L^q-PERTURBATIONS OF LEADING COEFFICIENTS OF ELLIPTIC OPERATORS: ASYMPTOTICS OF EIGENVALUES

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We consider eigenvalues of elliptic boundary value problems, written in variational form, when the leading coefficients are perturbed by terms which are small in some integral sense. We obtain asymptotic formulae. The main specific of these formulae is that the leading term is different from that in the corresponding formulae when the perturbation is small in L^∞-norm.

1. Introduction

Here we consider eigenvalues of boundary value problems for elliptic partial differential equations in variational form with measurable coefficients. The main goal is to describe asymptotics of eigenvalues under small perturbations of coefficients. A specific of the problem is that we suppose that perturbations are small only in some integral sense. Such class of perturbations is quite natural in applications. For example, if coefficients take different values on different parts of the domain and we will study what happened if boundaries between these parts are changed slightly, then we have smallness of the perturbations in L^q-norm with $q < \infty$ but not in L^∞-norm and we cannot apply in this situation well-known classical results of perturbation theory, see Kato [5], [4, Chapter 8]. Moreover, it appears that even the main term in the asymptotic formula for an eigenvalue is different from the classical one. In order to explain the difference, let us consider the following eigenvalue problem for a symmetric matrix:

$$\begin{pmatrix} \lambda I + B & C \\ C^* & A + D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \mu \begin{pmatrix} u \\ v \end{pmatrix}, \tag{1.1}$$

where I is the unit matrix, B, A, and D are symmetric matrices. The matrices B, D, and C are considered as small perturbation matrices. Then, as is well known, an approximation to an eigenvalue μ located near λ is given by $\lambda+\nu$, where ν is an eigenvalue to the matrix B. Another possibility is to write (1.1) as two equations and then solve the second one with respect to ν and insert this in the first equation. We obtain the eigenvalue problem $(\lambda I+B-C(A+D-\mu)^{-1}C^*)u=\mu u$ with respect to μ, which nonlinearly depends on μ. Now, as an approximation to the eigenvalue μ located near λ, we can take $\lambda+\nu'$, where ν' is an eigenvalue of

$$B-C(A+D-\lambda)^{-1}C^*. \tag{1.2}$$

Usually $\nu-\nu'$ gives a higher-order approximation to the eigenvalue μ. But, it appears that in the class of problems under consideration, ν and $\nu-\nu'$ may have the same order.

In Section 2, we present an abstract version of our asymptotic result. We consider two closed, positively definite forms a and b in a Hilbert space H with domains H_a and H_b with H_b densely imbedded in H_a. The main assumptions on the forms a and b are H_a is compactly imbedded into H, all eigenvectors corresponding to a belong to H_b, and that $c_0 a(u,u)\le b(u,u)$ for all $u\in H_b$. Then under a certain smallness assumption on $b-a$, see (2.9), we obtain the asymptotic formula (2.18) for all eigenvalues of the form b which are located near a fixed eigenvalue λ_m of the form a. The asymptotic parameters in this asymptotic formula are numbers ρ_m and σ_m defined by (2.16) and (2.17). We obtain also an asymptotic formula for corresponding eigenvectors.

In Section 3, we present our main application of the above asymptotic formula. We consider an elliptic quadratic form

$$a(u,v)=\sum_{|\alpha|=|\beta|=m}\int_\Omega\left(A_{\alpha\beta}(x)\partial_x^\beta u,\partial_x^\alpha v\right)dx \tag{1.3}$$

defined on $(\overset{\circ}{W}{}^{m,2}(\Omega))^d$, where Ω is a bounded domain in $\mathbb{R}^n$. The only assumptions on the coefficients $A_{\alpha\beta}$ and the domain Ω are the ellipticity condition (3.1) and that the eigenfunctions corresponding to these forms belong to $(W^{m,q}(\Omega))^d$ with some $q\ge 2$. Certainly, the last assumption is true for $q=2$. If $m=d=1$ and Ω is sufficiently smooth, then, as it follows from [1, 2] (for $n=2$) and from [9] (for arbitrary n), the eigenfunctions belong always to $W^{1,q}(\Omega)$ with a certain $q>2$ depending only on the ellipticity constants. Other cases of validity of the above property for operators with discontinuous coefficients are discussed in Remarks 3.1 and 3.2. Parallel to (1.3), we consider the form

$$b(u,v)=\sum_{|\alpha|=|\beta|=m}\int_\Omega\left(B_{\alpha\beta}(x)\partial_x^\beta u,\partial_x^\alpha v\right)dx. \tag{1.4}$$

The main assumptions on the form b are the ellipticity condition (3.3) and the smallness of the constant

$$\varkappa=\left(\sum_{|\alpha|=|\beta|=m}\int_\Omega\left|B_{\alpha\beta}(x)-A_{\alpha\beta}(x)\right|^{q/(q-2)}dx\right)^{(q-2)/q}. \tag{1.5}$$

Under these conditions, we show that the asymptotic formula (3.16) is valid for eigenvalues of the form b located near a fixed eigenvalue λ_m of the form a.

We also consider the case when the coefficients $B_{\alpha\beta}$ are bounded. Under some natural assumption on solutions to the problem $b(u,v) = (f,v)$, we simplify the general asymptotic formula for eigenvalues. At the end of Section 3, we give an example demonstrating that the leading term in the formulae (2.18) and (3.16), which differs from the classical one for L^∞-perturbations, is proper for such class of problems.

2. An abstract version of asymptotic formula for eigenvalues

Let H be a Hilbert space with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$ and let $a(\cdot,\cdot)$ be a sesquilinear, positive definite, closed form with the domain H_a, which is supposed to be dense and compactly imbedded in H. Consider the eigenvalue problem

$$a(u,v) = \lambda(u,v) \quad \forall v \in H_a \tag{2.1}$$

and denote by $0 < \lambda_1 < \lambda_2 < \cdots$ the eigenvalues of problem (2.1). Clearly, they have finite multiplicities and $\lambda_j \to \infty$ as $j \to \infty$. We denote by X_1, X_2,... corresponding eigenspaces and set $J_j = \dim X_j$.

Consider also a sesquilinear, positive definite, closed form $b(\cdot,\cdot)$ with domain $H_b \subset H_a$. We suppose that linear combinations of vectors from X_j belong to H_b and are dense there and that there exists a positive constant c_0 such that

$$c_0 a(u,u) \le b(u,u) \quad \forall u \in H_b. \tag{2.2}$$

Our main concern is the following spectral problem:

$$b(U,v) = \mu(U,v) \quad \forall v \in H_b. \tag{2.3}$$

Clearly, the spectrum of this problem consists of isolated eigenvalues of finite multiplicity. Let us fix an index m. Our goal is to describe eigenvalues of problem (2.3) located in a neighborhood of λ_m.

We choose N such that $N \ge m$ and

$$\lambda_{m+1} \le c_0 \lambda_{N+1} \tag{2.4}$$

and introduce spaces

$$\mathcal{X}_N = \sum_{j=1}^{N} X_j \tag{2.5}$$

and $\mathcal{Y}_N$ which are equal to the closure in H_b of all linear combinations of vectors from X_j, $j \ge N+1$. Clearly, H_b is the direct sum of $\mathcal{X}_N$ and $\mathcal{Y}_N$.

We supply the Hilbert spaces H_a and H_b with the innner products $(\cdot,\cdot)_a = a(\cdot,\cdot)$ and $(\cdot,\cdot)_b = b(\cdot,\cdot)$, respectively, and the corresponding norms are denoted by $\|\cdot\|_a$ and $\|\cdot\|_b$. Since H_b is a direct sum of the finite dimensional space $\mathcal{X}_N$ and the infinite dimensional space $\mathcal{Y}_N$, we can introduce an equivalent norm

$$\|w\|_N = \|w_1\|_a + \|w_2\|_b, \quad w = w_1 + w_2,\ w_1 \in \mathcal{X}_N,\ w_2 \in \mathcal{Y}_N. \tag{2.6}$$

Proposition 2.1. *Let*

$$p(u,v) = b(u,v) - a(u,v) \quad \textit{for } u,v \in H_b. \tag{2.7}$$

Put

$$\varepsilon_m = \min\left\{\frac{\lambda_m - \lambda_{m-1}}{\lambda_m + \lambda_{m+1}}, \frac{\lambda_{m+1} - \lambda_m}{\lambda_m + 2\lambda_{m+1}}\right\}. \tag{2.8}$$

If

$$|p(u,v)| \le \varepsilon \|u\|_a \|v\|_N \quad \forall u \in \mathscr{X}_N,\ v \in H_b \tag{2.9}$$

with $\varepsilon < \varepsilon_m$, then the interval $((1+\varepsilon)\lambda_{m-1}, (1-2\varepsilon)\lambda_{m+1})$ contains exactly J_m eigenvalues of the spectral problem (2.3). Moreover, these eigenvalues lie in the interval $[\lambda_m(1-\varepsilon), \lambda_m(1+\varepsilon)]$. If $m = 1$, then one should put in the above formulae $\lambda_0 = 0$.

Proof. (i) First, we show that

$$b(u,u) \ge (1-2\varepsilon)\|u\|^2 \tag{2.10}$$

for all $u \in \mathscr{Y}_m$. We write u as $u = V + W$, where $V \in \sum_{j=m+1}^{N} X_j$ and $W \in \mathscr{Y}_N$. Since $b(u,u) = b(W,W) + b(V,V) + 2\Re p(V,W)$, it follows from (2.9) and from the definition (2.6) that

$$\begin{aligned} b(u,u) &\ge \|W\|_b^2 + \|V\|_a^2 - \varepsilon\|V\|_a^2 - 2\varepsilon\|V\|_a\|W\|_b \\ &\ge (1-\varepsilon)\|W\|_b^2 + (1-2\varepsilon)\|V\|_a^2. \end{aligned}$$

This together with (2.2) and (2.4) gives

$$b(u,u) \ge (1-\varepsilon)c_0\lambda_{N+1}\|W\|^2 + (1-2\varepsilon)\lambda_{m+1}\|V\|^2 \ge (1-2\varepsilon)\lambda_{m+1}\|u\|^2, \tag{2.11}$$

which gives (2.10) provided $\varepsilon \le \varepsilon_0$, where

$$(1-2\varepsilon_0)\lambda_{m+1} \ge \frac{\lambda_m + \lambda_{m+1}}{2}. \tag{2.12}$$

This implies that the interval $(0, (1-2\varepsilon)\lambda_{m+1})$ contains at most $J_1 + \cdots + J_m$ eigenvalues of problem (2.3).

(ii) Let $u \in X_m$. Then

$$b(u,u) = \lambda_m\|u\|^2 + p(u,u) \tag{2.13}$$

and by (2.9),

$$|b(u,u) - \lambda_m\|u\|^2| \le \lambda_m\varepsilon\|u\|^2, \tag{2.14}$$

which implies that the interval $[\lambda_m(1-\varepsilon), \lambda_m(1+\varepsilon)]$ contains at least J_m eigenvalues of problem (2.3).

(iii) Finally, let $u \in X_1 + \cdots + X_{m-1}$. Then

$$b(u,u) \leq \lambda_{m-1}\|u\|^2 + \varepsilon\|u\|_a^2 \leq (1+\varepsilon)\lambda_{m-1}\|u\|^2, \tag{2.15}$$

which shows that the interval $(0,(1+\varepsilon)\lambda_{m-1})$ contains at least $J_1 + \cdots + J_{m-1}$ eigenvalues of problem (2.3). Now the required assertions follow from (i)–(iii).

In order to describe more explicitly the asymptoics of the eigenvalues of problem (2.3) located near λ_m, we introduce the numbers ρ_m and σ_m as follows. Let Y_m be the closure in H_b of all linear combinations of vectors from X_k with $k \neq m$. Then ρ_m and σ_m are defined as the best constants in the inequalities

$$|\,p(u,v)\,| \leq \rho_m\|u\|_a\|v\|_a \quad \forall u,v \in X_m, \tag{2.16}$$

$$|\,p(u,w)\,| \leq \sigma_m\|u\|_a\|w\|_N \quad \forall u \in X_m,\ w \in Y_m. \tag{2.17}$$

Clearly, $\rho_m \leq \varepsilon$ and $\sigma_m \leq \varepsilon$, where ε is the constant in (2.9). □

Theorem 2.2. *Let (2.9) be satisfied with $\varepsilon < \varepsilon_m$. Then the following assertions are valid.*

(i) *There exists a positive constant c depending only on the form such that the interval $(\lambda_m - c(\rho_m + \sigma_m^2), \lambda_m + c(\rho_m + \sigma_m^2))$ contains exactly J_m eigenvalues μ_{mj}, $j = 1,\ldots,J_m$, of problem (2.3). Moreover,*

$$\mu_{mj} = \lambda_m + \nu_{mj} + O\big(\sigma_m^2(\rho_m + \sigma_m^2)\big), \tag{2.18}$$

where $\{\nu_{mj}\}_{j=1}^{J_m}$ are eigenvalues of the sesquilinear form

$$X_m \ni V \longrightarrow p(V,V) + p\big(W_m(V),V\big), \tag{2.19}$$

where $W_m = W_m(V) \in Y_m$ is the solution of the equation

$$b(W_m,w) - \lambda_m(W_m,w) = -p(V,w) \quad \forall w \in Y_m. \tag{2.20}$$

This solution satisfies the estimate

$$\|W_m\|_N \leq c\sigma_m\|V\|_a. \tag{2.21}$$

(ii) *Let the numbers $\nu_{m1},\ldots,\nu_{mJ_m}$ be different and*

$$|\,\nu_{mj} - \nu_{mk}\,| \geq h(\rho_m + \sigma_m^2)\sigma_m^2(\rho_m + \sigma_m^2) \quad \text{for } j \neq k \tag{2.22}$$

with $h(s) \to \infty$ as $s \to 0$, then the corresponding eigenvectors to μ_{mj} are given by

$$\Psi_{mj} = \Phi_{mj} + W_{mj} + R_{mj}, \tag{2.23}$$

where Φ_{mj} is the eigenvector of the sesquilinear form (2.20) corresponding to the eigenvalue ν_{mj} normed by $\|\Phi_{mj}\|_H = 1$ and $W_{mj} \in Y_m$ solves (2.20) with $V = \Phi_{mj}$. The remainder R_{mj} satisfies

$$||R_{mj}||_N \le c\left(\frac{1}{h(\rho_m + \sigma_m^2)} + \sigma_m(\rho_m + \sigma_m^2)\right). \tag{2.24}$$

Proof. We represent a solution to problem (2.3) in the form

$$U = V + W \quad \text{with } V \in X_m,\ W \in Y_m \tag{2.25}$$

and split (2.3) into two equations

$$a(V,v) + p(V,v) + p(W,v) = \mu(V,v) \quad \forall v \in X_m, \tag{2.26}$$

$$b(W,w) - \mu(W,w) = -p(V,w) \quad \forall w \in Y_m. \tag{2.27}$$

(1) Let us show that the equation

$$b(W,w) - \mu(W,w) = f(w) \quad \forall w \in Y_m \tag{2.28}$$

has a unique solution $W \in Y_m$, where f is an antilinear, bounded functional on Y_m, provided μ is sufficiently close to λ_m. Let $X' = X_1 + \cdots + X_{m-1} + X_m + \cdots + X_N$. Then Y_m is the direct sum of X' and $\mathcal{Y}_N$ and we can represent W as $W_1 + W_2$ with $W_1 \in X'$ and $W_2 \in \mathcal{Y}_N$. Now, (2.28) is equivalent to

$$a(W_1,w_1) - \mu(W_1,w_1) + p(W_1 + W_2,w_1) = f(w_1) \quad \forall w_1 \in X', \tag{2.29}$$

$$b(W_2,w_2) - \mu(W_2,w_2) + p(W_1,w_2) = f(w_2) \quad \forall w_2 \in \mathcal{Y}_N. \tag{2.30}$$

Consider first the equation

$$b(W_2,w_2) - \mu(W_2,w_2) = F(w_2) \quad \forall w_2 \in \mathcal{Y}_N, \tag{2.31}$$

where F is an antilinear functional on $\mathcal{Y}_N$. We suppose that $|F(w_2)| \le F_b\|w_2\|_b$. By (2.2) and (2.4),

$$b(w_2,w_2) - \mu||w_2||^2 \ge b(w_2,w_2) - \frac{\mu}{\lambda_{N+1}}a(w_2,w_2) \ge \left(1 - \frac{\mu}{\lambda_{m+1}}\right)b(w_2,w_2). \tag{2.32}$$

Assuming that $\mu \le (\lambda_m + \lambda_{m+1})/2$, we get

$$b(w_2,w_2) - \mu||w_2||^2 \ge \frac{1}{2}\left(1 - \frac{\lambda_m}{\lambda_{m+1}}\right)b(w_2,w_2). \tag{2.33}$$

Therefore, (2.31) is uniquely solvable, taking $w_2 = W_2$ in (2.31) and using (2.33), we obtain

$$||W_2||_b \le \frac{2\lambda_{m+1}}{\lambda_{m+1}-\lambda_m} F_b. \tag{2.34}$$

Furthermore, consider the equation

$$a(W_1, w_1) - \mu(W_1, w_1) = h(w_1) \quad \forall w_1 \in X', \tag{2.35}$$

where h is an antilinear functional on X' with the norm h_a, that is, $|h(w_1)| \le h_a \|w_1\|_a$ for all $w_1 \in X'$. Clearly, this problem is uniquely solvable for $\lambda_{m-1} < \mu < \lambda_{m+1}$ and

$$||W_1||_a \le \max\left(\frac{\lambda_{m-1}}{\mu - \lambda_{m-1}}, \frac{\lambda_{m+1}}{\lambda_{m+1}-\mu}\right) h_a. \tag{2.36}$$

By (2.6) and (2.9), the perturbations $p(W_1 + W_2, w_1)$ and $p(W_1, w_2)$ in (2.29) and (2.30) admit the estimates

$$|\,p(W_1 + W_2, w_1)\,| \le \varepsilon(||W_1||_a + ||W_2||_b)||w_1||_a, \qquad |\,p(W_1, w_2)\,| \le \varepsilon||W_1||_a||w_2||_b. \tag{2.37}$$

Having in mind estimates (2.34) and (2.36) for problems (2.31) and (2.35), which represent the leading terms in (2.29) and (2.30), we see that there exists a constant ε_0 depending only on λ_{m-1}, λ_m and λ_{m+1} such that if $\varepsilon \le \varepsilon_0$ and

$$|\,\mu - \lambda_m\,| \le \frac{1}{2}\min\left(\lambda_{m+1} - \lambda_m, \lambda_m - \lambda_{m-1}\right), \tag{2.38}$$

then the system (2.29), (2.30) has a unique solution (W_1, W_2), which is subject to the estimate

$$\|W\|_N = ||W_1||_a + ||W_2||_b \le c(f_a + f_b), \tag{2.39}$$

where $|f(w_1)| \le f_a \|w_1\|_a$ for all $w_1 \in X'$ and $|f(w_2)| \le f_b \|w_2\|_b$ for all $w_2 \in \mathcal{Y}_N$. Here c is a positive constant depending only on λ_{m-1}, λ_m and λ_{m+1}.

(2) Let us turn to (2.27). By (2.17),

$$|\,p(V, w_1)\,| \le \sigma_m \|V\|_a ||w_1||_a, \qquad |\,p(V, w_2)\,| \le \sigma_m \|V\|_a ||w_2||_b \tag{2.40}$$

for $w_1 \in X'$ and for $w_2 \in \mathcal{Y}_N$, respectively. Therefore, problem (2.27) has a unique solution and using (2.39), we obtain

$$\|W\|_N \le c\sigma_m \|V\|_a. \tag{2.41}$$

So, we can consider $W = W(V)$ as a linear bounded operator from X_m into Y_m. Assuming $\|V\|_a = 1$, taking $v = V$ in (2.26), and using (2.41) together with (2.16) and (2.17), we arrive at

$$\left|1 - \frac{\mu}{\lambda_m}\right| \le c(\rho_m + \sigma_m^2). \tag{2.42}$$

In order to derive an asymptotic representation for μ, we proceed as follows. We represent the solution W of (2.27) as $W_m + W_r$, where W_m satisfies (2.20) and

$$b(W_r, w) - \mu(W_r, w) = (\mu - \lambda_m)(W_m, w) \quad \forall w \in Y_m. \tag{2.43}$$

Similar to (2.41), we get (2.21). Therefore, $W_m = W_m(V)$ is a linear, bounded operator from X_m to Y_m. Moreover, taking in (2.20) $w = W_m$, we conclude that the form $p(W_m(V), V)$ is real valued on X_m. Using (2.42) together with (2.21), we obtain

$$||W_r||_N \le c(\rho_m + \sigma_m^2)||W_m||_N \le c\sigma_m(\rho_m + \sigma_m^2)\|V\|_a. \tag{2.44}$$

Now (2.26) can be written as

$$(\lambda_m - \mu)(V, v) + p(V, v) + p(W_m, v) = -p(W_r, v) \quad \forall v \in X_m. \tag{2.45}$$

By (2.44) and (2.17),

$$|p(W_r(V), v)| \le c\sigma_m^2(\rho_m + \sigma_m^2)\|V\|_a\|v\|_a. \tag{2.46}$$

Since the sesquilinear form $p(V, v) + p(W_m(V), v)$ corresponds to a selfadjoint operator on X_m, there exists an ON-basis $\Phi_{m1}, \dots, \Phi_{mJ_m}$ in X_m with respect to the inner product $(\cdot, \cdot)$, such that

$$p(\Phi_{mj}, \Phi_{mk}) + p(W_m(\Phi_{mj}), \Phi_{mk}) = \delta_j^k \nu_{mj} \quad \text{for } j \ne k. \tag{2.47}$$

Therefore, the eigenvalues μ_{mj}, $j = 1, \dots, J_m$, satisfy (2.18).

(3) Let us prove (ii). Let μ_{mj} be an eigenvalue of (2.3) satisfying (2.18) and let Ψ_{mj} be a corresponding eigenfunction subject to $\|\Psi_{mj}\| = 1$. We represent it as

$$\Psi_{mj} = \sum_{k=1}^{J_m} c_k(\Phi_{mk} + W_m(\Phi_{mk}) + W_r(\Phi_{mk})), \tag{2.48}$$

where W_m, W_r, and Φ_{mj} are defined in (2). We will suppose that the coefficient c_j is a positive number or zero. Taking in (2.45) $V = \sum_{k=1}^{J_m} c_k\Phi_{mk}$, $\mu = \mu_{mj}$, and $v = \Phi_{mk}$, we obtain

$$(\lambda_m + \nu_{mk} - \mu_{mj})c_k = -p(W_r(V), \Phi_{mk}), \quad k = 1, \dots, J_m, \tag{2.49}$$

where ν_{mk} is given by (2.47). From (2.22) and (2.46), it follows that

$$c_k = O\left(\frac{1}{h(\rho_m + \sigma_m^2)}\right) \quad \text{for } k \ne j. \tag{2.50}$$

Since $\|\Psi_{mj}\| = 1$, we get that

$$c_j = 1 + O\left(\frac{1}{h(\rho_m + \sigma_m^2)}\right). \tag{2.51}$$

Using these relations for the coefficients together with estimates (2.21) and (2.44), we derive from (2.48) representation (2.23) and estimate (2.24). The proof is complete. □

Remark 2.3. Let us make some comments on the asymptotic formula (2.18). The main ingredient here is the finite dimensional matrix (2.19), which is built through solving problem (2.20) with a known invertible operator and right-hand sides. Thus, the construction of the matrix (2.19) involves only solving a finite number of well-defined problems with known coefficients and right-hand sides and the eigenvalues of this symmetric matrix deliver the main asymptotic term in (2.18). The class of perturbations covered by Theorem 2.2 is quite large and one cannot expect an explicit asymptotic form for the leading term as we have in usual asymptotic formulae for perturbations of an individual eigenvalue. But if a class of perturbations is more restrictive, then one can use various asymptotic methods for solving asymptotically problem (2.20) and constructing asymptotically matrix (2.19), which will lead to more explicit asymptotic representation for μ_{mj}.

3. Perturbations of elliptic problems with discontinuous coefficients

(1) *Unbounded perturbations.* Let Ω be a bounded domain in $\mathbb{R}^n$, $n \ge 2$. Let also m and d be positive integers and let $H = (L^2(\Omega))^d$. We consider the sesquilinear form (1.3) on $H_a = (\overset{\circ}{W}{}^{m,2}(\Omega))^d$, where $A_{\alpha\beta}$ are bounded, measurable $d \times d$-matrices on Ω and $(\cdot,\cdot)$ is the standard inner product in $\mathbb{C}^N$. We suppose that $A_{\alpha\beta} = \overline{A}_{\beta\alpha}$, and that there exist constants C_1 and C_2, $0 < C_1 \le C_2$, such that

$$C_1 \sum_{|\alpha|=m} |\xi_\alpha|^2 \le \sum_{|\alpha|=|\beta|=m} (A_{\alpha\beta}(x)\xi_\beta, \xi_\alpha)\,dx \le C_2 \sum_{|\alpha|=m} |\xi_\alpha|^2 \tag{3.1}$$

for all complex numbers ξ_α and for all $x \in \Omega$. By this assumption, the form $a(u,v)$ defines an equivalent inner product on H_a. Consider the eigenvalues of the problem

$$a(u,v) = \lambda(u,v) \quad \forall v \in H_a. \tag{3.2}$$

We suppose that Ω satisfies the following condition: the embedding operator from H_a into $L_2(\Omega)^N$ is compact. This guarantees that the spectrum of problem (3.2) consists of isolated eigenvalues of finite multiplicities with the only limit point at infinity. Let us denote by $0 < \lambda_1 < \lambda_2 < \cdots$ eigenvalues of the problem (3.2) and by J_k the multiplicity of λ_k, and by X_k the eigenspace corresponding to λ_k. We assume that all eigenfunctions belong to $(\overset{\circ}{W}{}^{m,q}(\Omega))^d$ with some $2 \le q \le \infty$.

Remark 3.1. Certainly, if $\partial\Omega$ is smooth and the coefficients $A_{\alpha\beta}$ are smooth in $\overline{\Omega}$, then eigenfunctions are smooth also and we can take $q = \infty$. For second-order scalar elliptic operators with discontinuous coefficients, it is known that eigenfunctions belong to $W^{1,2+\varepsilon}(\Omega)$ with a certain $\varepsilon > 0$, see [1, 2, 9]. From [10], see also [3, 6, 11], it follows that the same is true for higher-order systems, provided the boundary has some smoothness.

We introduce also the form (1.4), where $B_{\alpha\beta}$ are measurable $d \times d$-matrices on Ω and $B_{\alpha\beta} = \overline{B}_{\beta\alpha}$. We assume that

$$C_1 \sum_{|\alpha|=m} |\xi_\alpha|^2 \le \sum_{|\alpha|=|\beta|=m} (B_{\alpha\beta}(x)\xi_\beta, \xi_\alpha) \tag{3.3}$$

for all complex numbers ξ_α and for all $x \in \Omega$. We will consider the form b as a small perturbation of a in the following sense. Let

$$\kappa_s = \left(\sum_{|\alpha|=|\beta|=m} \int_\Omega |B_{\alpha\beta}(x) - A_{\alpha\beta}(x)|^s dx \right)^{1/s} \tag{3.4}$$

if $s \in [1, \infty)$ and

$$k_\infty = \operatorname{esssup}_\Omega \sum_{|\alpha|=|\beta|=m} |B_{\alpha\beta}(x) - A_{\alpha\beta}(x)|. \tag{3.5}$$

Then we assume that the quantity

$$\varkappa = \kappa_{q/(q-2)} \tag{3.6}$$

is small.

The form $b(U,U)$ is well defined on all elements from $(C_0^\infty(\Omega))^d$ and we denote by H_b the closure of these elements in the norm $(b(U,U))^{1/2}$. Clearly, $H_b \subset H_a$ and all elements from $\sum_k X_k$ belong to H_b. In parallel to (3.2), we consider also the following eigenvalue problem:

$$b(U,v) = \mu(U,v) \quad \forall v \in H_b. \tag{3.7}$$

Since the embedding operator from H_b into $(L_2(\Omega))^d$ is also compact, the spectrum of this problem consists of isolated eigenvalues of finite multiplicities with the only limit point at infinity.

By (3.1) and (3.3),

$$c_0 a(u,u) \le b(u,u) \quad \forall u \in H_b \tag{3.8}$$

with $c_0 = C_1/C_2$. Our goal is to describe the eigenvalues of problem (3.7) situated near λ_m. We chose N according to (2.4) and put

$$p(u,v) = b(u,v) - a(u,v) = \sum_{|\alpha|=|\beta|=m} \int_\Omega ((B_{\alpha\beta}(x) - A_{\alpha\beta})\partial_x^\beta u, \partial_x^\alpha v)\, dx. \tag{3.9}$$

Since for almost every $x \in \Omega$,

$$\left| \sum_{|\alpha|=|\beta|=m} ((B_{\alpha\beta}(x) - A_{\alpha\beta})\xi_\alpha, \xi_\beta) \right| \le ch^{1/2}(x)\|\xi\| \left(\sum_{|\alpha|=|\beta|=m} ((B_{\alpha\beta}(x) + A_{\alpha\beta})\xi_\alpha, \xi_\beta) \right)^{1/2}, \tag{3.10}$$

where

$$h(x) = \sum_{|\alpha|=|\beta|=m} |B_{\alpha\beta}(x) - A_{\alpha\beta}(x)|, \tag{3.11}$$

we have that

$$|p(u,v)| \le c\varkappa^{1/2}\left(\sum_{|\alpha|=m}\int_\Omega |\partial_x^\alpha u|^p dx\right)^{1/p} (b(v,v))^{1/2} \tag{3.12}$$

for all $u \in \mathscr{X}_N$ and $v \in H_b$, where $\varkappa$ is introduced by (3.6). This implies that

$$|p(u,v)| \le C(N,q)\varkappa^{1/2}\|u\|_a\|v\|_b \tag{3.13}$$

for all $u \in \mathscr{X}_N$ and $v \in \mathscr{Y}_N$. If $u,v \in \mathscr{X}_N$, then using $u,v \in (W^{m,q}(\Omega))^d$, we obtain

$$|p(u,v)| \le C(N,q)\varkappa\|u\|_a\|v\|_a. \tag{3.14}$$

Since we have assumed that the number κ_q is small enough, it follows from Proposition 2.1 that there are exactly J_m eigenvalues $\{\mu_{mj}\}_{j=1}^{J_m}$ of problem (3.7) in a neighborhood of λ_m. By (3.13) and (3.14), the numbers ρ_m and σ_m in (2.16) and (2.17) admit the estimate

$$\rho_m + \sigma_m^2 \le c\varkappa. \tag{3.15}$$

Now, we are in a position to apply Theorem 2.2. By this theorem,

$$\mu_{mj} = \lambda_m + \nu_{mj} + O(\varkappa^2), \quad j = 1,\dots,J_m, \tag{3.16}$$

where ν_{mj} are eigenvalues of the form (2.19).

(2) *Bounded perturbations.* Let us consider the same b under the additional assumption

$$C_1 \sum_{|\alpha|=m} |\xi_\alpha|^2 \le \sum_{|\alpha|=|\beta|=m} (B_{\alpha\beta}(x)\xi_\beta,\xi_\alpha) \le C_2 \sum_{|\alpha|=m} |\xi_\alpha|^2. \tag{3.17}$$

Here and in (3.1), C_1 and C_2 are fixed constants. In this case, $H_b = H_a = (\overset{\circ}{W}{}^{m,2}(\Omega))^d$ and the corresponding norms are equivalent. The main assumption now is the following. Let $u \in H_a$ be the solution of the equation

$$b(u,v) = (f,v) \quad \forall v \in H_a, \tag{3.18}$$

where $f \in (W^{m,2}(\Omega))^d$. Then there exists $q > 2$ depending only on the ellipticity constants C_1 and C_2 such that $u \in (W^{m,q}(\Omega))^d$ and

$$\|u\|_{(W^{m,q}(\Omega))^d} \le c\|f\|_{(W^{m,2}(\Omega))^d}. \tag{3.19}$$

Remark 3.2. To establish such property for operators with discontinuous coefficients, one can use the following abstract interpolation result. Let $T : (X_0,X_1) \to (Y_0,Y_1)$ be a bounded linear operator acting on two Banach pairs. Let also $[X_0,X_1]_\theta$ and $[Y_0,Y_1]_\theta$, $\theta \in (0,1)$, be interpolation spaces. If

$$T : [X_0,X_1]_\theta \longrightarrow [Y_0,Y_1]_\theta \tag{3.20}$$

is invertible for some $\theta = \theta_0 \in (0,1)$, then there exists $\varepsilon > 0$ depending on the norms of T and the inversion to (3.20) for $\theta = \theta_0$, and on θ_0 such that the operator (3.20) is invertible for all $\theta \in [\theta_0 - \varepsilon, \theta_0 + \varepsilon]$. This theorem is due to Shneiberg [10], various generalizations can be found in [3, 6, 11] (see also references therein). Using this result, one can obtain that the solution of the problem (3.18) even with $f \in (W^{-m,q}(\Omega))^d$ belongs to $(W^{m,q}(\Omega))^d$ with a certain $q > 2$ depending on the ellipticity constants C_1 and C_2, provided the boundary $\partial\Omega$ has some smoothness, in order to apply interpolation results for $\mathring{W}^{m,p}$-spaces.

Certainly, the above regularity property should be valid for the form a because it also satisfies the estimates (3.1). Clearly, all eigenfuctions of (3.2) belong to $(W^{m,q}(\Omega))^d$ and we can apply previous result on asymptotics of eigenvalues of (3.7). But in this case, we can construct a simpler approximation to $W_m = W_m(V)$. Indeed, we represent it as

$$W_m = Pw_m + r_m, \tag{3.21}$$

where P is the orthogonal projector in H_a onto Y_m with respect to the inner product $a(\cdot,\cdot)$, w_m satisfies

$$b(w_m, w) = -p(V, w) \quad \forall w \in H_a \tag{3.22}$$

and $r_m \in Y_m$ is a solution of

$$b(r_m, w) - \lambda_m (r_m, w) = \lambda_m (Pw_m, w) \quad \forall w \in Y_m. \tag{3.23}$$

We can extend this relation for all $w \in H_a$ but then we should add an additional term (Φ, w) with a $\Phi \in X_m$,

$$b(r_m, w) - \lambda_m (r_m, w) = \lambda_m (Pw_m, w) + (\Phi, w) \quad \forall w \in H_a. \tag{3.24}$$

Taking in the last relation $w = \Phi$ and using orthogonality of Φ to Y_m with respect to the inner products $(\cdot,\cdot)$ and $a(\cdot,\cdot)$, we obtain $\|\Phi\|^2_{(L^2(\Omega))^d} = p(r_m, \Phi)$. By (3.19) applied to equation $a(\Phi, v) = \lambda_m(\Phi, v)$, we conclude that $\Phi \in (W^{m,q}(\Omega))^d$ and

$$\|\Phi\|_{(W^{m,q}(\Omega))^d} \le C\|\Phi\|_a = C\lambda_m^{1/2}\|\Phi\|_{(L^2(\Omega))^d}. \tag{3.25}$$

Moreover,

$$|\,p(r_m, \Phi)\,| \le C\varkappa \|r_m\|_{(W^{m,q}(\Omega))^d} \|\Phi\|_{(W^{m,q}(\Omega))^d}, \tag{3.26}$$

where $\varkappa$ is given by (3.6). Therefore,

$$\|\Phi\|_{(L^2(\Omega))^d} \le c\varkappa \|r_m\|_{(W^{m,q}(\Omega))^d}. \tag{3.27}$$

Applying this estimate together with (3.19) to (3.24) and using smallness of $\varkappa$, we obtain

$$||r_m||_{(W^{m,q}(\Omega))^d} \le c(||r_m||_a + ||w_m||_a). \tag{3.28}$$

Using estimate (2.39) for the solution of (2.28), we get

$$||r_m||_a \le c||w_m||_{(L^2(\Omega))^d}. \tag{3.29}$$

Therefore,

$$||r_m||_{(W^{m,q}(\Omega))^d} \le c||w_m||_a. \tag{3.30}$$

From (3.22), we derive

$$||w_m||_a \le C\varkappa^{1/2}\|V\|_a, \tag{3.31}$$

which together with (3.30) leads to

$$||r_m||_{(W^{m,q}(\Omega))^d} \le C\varkappa^{1/2}\|V\|_a. \tag{3.32}$$

Since estimate (3.26) with Φ replaced by V is valid for $p(r_m, V)$, we have

$$|\, p(r_m, V)\,| \le C\varkappa^{3/2}\|V\|_a^2. \tag{3.33}$$

Thus the formula (3.16) in this case can be written as

$$\mu_{mj} = \lambda_m + \nu'_{mj} + O(\varkappa^{3/2}), \tag{3.34}$$

where ν'_{mj} are eigenvalues of the form

$$X_m \ni V \longrightarrow p(V,V) + p(w_m(V),V), \tag{3.35}$$

where w_m solves (3.22).

We note that in order to get an asymptotic approximation of ν'_{mj}, it suffices to obtain an asymptotic representation for solution w_m to problem (3.22). For this goal, one can use various asymptotic methods, see, for example, [7, 8].

(3) *Example.* The eigenproblem for finding ν'_{mj} contains two terms, see (3.35). In the following example, we show that contribution of these terms to ν'_{mj} may have the same order and in general, one cannot neglect one of them. Let Ω be an interval $(0,1)$ in $\mathbb{R}$ and

$$a(u,v) = \int_0^1 A(t)u'(t)\overline{v}'(t)dt, \tag{3.36}$$

where A is a measurable function on $(0,1)$, $a_0 \le A(t) \le a_1$, with some positive constants a_0 and a_1. Let also

$$b(u,v) = \int_0^1 B(t)u'(t)\overline{v}'(t)dt, \tag{3.37}$$

where B is a measurable function such that $B(t) \geq A(t)$. One can check that in this case, estimate (3.19) is valid with $m = 1$ and $q = \infty$. Therefore,

$$\varkappa = \int_0^1 |(B-A)(t)|\,dt \tag{3.38}$$

is supposed to be small. Let λ be the first eigenvalue of (3.2). This eigenvalue is positive and simple and the corresponding eigenfunction Φ is positive (up to a constant factor) in Ω. We suppose that $\|\Phi\|_{L^2(0,1)} = 1$. Equation (3.22) for the function w takes the form

$$\frac{d}{dt}\left(B(t)\frac{d}{dt}w(t)\right) = \frac{d}{dt}\left((B-A)(t)\frac{d}{dt}\Phi(t)\right) \tag{3.39}$$

for $0 < t < 1$ and $w(0) = w(1) = 0$. The solution of (3.39) is

$$w(t) = \int_0^t \frac{B(\tau)-A(\tau)}{B(\tau)}\Phi'(\tau)d\tau - Q\int_0^t \frac{1}{B(\tau)}d\tau, \tag{3.40}$$

where

$$Q = \int_0^1 \frac{B(\tau)-A(\tau)}{B(\tau)}\Phi'(\tau)d\tau\left(\int_0^1 \frac{1}{B(\tau)}d\tau\right)^{-1}. \tag{3.41}$$

Therefore, formula (3.34) takes the form

$$\begin{aligned}\mu-\lambda &= \int_0^1 (B(\tau)-A(\tau))\Phi'^2(\tau)d\tau \\ &\quad + \int_0^1 \frac{B(\tau)-A(\tau)}{B(\tau)}\left((B(\tau)-A(\tau))\Phi'^2(\tau) - Q\Phi'(\tau)\right)d\tau + O(\varkappa^{3/2}).\end{aligned} \tag{3.42}$$

Assuming, for example, that $B(\tau) - A(\tau) = 1$ for $a < \tau < a+\varepsilon$ and $B(\tau) - A(\tau) = 0$ otherwise, where $a \in (0,1)$ is a fixed number and ε is a small positive number, we see that $\varkappa \sim \varepsilon$ and the second term in the right-hand side of (3.42) can be written as

$$\sim \Phi'^2(a)\int_a^{a+\varepsilon} \frac{1}{B(\tau)}d\tau\left(\int_0^1 \frac{1}{B(\tau)}d\tau\right)^{-1}\left(\int_0^1 - \int_a^{a+\varepsilon}\right)\frac{1}{B(\tau)}d\tau, \tag{3.43}$$

which shows that the first and the second terms in the right-hand side of (3.42) have the same order ε.

This example shows also that the form b in the left-hand side of (2.20) or (3.22) cannot be replaced by the form a in general case.

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