# A NONLINEAR SECOND ORDER PROBLEM WITH A NONLOCAL BOUNDARY CONDITION

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We study a nonlinear problem of pendulum-type for a $p$-Laplacian with nonlinear periodic-type boundary conditions. Using an extension of Mawhin's continuation theorem for nonlinear operators, we prove the existence of a solution under a Landesman-Lazer type condition. Moreover, using the method of upper and lower solutions, we generalize a celebrated result by Castro for the classical pendulum equation.

## 1. Introduction

The existence of periodic solutions for nonlinear second order ordinary differential equations have been widely studied, using the different techniques from nonlinear analysis, such as variational methods, topological methods and the method of upper and lower solutions.

In this work we consider an extension of the periodic problem for a nonlinear partial differential equation, namely:

$$\Delta_p u + g(u) = f(x) \quad \text{in } \Omega$$
$$u = c \quad \text{on } \partial\Omega \tag{1.1}$$
$$\int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial \eta} = h(c).$$

Here $c$ is a constant (whose value is unknown), $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, and $\Delta_p$ is the $p$-Laplacian ($1 < p < \infty$):

$$\Delta_p u = \text{div}\left(|\nabla u|^{p-2}\nabla u\right). \tag{1.2}$$

The boundary integral condition

$$\int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial \eta} = h(c) \tag{1.3}$$

will be understood in the sense, which holds for smooth functions, that

$$\int_\Omega \Delta_p u = h(c). \tag{1.4}$$

Some physical motivation for the study of this kind of problems (with $p = 2$) comes from [2], where the authors study a model describing the equilibrium of a plasma confined in a toroidal cavity. Under appropriate conditions this model can be reduced to the nonhomogeneous boundary-value problem

$$\begin{aligned}
\Delta u + h(x, u) &= 0 \quad \text{in } \Omega \\
u \mid_{\partial\Omega} &= \text{constant} \\
-\int_{\partial\Omega} \frac{\partial u}{\partial \nu} &= I.
\end{aligned} \tag{1.5}$$

The authors prove the existence of at least one solution $u \in H^2$ of the problem for any $h$ satisfying the following assumptions:

(A1) $h : \overline{\Omega} \times \mathbb{R} \to [0, +\infty)$ is continuous, nondecreasing on $u$, with $h(x, u) = 0$ for $u \leq 0$.

(A2) $\lim_{u \to +\infty} \int_\Omega h(x, u) dx > I$.

(A3) $\lim_{u \to +\infty} (h(x, u))/u^r = 0$ for some $r \in \mathbb{R}$ (with $r \leq n/(n-2)$ when $n > 2$).

In this work, we will show that some of the techniques that have been proved to be useful for the study of periodic solutions of ordinary differential equations can be applied to problem (1.1). For related results for Dirichlet boundary conditions, see, for example, [5].

First we consider the case in which $g, h : \mathbb{R} \to \mathbb{R}$ are continuous and $T$-periodic functions such that

$$\int_0^T g(t) dt = \int_0^T h(t) dt = 0. \tag{1.6}$$

Under these assumptions, the following theorem can be proved by an application of the variational method.

**THEOREM 1.1.** *Let (1.6) hold, and assume that $f \in L^{p'}(\Omega)$ satisfy $\overline{f} = 0$, where*

$$\overline{f} := \frac{1}{|\Omega|} \int_\Omega f. \tag{1.7}$$

*Then there exists at least one weak solution $u \in W_0^{1,p}(\Omega) + \mathbb{R}$ of problem (1.1).*

On the other hand, we will apply the method of upper and lower solutions in order to study problem (1.1).

*Definition 1.2.* We call $\alpha \in C^1(\overline{\Omega})$ a lower solution of (1.1) if

$$\Delta_p \alpha + g(\alpha) \geq f(x) \quad \text{in } \Omega$$
$$\alpha = c_\alpha \quad \text{on } \partial\Omega$$
$$\int_{\partial\Omega} |\nabla\alpha|^{p-2}\frac{\partial\alpha}{\partial\eta} \leq h(c_\alpha)$$

(1.8)

and $\beta \in C^1(\overline{\Omega})$ an upper solution of (1.1) if

$$\Delta_p \beta + g(\beta) \leq f(x) \quad \text{in } \Omega$$
$$\beta = c_\beta \quad \text{on } \partial\Omega$$
$$\int_{\partial\Omega} |\nabla\beta|^{p-2}\frac{\partial\beta}{\partial\eta} \geq h(c_\beta),$$

(1.9)

where the inequalities are understood in the weak sense. Then we have the following theorem (for a related result for the case $h \equiv 0$, see [10]).

THEOREM 1.3. *Let $\alpha, \beta \in C^1(\overline{\Omega})$, $\alpha \leq \beta$ be a lower solution and an upper solution as above, and assume that $f \in L^\infty(\Omega)$. Then there exists a solution $u \in C^{1,r}(\overline{\Omega})$ for some $r \in (0,1)$ of problem (1.1) such that $\alpha \leq u \leq \beta$.*

In particular, we may apply this result to an $n$-dimensional pendulum-like equation with nonlinear boundary conditions: assume that $g$ and $h$ are $T$-periodic, and let $f \in L^\infty(\Omega)$ be fixed. Consider the set

$$\mathscr{C} = \{(c_1, c_2) \in \mathbb{R}^2 : (1.11) \text{ is solvable}\},$$

(1.10)

where problem (1.11) is defined by

$$\Delta_p u + g(u) = f(x) + c_1 \quad \text{in } \Omega$$
$$u = c \quad \text{on } \partial\Omega$$
$$\int_{\partial\Omega} |\nabla u|^{p-2}\frac{\partial u}{\partial\eta} = h(c) - c_2.$$

(1.11)

The following result can be regarded as an extension of the well known result obtained by Castro in [3] for the classical pendulum equation, and related results in [1, 7].

THEOREM 1.4. *With the previous notations, assume that (1.6) hold, and that $(\underline{c_1}, \underline{c_2}), (\overline{c_1}, \overline{c_2}) \in \mathscr{C}$, with*

$$\underline{c_1} \leq \overline{c_1}, \qquad \underline{c_2} \leq \overline{c_2}.$$

(1.12)

*Then $(c_1, c_2) \in \mathscr{C}$ for any $(c_1, c_2)$ such that*

$$\underline{c_1} \leq c_1 \leq \overline{c_1}, \qquad \underline{c_2} \leq c_2 \leq \overline{c_2}.$$

(1.13)

Finally, we consider the case in which $g$ is a nonperiodic bounded function. More precisely, we have the following theorem, which asserts the existence of solutions under conditions of Landesman-Lazer type [9].

THEOREM 1.5. *Assume that $f \in L^\infty(\Omega)$, and $g$ is bounded. Further, assume that*

$$\limsup_{s \to +\infty} g(s) + \limsup_{s \to +\infty} \frac{h(s)}{|\Omega|} < \overline{f} < \liminf_{s \to -\infty} g(s) + \liminf_{s \to -\infty} \frac{h(s)}{|\Omega|} \tag{1.14}$$

*or*

$$\limsup_{s \to -\infty} g(s) + \limsup_{s \to -\infty} \frac{h(s)}{|\Omega|} < \overline{f} < \liminf_{s \to +\infty} g(s) + \liminf_{s \to +\infty} \frac{h(s)}{|\Omega|}. \tag{1.15}$$

*Then problem (1.1) admits at least one solution $u \in C^{1,r}(\overline{\Omega})$ for some $r > 0$.*

The proof of this theorem is based on a generalization of Mawhin coincidence degree theory for the case of quasi-linear operators. This generalization goes back to a paper by Manásevich and Mawhin [11], and was formulated in abstract form by Ge and Ren in [8].

It is worth to note that, as in the classical Landesman-Lazer result, if

$$\limsup_{s \to -\infty} g(s) < g(x) < \limsup_{s \to +\infty} g(s),$$
$$\limsup_{s \to -\infty} h(s) < h(x) < \limsup_{s \to +\infty} h(s) \tag{1.16}$$

then condition (1.14) is also necessary. This result follows immediately by integrating the equation. As it was shown in [6], a different situation occurs under Dirichlet conditions, where Landesman-Lazer conditions are no longer necessary.

Finally, we remark that the nonlinear character of the $p$-Laplacian when $p \neq 2$, introduces many differences with the case of the ordinary Laplacian considered in [1]. For instance, we prove a comparison principle suitable for the nonlinear case (Lemma 3.1), and we use the quasilinear version of the coincidence degree theory. Moreover, the optimal regularity of the solutions is different from the linear case. In general, it can only asserted that they are $C^{1,r}(\Omega)$, for some $r \in (0,1)$, and not in $W^{2,p}(\Omega)$ when $p \neq 2$.

This paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we prove Theorems 1.3 and 1.4. Finally, in Section 4, we prove Theorem 1.5.

## 2. Existence by variational methods

For a proof of Theorem 1.1, let us consider the following functional in the space $W_0^{1,p}(\Omega) + \mathbb{R}$:

$$I(u) = \int_\Omega \left( \frac{|\nabla u|^p}{p} - G(u) + f u \right) - H(u|_{\partial\Omega}), \tag{2.1}$$

where $G(s) = \int_0^s g(t)dt$ and $H(s) = \int_0^s h(t)dt$.

LEMMA 2.1. *Let $\Omega \subset \mathbb{R}^N$ a smooth bounded domain, and let*

$$\varphi_n(x) = \begin{cases} n \cdot d(x, \partial\Omega) & \text{if } d(x, \partial\Omega) \leq 1/n \\ 1 & \text{if } d(x, \partial\Omega) > 1/n. \end{cases} \tag{2.2}$$

*Then $\varphi_n \in W^{1,\infty}(\Omega)$ and if $v \in C(\overline{\Omega}, \mathbb{R}^N)$ then*

$$-\int_\Omega v \cdot \nabla \varphi_n \longrightarrow \int_{\partial\Omega} v \cdot \eta. \tag{2.3}$$

*Proof.* For $v \in C^1(\overline{\Omega}, \mathbb{R}^N)$, the result is immediate by the divergence theorem, and the general case follows by density. $\qquad\square$

LEMMA 2.2. *If $u$ is a critical point of $I$ in the space $W_0^{1,p}(\Omega) + \mathbb{R}$, then $u$ is a weak solution of problem (1.1).*

*Proof.* The derivative of $I$ is given by:

$$\langle I'(u), \varphi \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi - \int_\Omega g(u)\varphi + \int_\Omega f(x)\varphi - h(u\,|_{\partial\Omega})\varphi\,|_{\partial\Omega} \tag{2.4}$$

for all $\varphi \in W_0^{1,p}(\Omega) + \mathbb{R}$ Hence, if $u$ is a critical point of $I$, $u$ is a weak solution of

$$\Delta_p u + g(u) = f(x). \tag{2.5}$$

From regularity theory [4, 14], it follows that $u \in C^{1,r}(\overline{\Omega})$ for some $r > 0$. Furthermore, (by choosing $\varphi \equiv 1$) we see that

$$\int_\Omega g(u) = \int_\Omega f(x) - h(u\,|_{\partial\Omega}). \tag{2.6}$$

On the other hand, by choosing $\varphi = \varphi_n$ as in Lemma 2.1 and letting $n \to \infty$ we have that:

$$\int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial \eta} = \int_\Omega [g(u) - f(x)]. \tag{2.7}$$

Hence, we conclude that $u$ solves the weak formulation of problem (1.1). $\qquad\square$

*Proof of Theorem 1.1.* It is well known that the functional $I$ is weakly lower semicontinuous on $W_0^{1,p}(\Omega) + \mathbb{R}$, and bounded from below. In order to show that $I$ achieves a minimum, let us consider a minimizing sequence $\{u_n\} \subset W_0^{1,p}(\Omega) + \mathbb{R}$. We observe that since $G$ and $H$ are $T$-periodic then

$$I(u + T) = I(u) \quad \forall u \in W_0^{1,p}(\Omega) + \mathbb{R}. \tag{2.8}$$

Hence, we may assume that $u_n\,|_{\partial\Omega} \in [0, T]$ for every $n$. From Poincaré inequality,

$$\|u_n - u_n\,|_{\partial\Omega}\|_{L^p}^p \le c\|\nabla u_n\|_{L^p}^p \le c_1 I(u_n) + c_2\|u_n\|_{L^p} + c_3, \tag{2.9}$$

and it follows that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega) + \mathbb{R}$. By standard results, $I$ achieves a minimum, and the proof is complete. $\qquad\square$

## 3. The method of upper and lower solutions

In order to apply the method of upper and lower solutions to our problem, we will first prove an associated comparison principle.

LEMMA 3.1. *Let $\lambda > 0$, $\rho : \mathbb{R} \to \mathbb{R}$ a continuous nondecreasing function, and let $\Omega_1$ be an open subset of $\Omega$. Assume that $u, v \in C_0^{1,r}(\overline{\Omega}) + \mathbb{R}$ satisfy:*

$$\Delta_p u - \lambda |u|^{p-2}u \geq \Delta_p v - \lambda |v|^{p-2}v \quad in \ \Omega - \overline{\Omega_1} \tag{3.1}$$

*in weak sense,*

$$\int_{\partial\Omega} |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + \rho(u \mid_{\partial\Omega}) \leq \int_{\partial\Omega} |\nabla v|^{p-2}\frac{\partial v}{\partial \nu} + \rho(v \mid_{\partial\Omega}), \tag{3.2}$$
$$u \leq v \quad on \ \partial\Omega_1.$$

*Then $u \leq v$ in $\Omega - \Omega_1$.*

*Proof.* Let us consider the positive test function $(u - v)^+ \mid_{\Omega - \overline{\Omega_1}}$ and

$$\Omega^+ = \{x \in \Omega - \Omega_1 : u(x) > v(x)\}. \tag{3.3}$$

Then

$$\int_{\Omega^+} \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\nabla(u - v) + \lambda \int_{\Omega^+} \left(|u|^{p-2}u - |v|^{p-2}v\right)(u - v)$$
$$\leq (u \mid_{\partial\Omega} - v \mid_{\partial\Omega})^+ \int_{\partial\Omega} \left(|\nabla u|^{p-2}\frac{\partial u}{\partial \eta} - |\nabla v|^{p-2}\frac{\partial v}{\partial \eta}\right) \tag{3.4}$$
$$\leq (u \mid_{\partial\Omega} - v \mid_{\partial\Omega})^+ (\rho(v \mid_{\partial\Omega}) - \rho(u \mid_{\partial\Omega})) \leq 0.$$

On the other hand, from strict monotonicity of the $p$-Laplacian, if $|\Omega^+| > 0$ then

$$\int_{\Omega^+} \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) \cdot \nabla(u - v) > 0, \tag{3.5}$$

a contradiction. □

LEMMA 3.2. *Let $\lambda, \mu > 0$. Then, for every $\phi \in L^{p'}(\Omega)$ and $k \in \mathbb{R}$ the problem*

$$\Delta_p u - \lambda |u|^{p-2}u = \phi \quad in \ \Omega$$
$$u = c \quad on \ \partial\Omega \tag{3.6}$$
$$\int_{\partial\Omega} |\nabla u|^{p-2}\frac{\partial u}{\partial \eta} + \mu |c|^{p-2}c = k$$

*has a unique solution. Moreover, the mapping $(\phi, k) \mapsto u$ from $L^{p'}(\Omega) \times \mathbb{R}$ to $W_0^{1,p}(\Omega) + \mathbb{R}$ is compact.*

*Proof.* Let us consider the functional $I : W_0^{1,p}(\Omega) + \mathbb{R} \to \mathbb{R}$ given by:

$$I(u) = \int_\Omega \left( \frac{|\nabla u|^p}{p} + \lambda \frac{|u|^p}{p} + \phi u \right) - \left( ku \mid_{\partial\Omega} - \mu \mid u \mid_{\partial\Omega} \mid^p \right). \tag{3.7}$$

As before, it is easy to see that any critical point of $I$ is a solution of the problem. Moreover, $I$ is coercive, and the existence of a minimum of $I$ follows from standard results. Uniqueness follows from the comparison principle, and compactness follows from standard arguments (see, e.g., [5]) □

*Proof of Theorem 1.3.* Choose $\lambda, \mu > 0$ and define the function $P$ given by:

$$P(x, u) = \begin{cases} u & \text{if } \alpha(x) \le u \le \beta(x) \\ \beta(x) & \text{if } u > \beta(x) \\ \alpha(x) & \text{if } u < \alpha(x) \end{cases} \tag{3.8}$$

and consider the following fixed point operator $T : W_0^{1,p}(\Omega) + \mathbb{R} \to W_0^{1,p}(\Omega) + \mathbb{R}$. For fixed $u$, define $T(u)$ as the unique solution $v$ of the problem:

$$\Delta_p v - \lambda |v|^{p-2} v = f(x) - g(P(x, u)) - \lambda |P(x, u)|^{p-2} P(x, u)$$

$$v \mid_{\partial\Omega} = c \tag{3.9}$$

$$\int_{\partial\Omega} |\nabla v|^{p-2} \frac{\partial v}{\partial \eta} + \mu |c|^{p-2} c = h(P(x, u) \mid_{\partial\Omega}) + \mu |P(x, u) \mid_{\partial\Omega} |^{p-2} P(x, u) \mid_{\partial\Omega}.$$

From **Lemma 3.2**, $T$ is well-defined and compact. As the right-hand side term is bounded, it follows from Schauder theorem that $T$ has a fixed point $u$. We claim that $\alpha \le u \le \beta$, and hence $u$ is a solution of the problem. Indeed, if we define $\Omega_1 = \{x \in \Omega : u(x) < \beta(x)\}$ then for $x \in \Omega - \Omega_1$ it holds:

$$\Delta_p u - \lambda |u|^{p-2} u = f(x) - g(\beta) - \lambda |\beta|^{p-2} \beta \ge \Delta_p \beta - \lambda |\beta|^{p-2} \beta \tag{3.10}$$

and from the comparison principle it follows that $u \le \beta$ in $\Omega - \Omega_1$. In the same way, it follows that $u \ge \alpha$. From regularity theory, it follows that $u \in C^{1,r}(\overline{\Omega})$ for some $r \in (0, 1)$. □

*Proof of Theorem 1.4.* Let $\beta$ and $\alpha$ be solutions of (1.11) for $(\underline{c_1}, \underline{c_2})$ and $(\overline{c_1}, \overline{c_2})$ respectively. As $g$ is $T$-periodic and $\alpha, \beta \in C(\overline{\Omega})$, adding a term $kT$ ($k \in \mathbb{N}$) if necessary, we may suppose that $\alpha \le \beta$. From definition, it is clear that $\alpha$ is a lower solution and $\beta$ is an upper solution of (1.11) with $c_1$ and $c_2$, and the proof follows from **Theorem 1.3**. □

## 4. Coincidence degree methods

In this section we recall a continuation theorem due to Ge and Ren [8], which extends a classical result by Mawhin [12]. For convenience, we follow the version in [13].

Let $X$ and $Z$ be Banach spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Z$, respectively. A continuous operator $M : X \cap \operatorname{dom} M \to Z$ is said to be quasi-linear if

(i) $\operatorname{Im} M = M(X \cap \operatorname{dom} M)$ is a closed subset of $Z$;

(ii) $\operatorname{Ker} M = \{x \in X \cap \operatorname{dom} M : Mx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$.

Let $P : X \to X_1$ and $Q : Z \to Z_1$ be two projectors such that $\operatorname{Im} P = \operatorname{Ker} M$, $\operatorname{Ker} Q = \operatorname{Im} M$ and $X = X_1 \oplus X_2$, $Z = Z_1 \oplus Z_2$, where $X_1 = \operatorname{Ker} M$, $Z_2 = \operatorname{Im} M$ and $X_2$, $Z_1$ are respectively the complement space of $X_1$ in $X$, $Z_2$ in $Z$. If $U$ is an open and bounded subset of $X$ such that $\operatorname{dom} M \cap U \neq \varnothing$, the continuous operator $N_\lambda : \overline{U} \to Z$, $\lambda \in [0,1]$ will be called $M$-compact in $\overline{U}$ with respect to $M$ if

(iii) There is a subset $Z_1$ of $Z$ with $\dim Z_1 = \dim X_1$ and an operator $K : \operatorname{Im} M \to X_2$ with $K0 = 0$ such that for $\lambda \in [0,1]$,

$$(I - Q)N_\lambda(\bar{U}) \subset \operatorname{Im} M \subset (I - Q)Z,$$

$$(I - Q)N_0 = 0, \qquad QN_\lambda x = 0 \iff QNx = 0, \quad \lambda \in (0,1),$$

$$KM = I - P, \qquad K(I - Q)N_\lambda : \overline{U} \longrightarrow X_2 \subset X \text{ is compact,}$$

$$M[P + K(I - Q)N_\lambda] = (I - Q)N_\lambda.$$

(4.1)

**THEOREM 4.1 ([8]).** *Let $X$ and $Z$ be two Banach spaces with the norms $\| \cdot \|_X$ and $\| \cdot \|_Z$, respectively, and $U \subset X$ an open and bounded set. Suppose $M : X \cap \operatorname{dom} M \to Z$ is a quasi-linear operator and $N_\lambda : \overline{U} \to Z$ is M-compact with respect to M. In addition, if*

(C1) *$Mx \neq N_\lambda x$, $\lambda \in (0,1)$, $x \in \partial U$*

(C2) *$\deg(JQN, U \cap \operatorname{Ker} M, 0) \neq 0$, where $N = N_1$ and $J : Z_1 \to X_1$ is a homeomorphism with $J(0) = 0$;*

*then the abstract equation $Mx = Nx$ has at least one solution in $\overline{U}$.*

The proof of this continuation theorem is based on a Lyapunov-Schmidt reduction argument and an application of the Leray-Schauder degree theory.

In order to apply the continuation theorem to our problem, let us consider:

$$X = W_0^{1,p}(\Omega) + \mathbb{R} = \{u \in W^{1,p}(\Omega) : u_{|\partial\Omega} \text{ is constant}\}$$

$$Z = L^{p'}(\Omega) \times \mathbb{R}$$

$$M(u) = \left(\Delta_p u, \int_{\partial\Omega} \Delta_p u\right)$$

$$\operatorname{dom}(M) = \{u \in X : \Delta_p u \in L^{p'}(\Omega)\}.$$

(4.2)

Then

$$\operatorname{Ker}(M) = \mathbb{R},$$

$$\operatorname{Im}(M) = \left\{(f,c) \in Y : \int_\Omega f(x) = c\right\}.$$

(4.3)

We may define the projectors $P$ and $Q$ as

$$P(u) = \overline{u}$$

$$Q(f,c) = \left(\overline{f} - \frac{c}{|\Omega|}, 0\right).$$

(4.4)

Then

$$\mathrm{Im}(Q) = \{(c,0) : c \in \mathbb{R}\}, \tag{4.5}$$

and we may define $J(c,0) = c$. For $(f,c) \in \mathrm{Im}(M)$, we define $K(f,c)$ as the unique solution of the problem

$$\begin{aligned}
\Delta_p u &= f(x) \quad \text{in } \Omega \\
u &= c \quad \text{on } \partial\Omega \\
\bar{u} &= 0.
\end{aligned} \tag{4.6}$$

Finally, let us consider

$$N(u) = (f - g(u), h(u_{|\partial\Omega})). \tag{4.7}$$

It follows from the strong monotonicity of $M$ that $K(I - Q)N_\lambda$ is compact on $U$ for any open bounded subset $U \subset X$.

### 4.1. A priori bounds

PROPOSITION 4.2. *Let us assume that the conditions of Theorem 1.5 hold. Then there exists a constant $R_0 > 0$ such that if $u$ is a solution of*

$$\begin{aligned}
\Delta_p u &= \lambda[f(x) - g(u)] \quad \text{in } \Omega \\
u &= c \quad \text{on } \partial\Omega \\
\int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial \eta} &= \lambda h(c)
\end{aligned} \tag{4.8}$$

*with $\lambda \in (0,1]$, then $\|u\|_{W^{1,p}} \le R_0$.*

*Proof.* It suffices to consider only the case in which (1.14) holds, since the other case is similar. Assume by contradiction that we have a sequence $(u_n)$ of solutions of

$$\begin{aligned}
\Delta_p u_n &= \lambda_n[f(x) - g(u)] \quad \text{in } \Omega \\
u_n &= c_n \quad \text{on } \partial\Omega \\
\int_{\partial\Omega} |\nabla u_n|^{p-2} \frac{\partial u_n}{\partial \eta} &= \lambda_n h(c_n)
\end{aligned} \tag{4.9}$$

such that $\|u_n\|_{W^{1,p}} \to +\infty$. Set $v_n = u_n - c_n$. Then we have that

$$\begin{aligned}
\Delta_p v_n &= \lambda_n[f(x) - g(v_n + c_n)] \quad \text{in } \Omega \\
v_n &= 0 \quad \text{on } \partial\Omega \\
\int_{\partial\Omega} |\nabla v_n|^{p-2} \frac{\partial v_n}{\partial \eta} &= \lambda_n h(c_n).
\end{aligned} \tag{4.10}$$

As $g$ is bounded, we obtain:

$$\|v_n\|_{W^{1,p}} \le k_1 \|\lambda_n(f(x) - g(v_n + c_n))\|_{L^{p'}} \le k_2. \tag{4.11}$$

It follows that $c_n$ is unbounded, and taking a subsequence we may assume that $c_n \to +\infty$, or that $c_n \to -\infty$.

Since the imbedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, we can extract a subsequence $v_{n_k}$ such that $v_{n_k} \to v$ for the $L^p$-norm and $v_{n_k}(x) \to v(x)$ a.e.

If $c_{n_k} \to +\infty$,

$$\limsup_{k\to\infty} g(v_{n_k}(x) + c_{n_k}) \le \limsup_{s\to+\infty} g(s) \tag{4.12}$$

a.e. in $\Omega$. Thus,

$$\frac{1}{|\Omega|} \int_\Omega \limsup_{k\to\infty} g(v_{n_k}(x) + c_{n_k}) \le \limsup_{s\to+\infty} g(s) \tag{4.13}$$

and from Fatou's lemma:

$$\limsup_{k\to\infty} \frac{1}{|\Omega|} \int_\Omega g(v_{n_k}(x) + c_{n_k}) \le \lim_{s\to+\infty} \sup g(s). \tag{4.14}$$

By Lemma 2.1:

$$\int_{\partial\Omega} |\nabla u_{n_k}|^{p-2} \frac{\partial u_{n_k}}{\partial\eta} + \lambda_{n_k} \int_\Omega g(u_{n_k}) = \lambda_{n_k} \int_\Omega f(x)dx \tag{4.15}$$

or

$$\overline{f} = \frac{h(c_{n_k})}{|\Omega|} + \frac{1}{|\Omega|} \int_\Omega g(v_{n_k}(x) + c_{n_k}). \tag{4.16}$$

Then

$$\overline{f} \le \limsup_{s\to+\infty} g(s) + \limsup_{s\to+\infty} \frac{h(s)}{|\Omega|}, \tag{4.17}$$

a contradiction. In a similar way, we see that if $c_n \to -\infty$, then

$$\liminf_{s\to-\infty} g(s) + \liminf_{s\to-\infty} \frac{h(s)}{|\Omega|} \le \overline{f} \tag{4.18}$$

and the proof follows. □

*Proof of Theorem 1.5.* As before, we assume that (1.14) holds. In order to prove Theorem 1.5, let us consider the bounded open set

$$U = \{u \in X : \|u\|_{W^{1,p}} < R\} \tag{4.19}$$

for some $R > R_0$ large enough, where $R_0$ is the bound given by Proposition 4.2.

It remains to show that $d_B(JQN, U \cap \mathrm{Ker}(L), 0)$ is well defined and different from zero. Let $k : \mathbb{R} \to \mathbb{R}$ be defined by:

$$k(t) = JQN(t) = \overline{p} - g(t) - \frac{h(t)}{|\Omega|}. \tag{4.20}$$

From condition (1.14)

$$k(R) > 0 > k(-R) \tag{4.21}$$

for $R > R_0$ large.

Then $U \cap \mathrm{Ker}(L) = (-R, R)$ and we conclude that $d_B(JQN, U \cap \mathrm{Ker}(L), 0) \ne 0$. Hence, all the conditions of Theorem 4.1 are fulfilled and the proof is complete. □

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