, that is,

$$\Delta^2 G_\varepsilon(x,\cdot) = c\delta_x \quad \text{in } B_\varepsilon; \qquad \Delta G_\varepsilon(x,\cdot) = G(x,\cdot) = 0 \quad \text{on } \partial B_\varepsilon, \tag{2.21}$$

where $c = 3w_5$. Therefore ψ is given by

$$\psi(y) = \int_{B_\varepsilon} G_\varepsilon(x,y)\delta_\varepsilon^8\left(\delta_\varepsilon^{-\varepsilon} - \frac{1}{c_0^\varepsilon \lambda_\varepsilon^{\varepsilon/2}}\right)\frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k}, \quad y \in B_\varepsilon \tag{2.22}$$

and its normal derivative by

$$\frac{\partial \psi}{\partial \nu}(y) = \int_{B_\varepsilon} \frac{\partial G_\varepsilon}{\partial \nu}(x,y)\delta_\varepsilon^8\left(\delta_\varepsilon^{-\varepsilon} - \frac{1}{c_0^\varepsilon \lambda_\varepsilon^{\varepsilon/2}}\right)\frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k}, \quad y \in \partial B_\varepsilon. \tag{2.23}$$

Notice that for $y \in \partial B_\varepsilon$ we have the following estimates: for $x \in B_\varepsilon \setminus B(y, d_\varepsilon/2)$, we have

$$\frac{\partial G_\varepsilon}{\partial \nu}(x,y) = O\left(\frac{1}{d_\varepsilon^2}\right); \qquad \frac{\partial \Delta G_\varepsilon}{\partial \nu}(x,y) = O\left(\frac{1}{d_\varepsilon^4}\right) \tag{2.24}$$

for $x \in B_\varepsilon \cap B(y, d_\varepsilon/2)$, we have

$$\left|\frac{\partial G_\varepsilon}{\partial \nu}(x,y)\right| \le \frac{c}{|x-y|^2}; \qquad \left|\frac{\partial \Delta G_\varepsilon}{\partial \nu}(x,y)\right| \le \frac{c}{|x-y|^4} \tag{2.25}$$

for $x \in B_\varepsilon \cap B(y, d_\varepsilon/2)$, we have

$$\delta_\varepsilon^8\left(\delta_\varepsilon^{-\varepsilon} - \frac{1}{c_0^\varepsilon \lambda_\varepsilon^{\varepsilon/2}}\right)\frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k} = O\left(\frac{\varepsilon \log \lambda_\varepsilon d_\varepsilon}{(\lambda_\varepsilon d_\varepsilon)^9}\right), \tag{2.26}$$

for $x \in B_\varepsilon \setminus B(y, d_\varepsilon/2)$, we have

$$\delta_\varepsilon^8\left(\delta_\varepsilon^{-\varepsilon} - \frac{1}{c_0^\varepsilon \lambda_\varepsilon^{\varepsilon/2}}\right)\frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k} = O(\delta_\varepsilon^9 \varepsilon \log(1 + \lambda_\varepsilon^2 |x - a_\varepsilon|^2)). \tag{2.27}$$

Therefore

$$\left|\frac{\partial \psi}{\partial \nu}(y)\right| = O\left(\frac{\varepsilon}{\lambda_\varepsilon^{1/2} d_\varepsilon^2}\right). \tag{2.28}$$

In the same way, we have

$$\left|\frac{\partial \Delta \psi}{\partial \nu}(y)\right| = O\left(\frac{\varepsilon}{\lambda_\varepsilon^{1/2} d_\varepsilon^4}\right). \tag{2.29}$$

Using (2.20), (2.28), (2.29), we obtain

$$I_\varepsilon = O\left(\frac{\varepsilon}{\lambda_\varepsilon^{1/2} d_\varepsilon^2}\int_{\partial B_\varepsilon} |\Delta w_\varepsilon| + \frac{\varepsilon}{\lambda_\varepsilon^{1/2} d_\varepsilon^4}\int_{\partial B_\varepsilon} |w_\varepsilon|\right). \tag{2.30}$$

To estimate the right-hand side of (2.30), we introduce the following function

$$\bar{w}(X) = d_\varepsilon^{1/2} w_\varepsilon(a_\varepsilon + d_\varepsilon X), \qquad \bar{v}(X) = d_\varepsilon^{1/2} v_\varepsilon(a_\varepsilon + d_\varepsilon X) \quad \text{for } X \in B(0,1). \tag{2.31}$$

$\bar{w}$ satisfies

$$\Delta^2 \bar{w} = 0 \quad \text{in } B := B(0,1); \qquad \Delta\bar{w} = \Delta\bar{v}, \quad \bar{w} = \bar{v} \text{ on } \partial B. \tag{2.32}$$

We deduce that

$$\int_{\partial B} |\Delta\bar{w}| + \int_{\partial B} |\Delta\bar{w}| \le C\left(\int_B |\Delta\bar{v}|^2\right)^{1/2} = C\left(\int_{B_\varepsilon} |\Delta v_\varepsilon|^2\right)^{1/2}. \tag{2.33}$$

But, we have

$$\int_{\partial B} |\Delta\bar{w}| + \int_{\partial B} |\Delta\bar{w}| = \left(\frac{1}{d_\varepsilon}\right)^{3/2} \int_{\partial B_\varepsilon} |\Delta w_\varepsilon| + \left(\frac{1}{d_\varepsilon}\right)^{7/2} \int_{\partial B_\varepsilon} |w_\varepsilon|. \tag{2.34}$$

Using (2.30), (2.33) and (2.34), the lemma follows. □

Lemma 2.7. *For ε small, we have*
(i) $\int_{B_\varepsilon} \Delta((1/\lambda_\varepsilon)(\partial\Pi\delta_\varepsilon/\partial a_k))\Delta w_\varepsilon = O(\|v_\varepsilon\|/(\lambda_\varepsilon d_\varepsilon)^{3/2})$,
(ii) $\int_{B_\varepsilon} \delta_\varepsilon^{8-\varepsilon}\Pi v_\varepsilon^o w_\varepsilon = O(\|v_\varepsilon\|\|\Pi v_\varepsilon^o\|/(\lambda_\varepsilon d_\varepsilon)^{1/2})$.

Proof. Using (2.17), we obtain

$$\int_{B_\varepsilon} \Delta\left(\frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_k}\right)\Delta w_\varepsilon = \int_{\partial B_\varepsilon} \frac{\partial\psi_k}{\partial\nu}\Delta w_\varepsilon, \quad \text{with } \psi_k = \frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_k}. \tag{2.35}$$

Using an integral representation for ψ_k as in (2.23), we obtain for $y \in \partial B_\varepsilon$,

$$\frac{\partial\psi}{\partial\nu}(y) = \int_{B_\varepsilon} \frac{\partial G_\varepsilon}{\partial\nu}(x,y)\Delta^2\psi_k, \tag{2.36}$$

where G_ε is the Green's function defined in (2.21). Clearly, we have

$$\Pi\delta_\varepsilon(x) = \delta_\varepsilon(x) - \frac{c_0\lambda_\varepsilon^{1/2}}{(1+\lambda_\varepsilon^2 d_\varepsilon^2)^{1/2}} - \frac{c_\varepsilon(a_\varepsilon,d_\varepsilon)}{10}(|x-a_\varepsilon|^2 - d_\varepsilon^2), \tag{2.37}$$

with $c_\varepsilon(a_\varepsilon,d_\varepsilon) = \Delta\delta_{\varepsilon|\partial B_\varepsilon}$. Thus we deduce that

$$\frac{\partial\psi}{\partial\nu}(y) = 9\int_{B_\varepsilon} \frac{\partial G_\varepsilon}{\partial\nu}(x,y)\delta_\varepsilon^8 \frac{1}{\lambda_\varepsilon}\frac{\partial\delta_\varepsilon}{\partial a_k}. \tag{2.38}$$

In $B_\varepsilon \setminus B(a_\varepsilon, d_\varepsilon/2)$, we argue as in (2.28) and (2.25), we obtain

$$\int_{B_\varepsilon} \frac{\partial G_\varepsilon}{\partial\nu}(x,y)\delta_\varepsilon^8 \frac{1}{\lambda_\varepsilon}\frac{\partial\delta_\varepsilon}{\partial a_k} = O\left(\frac{1}{\lambda_\varepsilon^{9/2} d_\varepsilon^6}\right). \tag{2.39}$$

Furthermore, since

$$\left|\nabla \frac{\partial G_\varepsilon}{\partial \nu}(x,y)\right| = O\left(\frac{1}{d_\varepsilon^3}\right) \quad \text{for } (x,y) \in B(a_\varepsilon, d_\varepsilon/2) \times \partial B_\varepsilon, \tag{2.40}$$

we obtain

$$\left| \int_{B(a_\varepsilon,d_\varepsilon/2)} \frac{\partial G_\varepsilon}{\partial \nu}(x,y)\delta_\varepsilon^8 \frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k}\right| \le \frac{c}{d_\varepsilon^3}\int_{B(a_\varepsilon,d_\varepsilon/2)} \delta_\varepsilon^9 \left| x - a_\varepsilon \right| = O\left(\frac{1}{\lambda_\varepsilon^{3/2} d_\varepsilon^3}\right), \tag{2.41}$$

where we have used the evenness of δ_ε and the oddness of its derivative. Thus

$$\frac{\partial \psi_k}{\partial \nu}(y) = O\left(\frac{1}{\lambda_\varepsilon^{3/2} d_\varepsilon^3}\right). \tag{2.42}$$

Using (2.35) and (2.42), we obtain

$$\int_{B_\varepsilon} \Delta\left(\frac{1}{\lambda_\varepsilon}\frac{\partial \Pi\delta_\varepsilon}{\partial a_k}\right)\Delta w_\varepsilon \le \frac{c}{\lambda_\varepsilon^{3/2} d_\varepsilon^3}\int_{\partial B_\varepsilon} \left|\Delta w_\varepsilon\right|. \tag{2.43}$$

Arguing as in (2.34), claim (i) follows. To prove claim (ii), let ψ be such that

$$\Delta^2 \psi = \delta_\varepsilon^{8-\varepsilon}\Pi v_\varepsilon^o \quad \text{in } B_\varepsilon; \qquad \Delta\psi = \psi = 0 \quad \text{on } \partial B_\varepsilon. \tag{2.44}$$

We have

$$\int_{B_\varepsilon} \delta_\varepsilon^{8-\varepsilon}\Pi v_\varepsilon^o w_\varepsilon = \int_{\partial B_\varepsilon} \frac{\partial \Delta\psi}{\partial \nu} w_\varepsilon + \int_{\partial B_\varepsilon} \frac{\partial \psi}{\partial \nu}\Delta w_\varepsilon. \tag{2.45}$$

As before, we prove that, for $y \in \partial B_\varepsilon$

$$\frac{\partial \psi}{\partial \nu}(y) = O\left(\frac{||\Pi v_\varepsilon^o||}{\lambda_\varepsilon^{1/2} d_\varepsilon^2}\right), \qquad \frac{\partial \Delta\psi}{\partial \nu}(y) = O\left(\frac{||\Pi v_\varepsilon^o||}{\lambda_\varepsilon^{1/2} d_\varepsilon^4}\right). \tag{2.46}$$

Therefore

$$\int_{B_\varepsilon} \delta_\varepsilon^{8-\varepsilon}\Pi v_\varepsilon^o w_\varepsilon \le \frac{c||\Pi v_\varepsilon^o||}{\lambda_\varepsilon^{1/2} d_\varepsilon^4}\left(\frac{1}{\delta_\varepsilon^{3/2}}\int_{\partial B_\varepsilon} \left| w_\varepsilon \right| + \frac{1}{\delta_\varepsilon^{7/2}}\int_{\partial B_\varepsilon}\left|\Delta w_\varepsilon\right|\right) \le \frac{c||v_\varepsilon||\,||\Pi v_\varepsilon^o||}{(\lambda_\varepsilon d_\varepsilon)^{1/2}}. \tag{2.47}$$

The proof of the lemma is completed. □

LEMMA 2.8. *For ε small, we have*

(i) $\int_{B_\varepsilon} \delta_\varepsilon^{7-\varepsilon} v_\varepsilon (1/\lambda_\varepsilon)(\partial\delta_\varepsilon/\partial a_k) = O(\|\Pi v_\varepsilon^o\|/\lambda_\varepsilon^{1/2} + \|v_\varepsilon\|/\lambda_\varepsilon d_\varepsilon^{1/2})$,

(ii) $\int_{B_\varepsilon} \delta_\varepsilon^{7-\varepsilon}\theta_\varepsilon v_\varepsilon (1/\lambda_\varepsilon)(\partial\delta_\varepsilon/\partial a_k) = O(\|\Pi v_\varepsilon^o\|/\lambda_\varepsilon d_\varepsilon + \|v_\varepsilon\|/(\lambda_\varepsilon d_\varepsilon)^{3/2})$.

Proof. Claim (i) can be proved in the same way as Lemma 2.6, so we omit its proof. Claim (ii) follows from Proposition 2.1 and claim (i). □

Let us now compute the contribution of the following integral which involves v_ε^2.

Lemma 2.9. *For ε small, we have*

$$\int_{B_\varepsilon} \delta_\varepsilon^{7-\varepsilon} v_\varepsilon^2 \frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k} = O\left(||\Pi v_\varepsilon^o||\,||v_\varepsilon|| + \frac{||v_\varepsilon||^2}{(\lambda_\varepsilon d_\varepsilon)^{1/2}} \right). \tag{2.48}$$

Proof. Using (2.17) and the fact that the even part of v_ε^2 has no contribution to the integrals, we obtain

$$\int_{B_\varepsilon} \delta_\varepsilon^{7-\varepsilon} v_\varepsilon^2 \frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k} = \int_{B_\varepsilon} \delta_\varepsilon^{7-\varepsilon} \frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k}(2v_\varepsilon - w_\varepsilon) w_\varepsilon + O(||\Pi v_\varepsilon^o||\,||v_\varepsilon||). \tag{2.49}$$

Let Ψ be the solution of

$$\Delta^2 \Psi = \delta_\varepsilon^{7-\varepsilon} \frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k}(2v_\varepsilon - w_\varepsilon) \quad \text{in } B_\varepsilon; \qquad \Delta\Psi = \Psi = 0 \quad \text{on } \partial B_\varepsilon. \tag{2.50}$$

Thus, as in the proof of Lemma 2.6, we obtain for $y \in \partial B_\varepsilon$

$$\frac{\partial \Psi}{\partial \nu}(y) = O\left(\frac{||v_\varepsilon||}{\lambda_\varepsilon^{1/2} d_\varepsilon^2} \right), \qquad \frac{\partial \Delta\Psi}{\partial \nu}(y) = O\left(\frac{||v_\varepsilon||}{\lambda_\varepsilon^{1/2} d_\varepsilon^4} \right) \tag{2.51}$$

and therefore

$$\int_{B_\varepsilon} \delta_\varepsilon^{7-\varepsilon} \frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k}(2v_\varepsilon - w_\varepsilon) w_\varepsilon = O\left(\frac{||v_\varepsilon||^2}{(\lambda_\varepsilon d_\varepsilon)^{1/2}} \right). \tag{2.52}$$

Thus our lemma follows. □

Next we are going to estimate the integrals involving the odd part of v_ε with respect to $(x - a_\varepsilon)_k$, for $1 \le k \le 5$.

Lemma 2.10. *For ε small, we have*

$$\int_{B_\varepsilon} u_\varepsilon^{9-\varepsilon} \Pi v_\varepsilon^o = 9\int_{B_\varepsilon} \delta_\varepsilon^8 (\Pi v_\varepsilon^o)^2 + o(||\Pi v_\varepsilon^o||^2) + O\left(||\Pi v_\varepsilon^o|| \left(\varepsilon^{3/2} + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^{3/2}} \right) \right). \tag{2.53}$$

Proof. We have

$$\begin{aligned} \int_{B_\varepsilon} u_\varepsilon^{9-\varepsilon} \Pi v_\varepsilon^o &= \alpha_\varepsilon^{9-\varepsilon} \int_{B_\varepsilon} P\delta_\varepsilon^{9-\varepsilon} \Pi v_\varepsilon^o + (9-\varepsilon)\alpha_\varepsilon^{8-\varepsilon} \int_{B_\varepsilon} P\delta_\varepsilon^{8-\varepsilon} v_\varepsilon \Pi v_\varepsilon^o \\ &\quad + O\left(\int_{B_\varepsilon} P\delta_\varepsilon^{7-\varepsilon} |v_\varepsilon|^2 |\Pi v_\varepsilon^o| + \int_{B_\varepsilon} |v_\varepsilon|^{9-\varepsilon} |\Pi v_\varepsilon^o| \right) \\ &= \alpha_\varepsilon^{9-\varepsilon} \int_{B_\varepsilon} P\delta_\varepsilon^{9-\varepsilon} \Pi v_\varepsilon^o + (9-\varepsilon)\alpha_\varepsilon^{8-\varepsilon} \int_{B_\varepsilon} P\delta_\varepsilon^{8-\varepsilon} v_\varepsilon \Pi v_\varepsilon^o + O(||v_\varepsilon||^2 ||\Pi v_\varepsilon^o||). \end{aligned} \tag{2.54}$$

We estimate the two integrals on the right-hand side in (2.54). First, using Proposition 2.1 and the Holder inequality, we have

$$\int_{B_\varepsilon} P\delta_\varepsilon^{8-\varepsilon} v_\varepsilon \Pi v_\varepsilon^o = \int_{B_\varepsilon} \delta_\varepsilon^{8-\varepsilon} v_\varepsilon \Pi v_\varepsilon^o + O\left(\frac{||v_\varepsilon|| \, ||\Pi v_\varepsilon^o||}{\lambda_\varepsilon d_\varepsilon}\right) = \int_{B_\varepsilon} \delta_\varepsilon^{8-\varepsilon} (\Pi v_\varepsilon^o)^2 + \int_{B_\varepsilon} \delta_\varepsilon^{8-\varepsilon} \Pi v_\varepsilon^o w_\varepsilon, \tag{2.55}$$

where we have used in the last equality the evenness of δ_ε and Πv_ε^e and the oddness of Πv_ε^o. By Lemmas 2.3 and 2.7 we obtain

$$\int_{B_\varepsilon} P\delta_\varepsilon^{8-\varepsilon} v_\varepsilon \Pi v_\varepsilon^o = \int_{B_\varepsilon} \delta_\varepsilon^{8} (\Pi v_\varepsilon^o)^2 + O\left(\frac{||v_\varepsilon|| \, ||\Pi v_\varepsilon^o||}{(\lambda_\varepsilon d_\varepsilon)^{1/2}}\right). \tag{2.56}$$

Secondly, we write

$$\int_{B_\varepsilon} P\delta_\varepsilon^{9-\varepsilon} \Pi v_\varepsilon^o = \int_{B_\varepsilon} \delta_\varepsilon^{9-\varepsilon} \Pi v_\varepsilon^o - (9-\varepsilon)\int_{B_\varepsilon} \delta_\varepsilon^{8-\varepsilon} \theta_\varepsilon \Pi v_\varepsilon^o + O\left(\int_{B_\varepsilon} \delta_\varepsilon^{7-\varepsilon} \theta_\varepsilon^2 \, |\Pi v_\varepsilon^o|\right). \tag{2.57}$$

Thus, using the evenness of δ_ε, the oddness of Πv_ε^o and Holder inequality, we obtain

$$\int_{B_\varepsilon} P\delta_\varepsilon^{9-\varepsilon} \Pi v_\varepsilon^o = O\left(\frac{||\Pi v_\varepsilon^o||}{(\lambda_\varepsilon d_\varepsilon)^2}\right). \tag{2.58}$$

Using (2.54), (2.56), (2.58) and Propositions 2.2 and 2.4, we easily derive our lemma. □

Lemma 2.11. *For ε small, we have*

$$||\Pi v_\varepsilon^o|| = O\left(\varepsilon^{3/2} + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^{3/2}}\right). \tag{2.59}$$

Proof. We write

$$\Pi v_\varepsilon^o = \tilde{\Pi} v_\varepsilon^o + \alpha \Pi\delta_\varepsilon + \beta\lambda_\varepsilon \frac{\partial \Pi\delta_\varepsilon}{\partial\lambda} + \sum_{r=1}^{5} \gamma_r \frac{1}{\lambda_\varepsilon} \frac{\partial \Pi\delta_\varepsilon}{\partial a_r} \tag{2.60}$$

with

$$\langle \tilde{\Pi} v_\varepsilon^o, \Pi\delta_\varepsilon \rangle = \left\langle \tilde{\Pi} v_\varepsilon^o, \frac{\partial \Pi\delta_\varepsilon}{\partial\lambda} \right\rangle = \left\langle \tilde{\Pi} v_\varepsilon^o, \frac{\partial \Pi\delta_\varepsilon}{\partial a_r} \right\rangle = 0 \quad \text{for each } r \in \{1,2,3,4,5\}. \tag{2.61}$$

Taking the scalar product in $H^2 \cap H_0^1(B_\varepsilon)$ of (2.60) with $\Pi\delta_\varepsilon$, $\lambda_\varepsilon \partial\Pi\delta_\varepsilon/\partial\lambda$, $\lambda_\varepsilon^{-1}\partial\Pi\delta_\varepsilon/\partial a_r$, $1 \le r \le 5$, provides us with the following invertible linear system in α, β, γ_r (with $1 \le r \le 5$)

$$\begin{aligned}
\langle \Pi\delta_\varepsilon, \Pi v_\varepsilon^o \rangle &= \alpha(C' + o(1)) + \beta\left\langle \Pi\delta_\varepsilon, \lambda_\varepsilon \frac{\partial\Pi\delta_\varepsilon}{\partial\lambda}\right\rangle + \sum_{r=1}^{5}\gamma_r\left\langle \Pi\delta_\varepsilon, \frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_r}\right\rangle \\
\left\langle \lambda_\varepsilon \frac{\partial\Pi\delta_\varepsilon}{\partial\lambda}, \Pi v_\varepsilon^o \right\rangle &= \alpha\left\langle \Pi\delta_\varepsilon, \lambda_\varepsilon \frac{\partial\Pi\delta_\varepsilon}{\partial\lambda}\right\rangle + \beta(C'' + o(1)) + \sum_{r=1}^{5}\gamma_r\left\langle \lambda_\varepsilon \frac{\partial\Pi\delta_\varepsilon}{\partial\lambda}, \frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_r}\right\rangle \\
\left\langle \frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_k}, \Pi v_\varepsilon^o \right\rangle &= \alpha\left\langle \Pi\delta_\varepsilon, \frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_k}\right\rangle + \beta\left\langle \lambda_\varepsilon \frac{\partial\Pi\delta_\varepsilon}{\partial\lambda}, \frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_k}\right\rangle \\
&\quad + \sum_{r=1}^{5}\gamma_r\left\langle \frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_k}, \frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_r}\right\rangle.
\end{aligned} \tag{S}$$

Observe that

$$\begin{aligned}
\left\langle \Pi\delta_\varepsilon, \lambda_\varepsilon \frac{\partial\Pi\delta_\varepsilon}{\partial\lambda}\right\rangle &= O\left(\frac{1}{\lambda_\varepsilon d_\varepsilon}\right); \\
\left\langle \lambda_\varepsilon \frac{\partial\Pi\delta_\varepsilon}{\partial\lambda}, \frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_r}\right\rangle &= O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^2}\right); \\
\left\langle \Pi\delta_\varepsilon, \frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_r}\right\rangle &= O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^2}\right); \\
\left\langle \frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_k}, \frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_r}\right\rangle &= (C''' + o(1))\delta_{kr} + O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^2}\right),
\end{aligned} \tag{2.62}$$

where δ_{kr} denotes the Kronecker symbol.

Now, because of the evenness of δ_ε and the oddness of Πv_ε^o with respect to $(x - a_\varepsilon)_k$ we obtain

$$\langle \Pi\delta_\varepsilon, \Pi v_\varepsilon^o \rangle = \int_{B_\varepsilon} \Delta\Pi\delta_\varepsilon \cdot \Delta\Pi v_\varepsilon^o = \int_{B_\varepsilon} \delta_\varepsilon^9 \Pi v_\varepsilon^o = 0. \tag{2.63}$$

In the same way we have

$$\left\langle \lambda_\varepsilon \frac{\partial\Pi\delta_\varepsilon}{\partial\lambda}, \Pi v_\varepsilon^o \right\rangle = \left\langle \frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_r}, \Pi v_\varepsilon^o \right\rangle = 0 \quad \text{for each } r \neq k. \tag{2.64}$$

We also have

$$\begin{aligned}
\left\langle \frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_k}, \Pi v_\varepsilon^o \right\rangle &= \int_{B_\varepsilon} \Delta\left(\frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_k}\right) \cdot \Delta(v_\varepsilon - \Pi v_\varepsilon^e - w_\varepsilon) \\
&= \int_{B_\varepsilon} \Delta\left(\frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_k}\right) \cdot \Delta v_\varepsilon - \int_{B_\varepsilon} \Delta\left(\frac{1}{\lambda_\varepsilon}\frac{\partial\Pi\delta_\varepsilon}{\partial a_k}\right) \cdot \Delta w_\varepsilon,
\end{aligned} \tag{2.65}$$

where we have used in the last equality the fact that Πv_ε^e is even with respect to $(x-a_\varepsilon)_k$. Using (2.37) and Holder inequality, we obtain

$$\int_{B_\varepsilon} \Delta\left(\frac{1}{\lambda_\varepsilon}\frac{\partial \Pi\delta_\varepsilon}{\partial a_k}\right)\cdot \Delta v_\varepsilon \le c\|v_\varepsilon\|\left(\int_{\Omega\setminus B_\varepsilon}\left|\Delta\frac{1}{\lambda_\varepsilon}\frac{\partial\delta_\varepsilon}{\partial a_k}\right|^2\right)^{1/2} = O\left(\frac{\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^{3/2}}\right). \tag{2.66}$$

Equation (2.66) and Lemma 2.7 imply that

$$\left\langle \frac{1}{\lambda_\varepsilon}\frac{\partial \Pi\delta_\varepsilon}{\partial a_k}, \Pi v_\varepsilon^o\right\rangle = O\left(\frac{\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^{3/2}}\right). \tag{2.67}$$

Inverting the linear system (S), we deduce from the above estimates

$$\begin{gathered} \alpha = O\left(\frac{\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^{7/2}}\right), \qquad \beta = O\left(\frac{\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^{7/2}}\right), \\ \gamma_k = O\left(\frac{\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^{3/2}}\right), \qquad \gamma_r = O\left(\frac{\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^{7/2}}\right), \quad r\neq k. \end{gathered} \tag{2.68}$$

This implies through (2.60)

$$\|\Pi v_\varepsilon^o - \widetilde{\Pi} v_\varepsilon^o\| = O\left(\frac{\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^{3/2}}\right), \qquad \|\Pi v_\varepsilon^o\|^2 = \|\widetilde{\Pi} v_\varepsilon^o\|^2 + O\left(\frac{\|v_\varepsilon\|^2}{(\lambda_\varepsilon d_\varepsilon)^{3}}\right). \tag{2.69}$$

We now turn to the last step, which consists in estimating $\|\widetilde{\Pi} v_\varepsilon^o\|$. Since u_ε is a solution of (P_ε), we have

$$\int_{B_\varepsilon} \Delta^2 u_\varepsilon \Pi v_\varepsilon^o = \int_{B_\varepsilon} u_\varepsilon^{9-\varepsilon}\Pi v_\varepsilon^o. \tag{2.70}$$

Because of the evenness of δ_ε and the oddness of Πv_ε^o with respect to $(x-a_\varepsilon)_k$, (2.70) becomes

$$\|\Pi v_\varepsilon^o\|^2 = \int_{B_\varepsilon} u_\varepsilon^{9-\varepsilon}\Pi v_\varepsilon^o. \tag{2.71}$$

By (2.69), (2.71) and Lemma 2.10, we obtain

$$\|\widetilde{\Pi v_\varepsilon^o}\|^2 - 9\int_{B_\varepsilon}\delta_\varepsilon^8 (\Pi v_\varepsilon^o)^2 + o(\|\widetilde{\Pi v_\varepsilon^o}\|^2) = O\left(\varepsilon^3 + \frac{1}{(\lambda_\varepsilon d_\varepsilon)^3}\right). \tag{2.72}$$

Using now (2.72) and the fact that the quadratic form

$$v \longmapsto \int_{B_\varepsilon} |\Delta v|^2 - 9\int_{B_\varepsilon}\delta_i^8 v^2 \tag{2.73}$$

is positive definite (see [6]) on the subset $[\mathrm{Span}(\Pi\delta_\varepsilon, \partial\Pi\delta_\varepsilon/\partial\lambda, \partial\Pi\delta_\varepsilon/\partial a_k\ 1 \leq k \leq 5)]^{\perp}_{H^2\cap H^1_0(B_\varepsilon)}$, we obtain

$$||\widetilde{\Pi v^o_\varepsilon}|| \leq C\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{3/2}} + \varepsilon^{3/2}\right). \tag{2.74}$$

Our lemma follows from (2.69) and (2.74). □

Before ending this section, let us prove the following estimate which will be needed later.

LEMMA 2.12. *For ε small, we have*

$$\left\langle \frac{\partial^2 P\delta_\varepsilon}{\partial\lambda\partial a_k}, v_\varepsilon \right\rangle = O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{3/2}} + \varepsilon^{3/2}\right). \tag{2.75}$$

Proof. We have

$$\begin{aligned}\int_\Omega \Delta\left(\frac{\partial^2 P\delta_\varepsilon}{\partial\lambda\partial a_k}\right)\Delta v_\varepsilon &= \int_{B_\varepsilon} \Delta^2\left(\frac{\partial^2 P\delta_\varepsilon}{\partial\lambda\partial a_k}\right)v_\varepsilon + O\left(\frac{||v_\varepsilon||}{(\lambda_\varepsilon d_\varepsilon)^{9/2}}\right)\\ &= \int_{B_\varepsilon} \Delta^2\left(\frac{\partial^2 P\delta_\varepsilon}{\partial\lambda\partial a_k}\right)\Pi v^o_\varepsilon + \int_{B_\varepsilon} \Delta^2\left(\frac{\partial^2 P\delta_\varepsilon}{\partial\lambda\partial a_k}\right)w_\varepsilon + O\left(\frac{||v_\varepsilon||}{(\lambda_\varepsilon d_\varepsilon)^{9/2}}\right).\end{aligned} \tag{2.76}$$

For the first integral on the right-hand side in (2.76), we have

$$\int_{B_\varepsilon} \Delta^2\left(\frac{\partial^2 P\delta_\varepsilon}{\partial\lambda\partial a_k}\right)\Pi v^o_\varepsilon = O(||\Pi v^o_\varepsilon||) = O\left(\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{3/2}} + \varepsilon^{3/2}\right), \tag{2.77}$$

where we have used in the last equality Lemma 2.11.

Now let ψ_4 be the solution of

$$\Delta^2\psi_4 = \Delta^2\left(\frac{\partial^2 P\delta_\varepsilon}{\partial\lambda\partial a_k}\right) \quad \text{in } B_\varepsilon, \qquad \Delta\psi_4 = \psi_4 = 0 \quad \text{on } \partial B_\varepsilon. \tag{2.78}$$

Thus, as in the proof of Lemma 2.6, we obtain for $y \in \partial B_\varepsilon$

$$\frac{\partial\psi_4}{\partial\nu}(y) = O\left(\frac{1}{\lambda_\varepsilon^{1/2} d_\varepsilon^2}\right), \qquad \frac{\partial\Delta\psi_4}{\partial\nu}(y) = O\left(\frac{1}{\lambda_\varepsilon^{1/2} d_\varepsilon^4}\right) \tag{2.79}$$

and therefore

$$\int_{B_\varepsilon} \Delta^2\left(\frac{\partial^2 P\delta_\varepsilon}{\partial\lambda\partial a_k}\right)w_\varepsilon = O\left(\frac{||v_\varepsilon||}{(\lambda_\varepsilon d_\varepsilon)^{1/2}}\right). \tag{2.80}$$

From (2.76), (2.77), (2.80) and Proposition 2.4, we easily deduce our lemma. □

3. Proof of theorems

Let us start by proving the following crucial result.

Proposition 3.1. *For $u_\varepsilon = \alpha_\varepsilon P\delta_{a_\varepsilon,\lambda_\varepsilon} + v_\varepsilon$ solution of (P_ε) with $\lambda_\varepsilon^\varepsilon = 1 + o(1)$ as ε goes to zero, we have the following estimates*

(a) $c_2\varepsilon + O(\varepsilon^2) - c_1(H(a_\varepsilon,a_\varepsilon)/\lambda_\varepsilon) + o(1/\lambda_\varepsilon d_\varepsilon) = 0$,

(b) $(c_3/\lambda_\varepsilon^2)(\partial H(a_\varepsilon,a_\varepsilon)/\partial a) + o(1/(\lambda_\varepsilon d_\varepsilon)^2) + O(\varepsilon^{5/2} + \varepsilon\log(\lambda_\varepsilon)/(\lambda_\varepsilon d_\varepsilon)^2 + 1/(\lambda_\varepsilon d_\varepsilon)^{5/2}) = 0$,

where c_1, c_2 are the constants defined in Theorem 1.1, and where $c_3 > 0$.

Proof. Since claim (a) was proved in [5], we only need to prove claim (b). Multiplying (P_ε) by $(1/\lambda_\varepsilon)(\partial P\delta_\varepsilon/\partial a_k)$ and integrating on Ω, we obtain for $1 \le k \le 5$

$$\begin{aligned} 0 &= \int_\Omega \Delta^2 u_\varepsilon \frac{1}{\lambda_\varepsilon}\frac{\partial P\delta_\varepsilon}{\partial a_k} - \int_\Omega u_\varepsilon^{9-\varepsilon}\frac{1}{\lambda_\varepsilon}\frac{\partial P\delta_\varepsilon}{\partial a_k} \\ &= \alpha_\varepsilon \int_\Omega \delta_\varepsilon^9 \frac{1}{\lambda_\varepsilon}\frac{\partial P\delta_\varepsilon}{\partial a_k} - \int_\Omega \Big[(\alpha_\varepsilon P\delta_\varepsilon)^{9-\varepsilon} + (9-\varepsilon)(\alpha_\varepsilon P\delta_\varepsilon)^{8-\varepsilon} v_\varepsilon \\ &\qquad + \frac{(9-\varepsilon)(8-\varepsilon)}{2}(\alpha_\varepsilon P\delta_\varepsilon)^{7-\varepsilon} v_\varepsilon^2 \Big] \frac{1}{\lambda_\varepsilon}\frac{\partial P\delta_\varepsilon}{\partial a_k} + O(\|v_\varepsilon\|^3). \end{aligned} \tag{3.1}$$

We estimate each term on the right-hand side in (3.1). First, by Proposition 2.1 and the Holder inequality, we have

$$\int_\Omega P\delta_\varepsilon^{7-\varepsilon} v_\varepsilon^2 \frac{1}{\lambda_\varepsilon}\frac{\partial P\delta_\varepsilon}{\partial a_k} = \int_\Omega \delta_\varepsilon^{7-\varepsilon} v_\varepsilon^2 \frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k} + O\left(\frac{\|v_\varepsilon\|^2}{\lambda_\varepsilon d_\varepsilon}\right). \tag{3.2}$$

Secondly, we compute

$$\begin{aligned} \int_\Omega P\delta_\varepsilon^{8-\varepsilon} v_\varepsilon \frac{1}{\lambda_\varepsilon}\frac{\partial P\delta_\varepsilon}{\partial a_k} &= \int_\Omega \delta_\varepsilon^{8-\varepsilon} v_\varepsilon \frac{1}{\lambda_\varepsilon}\frac{\partial P\delta_\varepsilon}{\partial a_k} + (8-\varepsilon)\int_\Omega \delta_\varepsilon^{7-\varepsilon}\theta_\varepsilon v_\varepsilon \frac{1}{\lambda_\varepsilon}\frac{\partial P\delta_\varepsilon}{\partial a_k} + O\Big(\int_\Omega \delta_\varepsilon^{7-\varepsilon}\theta_\varepsilon^2 |v_\varepsilon|\Big) \\ &= \int_\Omega \delta_\varepsilon^{8-\varepsilon} v_\varepsilon \frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k} + O\Big(\int_\Omega \delta_\varepsilon^{8-\varepsilon}|v_\varepsilon| \Big|\frac{1}{\lambda_\varepsilon}\frac{\partial \theta_\varepsilon}{\partial a_k}\Big|\Big) \\ &\quad + (8-\varepsilon)\int_\Omega \delta_\varepsilon^{7-\varepsilon}\theta_\varepsilon v_\varepsilon \frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k} \\ &\quad + O\Big(\int_\Omega \delta_\varepsilon^{7-\varepsilon}\theta_\varepsilon |v_\varepsilon| \Big|\frac{1}{\lambda_\varepsilon}\frac{\partial \theta_\varepsilon}{\partial a_k}\Big|\Big) + O\Big(\int_\Omega \delta_\varepsilon^{7-\varepsilon}\theta_\varepsilon^2 |v_\varepsilon|\Big). \end{aligned} \tag{3.3}$$

By Proposition 2.1 and the Holder inequality, we obtain

$$\begin{gathered} \int_\Omega \delta_\varepsilon^{7-\varepsilon}\theta_\varepsilon^2 |v_\varepsilon| = O\left(\frac{\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^2}\right), \qquad \int_\Omega \delta_\varepsilon^{7-\varepsilon}\theta_\varepsilon |v_\varepsilon| \left|\frac{1}{\lambda_\varepsilon}\frac{\partial \theta_\varepsilon}{\partial a_k}\right| = O\left(\frac{\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^3}\right), \\ \int_\Omega \delta_\varepsilon^{8-\varepsilon}|v_\varepsilon| \left|\frac{1}{\lambda_\varepsilon}\frac{\partial \theta_\varepsilon}{\partial a_k}\right| = O\left(\frac{\|v_\varepsilon\|}{(\lambda_\varepsilon d_\varepsilon)^2}\right). \end{gathered} \tag{3.4}$$

We also have by Proposition 2.2

$$\begin{aligned}\int_\Omega \delta_\varepsilon^{8-\varepsilon} v_\varepsilon \frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k} &= \int_\Omega \delta_\varepsilon^{8}\left(\delta_\varepsilon^{-\varepsilon} - \frac{c_0^{-\varepsilon}}{\lambda_\varepsilon^{\varepsilon/2}}\right) v_\varepsilon \frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k} \\ &= \int_{B_\varepsilon} \delta_\varepsilon^{8}\left(\delta_\varepsilon^{-\varepsilon} - \frac{c_0^{-\varepsilon}}{\lambda_\varepsilon^{\varepsilon/2}}\right) v_\varepsilon \frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k} + O\left(\frac{||v_\varepsilon||}{(\lambda_\varepsilon d_\varepsilon)^{9/2}}\right).\end{aligned} \tag{3.5}$$

Using (2.17), Lemma 2.3 and the Holder inequality, we derive that

$$\begin{aligned}\int_\Omega \delta_\varepsilon^{8-\varepsilon} v_\varepsilon \frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k} &= \int_{B_\varepsilon} \delta_\varepsilon^{8}\left(\delta_\varepsilon^{-\varepsilon} - \frac{c_0^{-\varepsilon}}{\lambda_\varepsilon^{\varepsilon/2}}\right) \frac{1}{\lambda_\varepsilon}\frac{\partial \delta_\varepsilon}{\partial a_k} w_\varepsilon + O\left(\varepsilon||\Pi v_\varepsilon^o|| + \frac{||v_\varepsilon||}{(\lambda_\varepsilon d_\varepsilon)^{9/2}}\right) \\ &= O\left(\frac{\varepsilon||v_\varepsilon||}{(\lambda_\varepsilon d_\varepsilon)^{1/2}} + \varepsilon||\Pi v_\varepsilon^o|| + \frac{||v_\varepsilon||}{(\lambda_\varepsilon d_\varepsilon)^{9/2}}\right),\end{aligned} \tag{3.6}$$

where we have used Lemma 2.6 in the last equality.

Using (3.2)–(3.6), Lemmas 2.5, 2.8, 2.9, Proposition 2.2 and the fact that $\lambda_\varepsilon^\varepsilon = 1 + O(\varepsilon \log \lambda_\varepsilon)$, we easily derive our result. □

We are now able to prove Theorem 1.1.

Proof of Theorem 1.1. Let (u_ε) be a solution of (P_ε) which satisfies (H). Then, using Proposition 2.2, $u_\varepsilon = \alpha_\varepsilon P\delta_{a_\varepsilon,\lambda_\varepsilon} + v_\varepsilon$ with $\alpha_\varepsilon \to 1$, $\lambda_\varepsilon^\varepsilon \to 1$, $\lambda_\varepsilon d(a_\varepsilon, \partial\Omega) \to \infty$, v_ε satisfies (V_0) and $\|v_\varepsilon\| \to 0$. Now, using claim (a) of Proposition 3.1, we derive that

$$\varepsilon = \frac{c_1}{c_2}\frac{H(a_\varepsilon, a_\varepsilon)}{\lambda_\varepsilon} + o\left(\frac{1}{\lambda_\varepsilon d_\varepsilon}\right) = O\left(\frac{1}{\lambda_\varepsilon d_\varepsilon}\right). \tag{3.7}$$

Therefore, it follows from claim (b) of Proposition 3.1 that

$$\frac{\partial H(a_\varepsilon, a_\varepsilon)}{\partial a} = o\left(\frac{1}{d_\varepsilon^2}\right). \tag{3.8}$$

Using (3.8) and the fact that for a near the boundary $(\partial H/\partial a)(a_\varepsilon, a_\varepsilon) \sim c d(a_\varepsilon, \partial\Omega)^{-2}$, we derive that a_ε is away from the boundary and it converges to a critical point x_0 of φ.

Finally, using (3.7), we obtain

$$\varepsilon\lambda_\varepsilon \longrightarrow \frac{c_1}{c_2}\varphi(x_0) \quad \text{as } \varepsilon \longrightarrow 0. \tag{3.9}$$

By Proposition 2.2, we have

$$||u_\varepsilon||_{L^\infty}^2 \sim c_0^2 \lambda_\varepsilon \quad \text{as } \varepsilon \longrightarrow 0. \tag{3.10}$$

This concludes the proof of Theorem 1.1. □

The sequel of this section is devoted to the proof of Theorem 1.2.

Proof of Theorem 1.2. Let x_0 be a nondegenerate critical point of φ. It is easy to see that $d(a,\partial\Omega) > d_0 > 0$ for a near x_0. We will take a function $u = \alpha P\delta_{(a,\lambda)} + v$ where $(\alpha - \alpha_0)$ is very small, λ is large enough, $\|v\|$ is very small, a is close to x_0 and $\alpha_0 = S^{-5/8}$ and we will prove that we can choose the variables (α,λ,a,v) so that u is a critical point of J_ε with $\|u\| = 1$. Here J_ε denotes the functional corresponding to problem (P_ε) defined by

$$J_\varepsilon(u) = \left(\int_\Omega |\Delta u|^2\right)\left(\int_\Omega |u|^{10-\varepsilon}\right)^{-2/(10-\varepsilon)}. \tag{3.11}$$

Let

$$\begin{aligned} M_\varepsilon = \{&(\alpha,\lambda,a,v) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \Omega \times \mathcal{H}(\Omega) / |\alpha - \alpha_0| < \nu_0, \\ &d_a > d_0,\ \lambda > \nu_0^{-1},\ \varepsilon\log\lambda < \nu_0,\ \|v\| < \nu_0 \text{ and } v \in E_{(a,\lambda)}\}, \end{aligned} \tag{3.12}$$

where ν_0 and d_0 are two suitable positive constants and where $d_a = d(a,\partial\Omega)$.

Let us define the functional

$$K_\varepsilon : M_\varepsilon \longrightarrow \mathbb{R}, \quad K_\varepsilon(\alpha,a,\lambda,v) = J_\varepsilon(\alpha P\delta_{(a,\lambda)} + v). \tag{3.13}$$

It is known that (α,λ,a,v) is a critical point of K_ε if and only if $u = \alpha P\delta_{(a,\lambda)} + v$ is a critical point of J_ε on $\mathcal{H}(\Omega)$. So this fact allows us to look for critical points of J_ε by successive optimizations with respect to the different parameters on M_ε.

First, arguing as in [25, Proposition 4] we see that the following problem

$$\min\{J_\varepsilon(\alpha P\delta_{(a,\lambda)} + v),\ v \text{ satisfying } (V_0) \text{ and } \|v\| < \nu_0\} \tag{3.14}$$

is achieved by a unique function $\bar{v}$ which satisfies the estimate of Proposition 2.4. This implies that there exist A, B and C_i's such that

$$\frac{\partial K_\varepsilon}{\partial v}(\alpha,\lambda,a,\bar{v}) = \nabla J_\varepsilon(\alpha P\delta_{(a,\lambda)} + \bar{v}) = AP\delta_{(a,\lambda)} + B\frac{\partial}{\partial\lambda}P\delta_{(a,\lambda)} + \sum_{i=1}^{5} C_i \frac{\partial}{\partial a_i}P\delta_{(a,\lambda)}, \tag{3.15}$$

where a_i is the ith component of a.

According to [5], we have that

$$A = O\left(\varepsilon\log\lambda + |\beta| + \frac{1}{\lambda}\right), \qquad B = O(\lambda\varepsilon + 1), \qquad C_j = O\left(\frac{\varepsilon^2}{\lambda} + \frac{1}{\lambda^3}\right). \tag{3.16}$$

To find critical points of K_ε, we have to solve the following system

$$\begin{aligned} \frac{\partial K_\varepsilon}{\partial\alpha} &= 0 \\ \frac{\partial K_\varepsilon}{\partial\lambda} &= B\left\langle \frac{\partial^2 P\delta}{\partial\lambda^2}, \bar{v}\right\rangle + \sum_{i=1}^{5} C_i \left\langle \frac{\partial^2 P\delta}{\partial\lambda\partial a_i}, \bar{v}\right\rangle \\ \frac{\partial K_\varepsilon}{\partial a_j} &= B\left\langle \frac{\partial^2 P\delta}{\partial\lambda\partial a_j}, \bar{v}\right\rangle + \sum_{i=1}^{5} C_i \left\langle \frac{\partial^2 P\delta}{\partial a_i\partial a_j}, \bar{v}\right\rangle, \quad \text{for each } j = 1,\dots,5. \end{aligned} \tag{E_1}$$

Observe that for $\psi = P\delta_{(a,\lambda)}$, $\partial P\delta_{(a,\lambda)}/\partial\lambda$, $\partial P\delta_{(a,\lambda)}/\partial a_i$ with $i = 1,\dots,5$ and for $u = \alpha P\delta_{(a,\lambda)} + \overline{v}$, we have

$$\langle \nabla J_\varepsilon(u),\psi\rangle = 2J_\varepsilon(u)\left(\alpha\langle P\delta_{(a,\lambda)},\psi\rangle - J_\varepsilon(u)^{5-\varepsilon/2}\int_\Omega |u|^{8-\varepsilon}u\psi\right). \tag{3.17}$$

We also have (see [5])

$$J_\varepsilon\left(\alpha P\delta_{(a,\lambda)} + \overline{v}\right) = S + O\left(\varepsilon\log\lambda + \frac{1}{\lambda}\right), \tag{3.18}$$

$$\frac{\partial K_\varepsilon}{\partial\alpha} = \langle \nabla J_\varepsilon(\alpha P\delta + \overline{v}), P\delta\rangle = 2J_\varepsilon(u)\left(\alpha S^{5/4}(1-\alpha^8 S^5) + O\left(\varepsilon\log\lambda + \frac{1}{\lambda}\right)\right), \tag{3.19}$$

$$\begin{aligned}\lambda\frac{\partial K_\varepsilon}{\partial\lambda} &= \left\langle \nabla J_\varepsilon(\alpha P\delta + \overline{v}), \lambda\frac{\partial P\delta}{\partial\lambda}\right\rangle = J_\varepsilon(u)\left(\alpha c_1\frac{H(a,a)}{\lambda}(1-2\alpha^8S^5) + c_2S^5\alpha^9\varepsilon\right.\\ &\qquad \left. + O\left(\varepsilon^2\log\lambda + \frac{\varepsilon\log\lambda}{\lambda} + \frac{1}{\lambda^3}\right)\right).\end{aligned} \tag{3.20}$$

Following the proof of claim (b) of Proposition 3.1, we obtain, for each $j = 1,\dots,5$,

$$\frac{1}{\lambda}\frac{\partial K_\varepsilon}{\partial a_j} = \left\langle \nabla J_\varepsilon(\alpha P\delta + \overline{v}), \frac{1}{\lambda}\frac{\partial P\delta}{\partial a_j}\right\rangle = -\frac{c\alpha}{2\lambda^2}\frac{\partial H(a,a)}{\partial a}(1-2\alpha^8S^5) + O\left(\varepsilon^{5/2} + \frac{\varepsilon\log\lambda}{\lambda^2} + \frac{1}{\lambda^{5/2}}\right). \tag{3.21}$$

On the other hand, one can easily verify that

$$\left\|\frac{\partial^2 P\delta}{\partial\lambda^2}\right\| = O\left(\frac{1}{\lambda^2}\right), \qquad \left\|\frac{\partial^2 P\delta}{\partial a_i\partial a_j}\right\| = O(\lambda^2). \tag{3.22}$$

Now, we take the following change of variables:

$$\alpha = \alpha_0 + \beta, \qquad a = x_0 + \xi, \qquad \frac{1}{\lambda^{1/2}} = \sqrt{\frac{c_2}{c_1}}\left(\frac{1}{\sqrt{H(x_0,x_0)}} + \rho\right)\sqrt{\varepsilon}. \tag{3.23}$$

Then, using estimates (3.18)–(3.22), Lemma 2.12, Proposition 2.4 and the fact that x_0 is a nondegenerate critical point of φ, the system (E_1) becomes

$$\begin{aligned}\beta &= O(\varepsilon|\log\varepsilon| + |\beta|^2)\\ \rho &= O(\varepsilon|\log\varepsilon| + |\beta|^2 + |\xi|^2 + \rho^2)\\ \xi &= O(|\beta|^2 + |\xi|^2 + \varepsilon^{1/2}).\end{aligned} \tag{E_2}$$

Thus Brower's fixed point theorem shows that the system (E_2) has a solution $(\beta_\varepsilon,\rho_\varepsilon,\xi_\varepsilon)$ for ε small enough such that

$$\beta_\varepsilon = O(\varepsilon|\log\varepsilon|), \qquad \rho_\varepsilon = O(\varepsilon|\log\varepsilon|), \qquad \xi_\varepsilon = O(\varepsilon^{1/2}). \tag{3.24}$$

By construction, the corresponding u_ε is a critical point of J_ε that is $w_\varepsilon = J_\varepsilon(u_\varepsilon)^{(5-\varepsilon/2)/(8-\varepsilon)} u_\varepsilon$ satisfies

$$\Delta^2 w_\varepsilon = |w_\varepsilon|^{8-\varepsilon} w_\varepsilon \quad \text{in } \Omega,\ w_\varepsilon = \Delta w_\varepsilon = 0 \text{ on } \partial\Omega \tag{3.25}$$

with $|w_\varepsilon^-|_{L^{10}(\Omega)}$ very small, where $w_\varepsilon^- = \max(0, -w_\varepsilon)$.

As in [7, Proposition 4.1], we prove that $w_\varepsilon^- = 0$. Thus, since w_ε is a non-negative function which satisfies (3.25), the strong maximum principle ensures that $w_\varepsilon > 0$ on Ω and then w_ε is a solution of (P_ε), which blows up at x_0 as ε goes to zero. This ends the proof of Theorem 1.2. □

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Khalil El Mehdi: Faculté des Sciences et Techniques, Université de Nouakchott, BP 5026, Nouakchott, Mauritania
E-mail address: khalil@univ-nkc.mr
Current address: The Abdus Salam ICTP, 34014 Trieste, Italy