EXISTENCE OF POSITIVE SOLUTIONS FOR SOME POLYHARMONIC NONLINEAR EQUATIONS IN $\mathbb{R}^n$

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We will study the following polyharmonic nonlinear elliptic equation $(-\Delta)^m u + f(\cdot,u) = 0$ in $\mathbb{R}^n$, $n > 2m$. Under appropriate conditions on the nonlinearity $f(x,t)$, related to a class of functions called m-Green-tight functions, we give some existence results for the above equation.

1. Introduction

In this paper, we deal with the higher order elliptic equation

$$(-\Delta)^m u = f(\cdot,u), \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where m is a positive integer such that $n > 2m$.

In the case $m = 1$, (1.1) contains several well-known types which have been studied extensively by many authors (see for example [1–3, 8, 9, 11, 12, 14] and the references therein). Their basic tools are essentially some properties of functions belonging to the classical Kato class $K_n(\mathbb{R}^n)$ and the subclass of Green-tight functions $K_n^\infty(\mathbb{R}^n)$ (some properties pertaining to these classes can be found in [1, 4, 14]).

In this paper, we are concerned with the high order. Our purpose is two folded. One is to extend the Kato class $K_n(\mathbb{R}^n)$ and the subclass $K_n^\infty(\mathbb{R}^n)$ to the order $m \geq 2$. The second purpose is to investigate the existence of positive solutions for (1.1). The outline of the paper is as follows. The existence results are given in Sections 3, 4 and 5. In Section 2, we give the explicit formula of the Green function $G_{m,n}(x,y)$ of $(-\Delta)^m$ in $\mathbb{R}^n$. Namely, for each x,y in $\mathbb{R}^n$

$$G_{m,n}(x,y) = k_{m,n}\frac{1}{|x-y|^{n-2m}}, \tag{1.2}$$

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where $k_{m,n}$ is a positive constant which will be precised later. The 3G-Theorem proved in [13] for the case $m = 1$, is also valid for every m. Indeed, for each x, y, z in $\mathbb{R}^n$, we have

$$\frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} \leq 2^{n-2m-1}\left[G_{m,n}(x,z)+G_{m,n}(z,y)\right]. \tag{1.3}$$

This 3G-Theorem will be useful to state our existence results.

Next, we study the Kato class $K_{m,n}(\mathbb{R}^n)$ defined as follows.

Definition 1.1. A Borel measurable function φ in $\mathbb{R}^n$ ($n > 2m$), belongs to the Kato class $K_{m,n}(\mathbb{R}^n)$ if

$$\lim_{\alpha\to 0}\left(\sup_{x\in\mathbb{R}^n}\int_{|x-y|\leq\alpha}\frac{|\varphi(y)|}{|x-y|^{n-2m}}dy\right) = 0. \tag{1.4}$$

Indeed, first we prove some properties of functions belonging to this class similar to those established in [1, 4]. In particular, we have the following characterization

$$\varphi \in K_{m,n}(\mathbb{R}^n) \Longleftrightarrow \lim_{t\to 0}\left(\sup_{x\in\mathbb{R}^n}\int_0^t s^{m-1}\int_{\mathbb{R}^n} p(s,x,y)|\varphi(y)|\,dy\,ds\right) = 0, \tag{1.5}$$

where $p(t,x,y) = (1/(4\pi t)^{n/2})\exp(-|x-y|^2/4t)$, for $t \in (0,\infty)$ and $x, y \in \mathbb{R}^n$, is the density of the Gauss semi-group on $\mathbb{R}^n$.

Secondly, we study a subclass of $K_{m,n}(\mathbb{R}^n)$ denoted by $K^{\infty}_{m,n}(\mathbb{R}^n)$ and defined by the following.

Definition 1.2. A Borel measurable function φ belongs to the class $K^{\infty}_{m,n}(\mathbb{R}^n)$ and it is called m-Green-tight function if $\varphi \in K_{m,n}(\mathbb{R}^n)$ and satisfies

$$\lim_{M\to\infty}\left(\sup_{x\in\mathbb{R}^n}\int_{|y|\geq M}\frac{|\varphi(y)|}{|x-y|^{n-2m}}dy\right) = 0. \tag{1.6}$$

In particular, we characterize the class $K^{\infty}_{m,n}(\mathbb{R}^n)$ as follows.

Theorem 1.3. *Let $\varphi \in \mathcal{B}^+(\mathbb{R}^n)$, ($n > 2m$). Then the following assertions are equivalent*

(1) $\varphi \in K^{\infty}_{m,n}(\mathbb{R}^n)$.

(2) *The m-potential of φ, $V\varphi(x) := \int_{\mathbb{R}^n} G_{m,n}(x,y)\varphi(y)dy$ is in $C_0^+(\mathbb{R}^n)$.*

This Theorem improves the result of Zhao in [14], for the case $m = 1$. A more fine characterization will be given in the radial case.

One can easily check that $L^1(\mathbb{R}^n) \cap K_{m,n}(\mathbb{R}^n) \subset K^{\infty}_{m,n}(\mathbb{R}^n)$. Also we show that for $p > n/2m$ and $\lambda < 2m - n/p < \mu$, we have

$$\frac{L^p(\mathbb{R}^n)}{(1+|\cdot|)^{\mu-\lambda}|\cdot|^{\lambda}} \subset K^{\infty}_{m,n}(\mathbb{R}^n), \tag{1.7}$$

and we precise the behaviour of the m-potential of functions in this class.

In Section 3, we are interested in the following polyharmonic problem

$$(-\triangle)^m u + u\varphi(\cdot,u) = 0, \quad \text{in } \mathbb{R}^n \text{ (in the sense of distributions)}$$
$$\lim_{|x|\to\infty} u(x) = c > 0. \tag{1.8}$$

The function φ is required to verify the following assumptions.

(H_1) φ is a nonnegative measurable function on $\mathbb{R}^n \times (0,\infty)$.

(H_2) For each $\lambda > 0$, there exists a nonnegative function $q_\lambda \in K^\infty_{m,n}(\mathbb{R}^n)$ with $\alpha_{q_\lambda} \le 1/2$ (see (1.24)) and such that for each $x \in \mathbb{R}^n$, the mapping $t \to t(q_\lambda(x) - \varphi(x,t))$ is continuous and nondecreasing on $[0,\lambda]$.

Under these hypotheses, we give an existence result for the problem (1.8). In fact, we will prove the following theorem.

Theorem 1.4. *Assume (H_1) and (H_2). Then the problem (1.8) has a positive continuous solution u in $\mathbb{R}^n$ satisfying for each $x \in \mathbb{R}^n$, $c/2 \le u(x) \le c$.*

To establish this result, we use a potential theory approach. In particular, we prove that if the function $q \in K^\infty_{m,n}(\mathbb{R}^n)$ is sufficiently small and f is a nonnegative function on $\mathbb{R}^n$, then the equation

$$(-\triangle)^m u + qu = f, \tag{1.9}$$

has a positive solution on $\mathbb{R}^n$. In [6], Grunau and Sweers gave a similar result in the unit ball of $\mathbb{R}^n$, with operators perturbed by small lower order terms:

$$(-\triangle)^m u + \sum_{|k|<2m} a_k(u) D^k u = f. \tag{1.10}$$

In the case $m = 1$, the problem (1.8) has been studied by Mâagli and Masmoudi in [7, 8], where they gave an existence and an uniqueness result in both bounded and unbounded domain Ω.

In Section 4, we are concerned with the following polyharmonic problem

$$(-\triangle)^m u = f(\cdot,u), \quad \text{in } \mathbb{R}^n \text{ (in the sense of distributions)}$$
$$\lim_{|x|\to\infty} u(x) = 0. \tag{1.11}$$

Here f is required to satisfy the following assumptions.

(H_3) f is a nonnegative measurable function on $\mathbb{R}^n \times (0,\infty)$, continuous with respect to the second variable.

(H_4) There exist a nonnegative function p in $\mathbb{R}^n$ such that

$$0 < \alpha_0 := \int_{\mathbb{R}^n} \frac{p(y)}{(|y|+1)^{2(n-2m)}} dy < \infty \tag{1.12}$$

and a nonnegative function $q \in K^\infty_{m,n}(\mathbb{R}^n)$ such that for $x \in \mathbb{R}^n$ and $t > 0$

$$p(x)h(t) \le f(x,t) \le q(x)g(t), \tag{1.13}$$

where h is a nonnegative nondecreasing measurable function on $[0,\infty)$ satisfying

$$m_0 := \frac{1}{k_{m,n}\alpha_0} < h_0 := \liminf_{t\to 0^+} \frac{h(t)}{t} \leq \infty \tag{1.14}$$

and g is a nonnegative measurable function locally bounded on $[0,\infty)$ satisfying

$$0 \leq g^\infty := \limsup_{t\to\infty} \frac{g(t)}{t} < M_0 := \frac{1}{\|Vq\|_\infty}. \tag{1.15}$$

By using a fixed point argument, we will state the following existence result.

Theorem 1.5. *Assume (H_3) and (H_4). Then the problem (1.11) has a positive continuous solution u in $\mathbb{R}^n$ satisfying for each $x \in \mathbb{R}^n$,*

$$\frac{a}{(|x|+1)^{n-2m}} \leq u(x) \leq bVq(x), \tag{1.16}$$

where a, b are positive constants.

This result follows up the one of Dalmasso (see [5]), who studied the problem (1.11) in the unit ball B, with more restrictive conditions on the function f. Indeed, he assumed that f is nondecreasing with respect to the second variable and satisfies

$$\lim_{t\to 0^+} \min_{x\in\overline{B}} \frac{f(x,t)}{t} = +\infty, \qquad \lim_{t\to+\infty} \max_{x\in\overline{B}} \frac{f(x,t)}{t} = 0. \tag{1.17}$$

He proved the existence of a positive solution and he gave also an uniqueness result for positive radial solution when $f(x,t) = f(|x|,t)$.

When $m = 1$, similar conditions, but more restrictive, on the nonlinearity f have been adopted by Mâagli and Masmoudi in [8]. In fact in [8], the authors studied (1.11) in an unbounded domain D of $\mathbb{R}^n$, $n \geq 3$, with compact nonempty boundary ∂D and gave an existence result as Theorem 1.5.

On the other hand, Brezis and Kamin proved in [3], the existence and the uniqueness of a positive solution for the problem

$$\begin{gathered} -\Delta u = \rho(x)u^\alpha \quad \text{in } \mathbb{R}^n, \\ \liminf_{|x|\to\infty} u(x) = 0, \end{gathered} \tag{1.18}$$

with $0 < \alpha < 1$ and ρ is a nonnegative measurable function satisfying some appropriate conditions. We improve in this section the result of Brezis and Kamin in [3] and the one of Mâagli and Masmoudi in [8].

In Section 5, we will study the existence of solutions to the following polyharmonic problem

$$\begin{gathered} (-\triangle)^m u = f(\cdot,u), \quad \text{in } \mathbb{R}^n \text{ (in the sense of distributions)} \\ u(x) > 0, \quad \text{in } \mathbb{R}^n, \end{gathered} \tag{1.19}$$

under the following assumptions on the nonlinearity f.

(H_5) f is a nonnegative measurable function on $\mathbb{R}^n \times (0,\infty)$, continuous with respect to the second variable on $(0,\infty)$.

(H_6) $f(x,t) \le q(x,t)$, where q is a nonnegative measurable function on $\mathbb{R}^n \times (0,\infty)$ such that the function $t \to q(x,t)$ is nondecreasing on $(0,\infty)$.

(H_7) There exists a constant $c > 0$ such that $q(\cdot,c) \in K^{\infty}_{m,n}(\mathbb{R}^n)$ and

$$||V(q(\cdot,c))||_{\infty} < c. \tag{1.20}$$

Put $c^* = c - \|V(q(\cdot,c))\|_{\infty}$. We give in this section the following existence result.

THEOREM 1.6. *Assume (H_5), (H_6), and (H_7). Then for each $\delta \in (0,c^*]$, the problem (1.19) has a positive continuous solution u in $\mathbb{R}^n$ satisfying for each $x \in \mathbb{R}^n$*

$$\begin{gathered} \delta \le u(x) \le c, \\ \lim_{|x|\to\infty} u(x) = \delta. \end{gathered} \tag{1.21}$$

If $m = 1$, Yin gave in [11] an existence result of the following problem

$$\begin{gathered} \triangle u + f(x,u) = 0, \quad \text{in } G_B, \\ u(x) > 0, \end{gathered} \tag{1.22}$$

where $G_B = \{x \in \mathbb{R}^n, |x| > B\}$, for some $B \ge 0$. His method relies on the technique of radial super/subsolutions. Our approach is different, in fact we will use a fixed point argument. We improve the result of Yin under more general assumptions (see Remark 5.3).

In order to simplify our statements, we define some convenient notations.

Notations.

(i) $\mathcal{B}(\mathbb{R}^n)$ denotes the set of Borel measurable functions in $\mathbb{R}^n$ and $\mathcal{B}^+(\mathbb{R}^n)$ the set of nonnegative ones.

(ii) $C_0(\mathbb{R}^n) := \{w$ continuous on $\mathbb{R}^n$ and $\lim_{|x|\to\infty} w(x) = 0\}$ and $C_0^+(\mathbb{R}^n)$ the set of nonnegative ones.

(iii) For $\varphi \in \mathcal{B}^+(\mathbb{R}^n)$, we put the m-potential of φ on $\mathbb{R}^n$ by

$$V\varphi(x) := V_{m,n}\varphi(x) = \int_{\mathbb{R}^n} G_{m,n}(x,y)\varphi(y)dy = k_{m,n}\int_{\mathbb{R}^n} \frac{\varphi(y)}{|x-y|^{n-2m}}dy. \tag{1.23}$$

(iv) For $\varphi \in \mathcal{B}^+(\mathbb{R}^n)$, we put

$$\alpha_{\varphi} = \sup_{x,y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} |\varphi(z)| dz. \tag{1.24}$$

(v) Let $\lambda \in \mathbb{R}$, we denote by $\lambda^+ = \max(\lambda,0)$.

(vi) Let f and g be two positive functions on a set S.

We call $f \sim g$, if there is $c > 0$ such that

$$\frac{1}{c}g(x) \le f(x) \le cg(x) \quad \forall x \in S. \tag{1.25}$$

We call $f \preceq g$, if there is $c > 0$ such that

$$f(x) \leq cg(x) \quad \forall x \in S. \tag{1.26}$$

The following properties will be used several times: for $s,t \geq 0$, we have

$$\begin{gathered} \min(s,t) = s \wedge t \sim \frac{st}{s+t}, \\ (s+t)^p \sim s^p + t^p, \quad p \in \mathbb{R}^+. \end{gathered} \tag{1.27}$$

2. Properties of the Kato class

In this section, we characterize functions belonging to the Kato class $K_{m,n}(\mathbb{R}^n)$ and the subclass $K^{\infty}_{m,n}(\mathbb{R}^n)$ of m-Green-tight functions and we prove Theorem 1.3. We recall that throughout this paper, we are concerned with $n > 2m$.

We set $p(t,x,y) = (1/(4\pi t)^{n/2})\exp(-|x-y|^2/4t)$, for $t \in (0,\infty)$ and $x,y \in \mathbb{R}^n$, the density of the Gauss semi-group on $\mathbb{R}^n$. By a simple computation, we obtain that the Green function of $(-\Delta)^m$ in $\mathbb{R}^n$, for each $m \geq 1$, is given by

$$G_{m,n}(x,y) = \frac{1}{(m-1)!}\int_0^{\infty} s^{m-1} p(s,x,y)\,ds, \quad \text{for } x,y \text{ in } \mathbb{R}^n. \tag{2.1}$$

Then we have the following explicit expression

$$G_{m,n}(x,y) = k_{m,n}\frac{1}{|x-y|^{n-2m}}, \quad \text{for } x,y \text{ in } \mathbb{R}^n, \tag{2.2}$$

where $k_{m,n} = \Gamma(n/2-m)/4^m\pi^{n/2}(m-1)!$.

2.1. The class $K_{m,n}(\mathbb{R}^n)$. We will study properties of functions belonging to $K_{m,n}(\mathbb{R}^n)$. First we remark the following comparison on the classes $K_{j,n}(\mathbb{R}^n)$, for $j \geq 1$.

Remark 2.1. Let $j,m \in \mathbb{N}$ such that $1 \leq j \leq m$, then we have for each $n > 2m$

$$K_n(\mathbb{R}^n) := K_{1,n}(\mathbb{R}^n) \subseteq K_{j,n}(\mathbb{R}^n) \subseteq K_{m,n}(\mathbb{R}^n), \tag{2.3}$$

where $K_n(\mathbb{R}^n)$ is the classical Kato class introduced in [1].

Example 2.2. Let $\varphi \in \mathcal{B}(\mathbb{R}^n)$. Suppose that for $p > n/2m$, we have

$$\sup_{x\in\mathbb{R}^n}\int_{|x-y|\leq 1} |\varphi(y)|^p\,dy < \infty. \tag{2.4}$$

Then by the Hölder inequality, we conclude that $\varphi \in K_{m,n}(\mathbb{R}^n)$.

In particular, we have that for $p > n/2m$, $L^p(\mathbb{R}^n) \subset K_{m,n}(\mathbb{R}^n)$.

To establish the characterization (1.5) of the Kato class $K_{m,n}(\mathbb{R}^n)$, we need the following lemmas.

Lemma 2.3. *For each $t > 0$ and $x, y \in \mathbb{R}^n$, we have*

$$\int_0^t s^{m-1} p(s,x,y)\,ds \preceq G_{m,n}(x,y). \tag{2.5}$$

Moreover, for $|x-y| \leq 2\sqrt{t}$, we have that

$$G_{m,n}(x,y) \preceq \int_0^t s^{m-1} p(s,x,y)\,ds. \tag{2.6}$$

Proof. Let $t > 0$ and $x, y \in \mathbb{R}^n$. Then (2.5) follows immediately from (2.1).

If we suppose further that $|x-y| \leq 2\sqrt{t}$, then we have

$$\begin{aligned} \int_0^t s^{m-1} p(s,x,y)\,ds &= c\int_0^t s^{m-n/2-1} \exp\left(-\frac{|x-y|^2}{4s}\right) ds \\ &= \frac{c}{|x-y|^{n-2m}} \int_{|x-y|^2/4t}^{\infty} r^{n/2-m-1} e^{-r}\,dr \\ &\geq \frac{c}{|x-y|^{n-2m}} \int_1^{\infty} r^{n/2-m-1} e^{-r}\,dr \\ &= c\,G_{m,n}(x,y), \end{aligned} \tag{2.7}$$

where the letter c is a positive constant which may vary from line to line. □

Lemma 2.4. *Let $\varphi \in K_{m,n}(\mathbb{R}^n)$. Then for each compact $L \subset \mathbb{R}^n$, we have*

$$\sup_{x\in\mathbb{R}^n} \int_{x+L} |\varphi(y)|\,dy < \infty. \tag{2.8}$$

In particular, we have $K_{m,n}(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$.

Proof. Let $\varphi \in K_{m,n}(\mathbb{R}^n)$, then by (1.4) there exists $\alpha > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \int_{|x-y|\leq\alpha} \frac{|\varphi(y)|}{|x-y|^{n-2m}}\,dy \leq 1. \tag{2.9}$$

Let $a_1,\dots,a_p \in L$ such that $L \subseteq \bigcup_{1\leq i\leq p} B(a_i,\alpha)$. Hence for each $x \in \mathbb{R}^n$, we have

$$\begin{aligned} \int_{x+L} |\varphi(y)|\,dy &\leq \sum_{i=1}^p \int_{B(x+a_i,\alpha)} |\varphi(y)|\,dy \\ &\leq \sum_{i=1}^p \alpha^{n-2m} \int_{B(x+a_i,\alpha)} \frac{|\varphi(y)|}{|x+a_i-y|^{n-2m}}\,dy \\ &\leq p\alpha^{n-2m}. \end{aligned} \tag{2.10}$$

So, $\sup_{x\in\mathbb{R}^n} \int_{x+L} |\varphi(y)|\,dy < \infty$. □

Proposition 2.5. *Let $\varphi \in K_{m,n}(\mathbb{R}^n)$. Then for each fixed $\alpha > 0$, we have*

$$\sup_{0\le t\le 1}\left(\sup_{x\in\mathbb{R}^n}\int_{|x-y|\ge\alpha} t^{m-1}p(t,x,y)\,|\,\varphi(y)\,|\,dy\right) := M(\alpha) < \infty. \tag{2.11}$$

Proof. Let $\varphi \in K_{m,n}(\mathbb{R}^n)$, $0 < t \le 1$. Let $\alpha > 0$, then we have that

$$\begin{aligned}\sup_{x\in\mathbb{R}^n}&\int_{|x-y|\ge\alpha} t^{m-1}p(t,x,y)\,|\,\varphi(y)\,|\,dy\\ &\preceq \frac{\exp\left(-\alpha^2/8t\right)}{t^{n/2-m+1}}\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\frac{|x-y|^2}{8}\right)|\,\varphi(y)\,|\,dy.\end{aligned} \tag{2.12}$$

So to prove (2.11), we need to show that

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\frac{|x-y|^2}{8}\right)|\,\varphi(y)\,|\,dy < \infty. \tag{2.13}$$

Indeed, using Lemma 2.4, we denote by

$$c := \sup_{x\in\mathbb{R}^n}\int_{x+B(0,1)}|\,\varphi(y)\,|\,dy < \infty. \tag{2.14}$$

On the other hand, since any ball $B(0,k)$ of radius $k \ge 1$ in $\mathbb{R}^n$ can be covered by $\alpha(n) := A_n k^n$ balls of radius 1, where A_n is a constant depending only on n (see [4, page 67]), then there exist $a_1, a_2, \dots, a_{\alpha(n)} \in B(0,k)$ such that

$$B(0,k) \subset \bigcup_{1\le i\le\alpha(n)} B(a_i,1). \tag{2.15}$$

Hence for each $x \in \mathbb{R}^n$, we have

$$\int_{x+B(0,k)}|\,\varphi(y)\,|\,dy \le \sum_{i=1}^{\alpha(n)}\int_{B(x+a_i,1)}|\,\varphi(y)\,|\,dy \le cA_n k^n, \tag{2.16}$$

which implies that for each $x \in \mathbb{R}^n$,

$$\begin{aligned}\int_{\mathbb{R}^n}\exp\left(-\frac{|x-y|^2}{8}\right)|\,\varphi(y)\,|\,dy &\le \sum_{k=0}^{\infty}\exp\left(-\frac{k^2}{8}\right)\int_{k\le|x-y|\le k+1}|\,\varphi(y)\,|\,dy\\ &\le cA_n\sum_{k=0}^{\infty}\exp\left(-\frac{k^2}{8}\right)(k+1)^n\\ &< \infty.\end{aligned} \tag{2.17}$$

Thus (2.13) holds. This ends the proof. □

Proposition 2.6. *Let $\varphi \in B(\mathbb{R}^n)$. Then $\varphi \in K_{m,n}(\mathbb{R}^n)$ if and only if*

$$\lim_{t\to 0}\left(\sup_{x\in\mathbb{R}^n}\int_0^t s^{m-1}\int_{\mathbb{R}^n} p(s,x,y)\,|\varphi(y)|\,dy\,ds\right)=0. \tag{2.18}$$

Proof. Suppose φ verifies (2.18), then from (2.6) we have that

$$\int_{|x-y|\le\alpha}\frac{|\varphi(y)|}{|x-y|^{n-2m}}dy \preceq \int_{\mathbb{R}^n}\int_0^{\alpha^2/4} s^{m-1}p(s,x,y)\,|\varphi(y)|\,ds\,dy, \tag{2.19}$$

which implies that the function φ satisfies (1.4).

Conversely, suppose that $\varphi \in K_{m,n}(\mathbb{R}^n)$. Let $\varepsilon > 0$, then by (1.4), there exists $\alpha > 0$ such that

$$\sup_{x\in\mathbb{R}^n}\int_{|x-y|\le\alpha}\frac{|\varphi(y)|}{|x-y|^{n-2m}}dy \le \varepsilon. \tag{2.20}$$

Thus from (2.5) and (2.11), we deduce that for each $x \in \mathbb{R}^n$ and $t \le 1$, we have

$$\begin{aligned}
&\int_0^t s^{m-1}\int_{\mathbb{R}^n} p(s,x,y)\,|\varphi(y)|\,dy\,ds\\
&\quad\preceq \int_{|x-y|\le\alpha}\int_0^t s^{m-1}p(s,x,y)\,|\varphi(y)|\,dy\,ds\\
&\qquad + \int_0^t\int_{|x-y|\ge\alpha} s^{m-1}p(s,x,y)\,|\varphi(y)|\,dy\,ds\\
&\quad\preceq \int_{|x-y|\le\alpha}\frac{|\varphi(y)|}{|x-y|^{n-2m}}dy + tM(\alpha)\\
&\quad\preceq \varepsilon + tM(\alpha).
\end{aligned} \tag{2.21}$$

This implies (2.18) and completes the proof. □

2.2. The class $K^{\infty}_{m,n}(\mathbb{R}^n)$. We will characterize the subclass of m-Green-tight functions $K^{\infty}_{m,n}(\mathbb{R}^n)$. In fact, we will prove Theorem 1.3 and we give in particular a more precise characterization in the radial case.

Example 2.7. Let $p > n/2m$. Then $L^p(\mathbb{R}^n)\cap L^1(\mathbb{R}^n) \subset K^{\infty}_{m,n}(\mathbb{R}^n)$.

Proof of Theorem 1.3. Let $\varphi \in \mathcal{B}^+(\mathbb{R}^n)$. First we suppose that $\varphi \in K^{\infty}_{m,n}(\mathbb{R}^n)$, then using similar arguments as in the proof [9, Proposition 6], we obtain easily that $V\varphi \in C_0^+(\mathbb{R}^n)$.

Conversely we suppose that $V\varphi \in C_0^+(\mathbb{R}^n)$. Then, we aim at proving that $\varphi \in K^{\infty}_{m,n}(\mathbb{R}^n)$. So we divide the proof into two steps.

Step 1. We will prove that φ satisfies (2.18). Indeed it is clear from (2.1), that for each $x \in \mathbb{R}^n$, we have that

$$\begin{aligned} V\varphi(x) &= \frac{1}{(m-1)!}\int_0^t s^{m-1}\int_{\mathbb{R}^n} p(s,x,y)\varphi(y)\,dy\,ds \\ &\quad + \frac{1}{(m-1)!}\int_t^\infty s^{m-1}\int_{\mathbb{R}^n} p(s,x,y)\varphi(y)\,dy\,ds \\ &= I_1(x)+I_2(x). \end{aligned} \tag{2.22}$$

From the properties of the density $p(s,x,y)$, we deduce that $x \to I_1(x)$ and $x \to I_2(x)$ are nonnegative lower semi-continuous functions in $\mathbb{R}^n$. Then using the fact that $V\varphi \in C_0^+(\mathbb{R}^n)$, we get that the function $x \to I_1(x)$ is also in $C_0^+(\mathbb{R}^n)$. So, for each $x \in \mathbb{R}^n$, the family $\{\int_0^t s^{m-1}\int_{\mathbb{R}^n} p(s,x,y)\varphi(y)\,dy\,ds,\ t>0\}$ is decreasing in $C_0^+(\mathbb{R}^n)$, which together with the fact that for each $x \in \mathbb{R}^n$,

$$\lim_{t\to 0}\int_0^t s^{m-1}\int_{\mathbb{R}^n} p(s,x,y)\varphi(y)\,dy\,ds = 0 \tag{2.23}$$

imply by Dini Lemma, that (2.18) is satisfied.

Step 2. We will prove that φ satisfies (1.6). Let $\varepsilon > 0$, then since $V\varphi \in C_0^+(\mathbb{R}^n)$, there exists $a > 0$ such that for $|x| \geq a$, we have that $V\varphi(x) \leq \varepsilon$.

Let $M \geq 2a$, then

$$\begin{aligned} \sup_{x\in\mathbb{R}^n}\int_{|y|\geq M}\frac{\varphi(y)}{|x-y|^{n-2m}}\,dy &\leq \sup_{|x|\geq a}\int_{\mathbb{R}^n}\frac{\varphi(y)}{|x-y|^{n-2m}}\,dy + \sup_{|x|\leq a}\int_{|y|\geq M}\frac{\varphi(y)}{|x-y|^{n-2m}}\,dy \\ &\preceq \varepsilon + \int_{|y|\geq M}\frac{\varphi(y)}{|y|^{n-2m}}\,dy. \end{aligned} \tag{2.24}$$

Now, since $V\varphi(0) < \infty$, we deduce that

$$\lim_{M\to\infty}\int_{|y|\geq M}\frac{\varphi(y)}{|y|^{n-2m}}\,dy = 0. \tag{2.25}$$

Then (1.6) holds and this ends the proof. □

For a nonnegative function ρ in $K_{m,n}^\infty(\mathbb{R}^n)$, we denote by

$$M_\rho := \{\varphi \in \mathcal{B}(\mathbb{R}^n),\ |\varphi| \preceq \rho\}. \tag{2.26}$$

Proposition 2.8. *For a nonnegative function ρ in $K_{m,n}^\infty(\mathbb{R}^n)$, the family of functions*

$$V(M_\rho) := \{V\varphi,\ \varphi \in M_\rho\} \tag{2.27}$$

is uniformly bounded and equicontinuous in $C_0(\mathbb{R}^n)$ and consequently it is relatively compact in $C_0(\mathbb{R}^n)$.

Proof. Let $\rho \in K_{m,n}^{\infty}(\mathbb{R}^n)$. Obviously, since each function φ in M_ρ is in $K_{m,n}^{\infty}(\mathbb{R}^n)$, we obtain by Theorem 1.3 that the family $V(M_\rho) \subset C_0(\mathbb{R}^n)$ and is uniformly bounded. Next, we prove the equicontinuity of functions in $V(M_\rho)$ on $\mathbb{R}^n \cup \{\infty\}$ by same arguments as in the proof of [9, Proposition 6]. Thus by Ascoli's Theorem the family $V(M_\rho)$ is relatively compact in $C_0(\mathbb{R}^n)$. This ends the proof. □

Remark 2.9. We recall (see [12, 14]) that for $m = 1$ and $n \geq 3$, a radial function is in $K_n^{\infty}(\mathbb{R}^n)$ if and only if $\int_0^\infty r|\varphi(r)|dr < \infty$.

Similarly, we will give in the sequel a characterization of radial functions belonging to $K_{m,n}^{\infty}(\mathbb{R}^n)$.

Proposition 2.10. *Let φ be a radial function in $\mathbb{R}^n$, then $\varphi \in K_{m,n}^{\infty}(\mathbb{R}^n)$ if and only if*

$$\int_0^\infty r^{2m-1}\,|\,\varphi(r)\,|\,dr < \infty. \tag{2.28}$$

In order to prove Proposition 2.10, we will use the following behaviour of the m-potential of radial functions on $\mathbb{R}^n$.

Proposition 2.11. *Let $\varphi \in \mathcal{B}^+(\mathbb{R}^n)$ be a radial function on $\mathbb{R}^n$, then for $x \in \mathbb{R}^n$, we have*

$$V\varphi(x) \sim \int_0^\infty \frac{r^{n-1}}{(|x| \vee r)^{n-2m}}\varphi(r)dr. \tag{2.29}$$

Proof. Let $\varphi \in \mathcal{B}^+(\mathbb{R}^n)$. First, we recall the well known results for $x, y \in \mathbb{R}^n$,

$$\begin{gathered}(n-2)k_{1,n}\int_{\mathbb{R}^n}\frac{\varphi(y)}{|x-y|^{n-2}}dy = \int_0^\infty \frac{r^{n-1}}{(|x| \vee r)^{n-2}}\varphi(r)dr,\\ \int_{\mathbb{R}^n}\frac{dz}{|x-z|^{n-2}|y-z|^{n-2}} = \frac{c_n}{|x-y|^{n-4}}.\end{gathered} \tag{2.30}$$

This implies that there exists a constant $c > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{\varphi(y)}{|x-y|^{n-4}}dy &= c\int_0^\infty r^{n-1}\varphi(r)\int_0^\infty \frac{t^{n-1}}{(|x| \vee t)^{n-2}(t \vee r)^{n-2}}dtdr\\ &\geq c\int_0^\infty r^{n-1}\varphi(r)\int_{|x|\vee r}^\infty \frac{1}{t^{n-3}}dtdr\\ &\geq \frac{c}{n-4}\int_0^\infty \frac{r^{n-1}\varphi(r)}{(|x| \vee r)^{n-4}}dr.\end{aligned} \tag{2.31}$$

Hence, we obtain by recurrence that

$$\int_0^\infty \frac{r^{n-1}}{(|x| \vee r)^{n-2m}}\varphi(r)dr \preceq \int_{\mathbb{R}^n}\frac{\varphi(y)}{|x-y|^{n-2m}}dy. \tag{2.32}$$

On the other hand, there exists a constant $\widetilde{c} > 0$ such that for each $x \in \mathbb{R}^n$,

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\varphi(y)}{|x-y|^{n-2m}} dy &= \widetilde{c} \int_0^\infty \int_0^\pi \frac{r^{n-1}\varphi(r)(\sin\theta)^{n-2}}{(|x|^2 + r^2 - 2r|x|\cos\theta)^{(n-2m)/2}} d\theta dr \\ &\le \widetilde{c} \int_0^\infty \int_0^\pi \frac{r^{n-1}\varphi(r)(\sin\theta)^{n-2}}{(|x| \vee r)^{n-2m}(\sin\theta)^{n-2m}} d\theta dr \\ &= \widetilde{c}\left(\int_0^\pi (\sin\theta)^{2m-2} d\theta\right)\left(\int_0^\infty \frac{r^{n-1}\varphi(r)}{(|x| \vee r)^{n-2m}} dr\right).\end{aligned} \tag{2.33}$$

Thus (2.29) holds. □

Proof of Proposition 2.10. Suppose that φ is a radial function in $K_{m,n}^\infty(\mathbb{R}^n)$, then by Theorem 1.3, $V\varphi(0) < \infty$ and so we deduce (2.28) from (2.29).

Conversely, suppose that φ satisfies (2.28). Let $\alpha > 0$ and $t = |x|$, then by (2.29), we have

$$\begin{aligned}\int_{|x-y|\le\alpha} \frac{|\varphi(y)|}{|x-y|^{n-2m}} dy &\preceq \int_{(t-\alpha)^+}^{t+\alpha} \frac{r^{n-1}}{(t \vee r)^{n-2m}} |\varphi(r)| dr \\ &\le \int_{(t-\alpha)^+}^{t+\alpha} r^{2m-1} |\varphi(r)| dr.\end{aligned} \tag{2.34}$$

Let $\phi(s) = \int_0^s r^{2m-1}|\varphi(r)|dr$, for $s \in [0, \infty]$. Using (2.28), we deduce that ϕ is a continuous function on $[0, \infty]$. This implies that

$$\int_{(t-\alpha)^+}^{t+\alpha} r^{2m-1} |\varphi(r)| dr = \phi(t+\alpha) - \phi((t-\alpha)^+), \tag{2.35}$$

converges to zero as $\alpha \to 0$ uniformly for $t \in [0, \infty]$. So φ verifies (1.4).

Next, we have by (2.29)

$$\int_{|y|\ge M} \frac{|\varphi(y)|}{|x-y|^{n-2m}} dy \preceq \int_M^\infty \frac{r^{n-1}}{(t \vee r)^{n-2m}} |\varphi(r)| dr \le \int_M^\infty r^{2m-1} |\varphi(r)| dr, \tag{2.36}$$

which, using (2.28), tends to zero as $M \to \infty$ and so φ verifies (1.6). This completes the proof. □

We close this section by giving a class of functions included in $K_{m,n}^\infty(\mathbb{R}^n)$ and we precise the behaviour of the m-potential of functions in this class. We need the following lemma.

Lemma 2.12. *Let $\alpha > 0$ and $a, b > 0$ such that $a+b < n$. Then*

$$\int_{|x-y|\leq\alpha} \frac{dy}{|y|^a|x-y|^b} \preceq \alpha^{n-(a+b)}. \tag{2.37}$$

Proof. Let $\alpha > 0$ and a, b be nonnegative real numbers such that $a+b<n$. Then

$$\begin{aligned}\int_{|x-y|\leq\alpha} \frac{dy}{|y|^a|x-y|^b} &\leq \int_{(|x-y|\leq\alpha)\cap(|x-y|\leq|y|)} \frac{dy}{|x-y|^{a+b}} + \int_{(|y|\leq|x-y|\leq\alpha)} \frac{dy}{|y|^{a+b}} \\ &\preceq \int_0^\alpha r^{n-1-(a+b)}dr \\ &\preceq \alpha^{n-(a+b)}.\end{aligned} \tag{2.38}$$

□

Proposition 2.13. *Let $p > n/2m$. Then for $\lambda < 2m - n/p < \mu$, we have*

$$\frac{L^p(\mathbb{R}^n)}{(1+|\cdot|)^{\mu-\lambda}|\cdot|^\lambda} \subset K^\infty_{m,n}(\mathbb{R}^n). \tag{2.39}$$

Proof. Let $p > n/2m$ and $q \geq 1$ such that $1/p + 1/q = 1$. Let a be a function in $L^p(\mathbb{R}^n)$ and $\lambda < 2m - n/p < \mu$. First, we will prove that the function $\varphi(x) := a(x)/(1+|x|)^{\mu-\lambda}|x|^\lambda$ satisfies (1.4). Let $\alpha > 0$, then by the Hölder inequality and Lemma 2.12, we have for $x \in \mathbb{R}^n$

$$\begin{aligned}\int_{|x-y|\leq\alpha} \frac{|\varphi(y)|}{|x-y|^{n-2m}}dy &\leq \|a\|_p\left(\int_{|x-y|\leq\alpha} \frac{dy}{(1+|y|)^{(\mu-\lambda)q}|y|^{\lambda q}|x-y|^{(n-2m)q}}\right)^{1/q} \\ &\leq \|a\|_p\left(\int_{|x-y|\leq\alpha} \frac{dy}{|y|^{q\lambda^+}|x-y|^{(n-2m)q}}\right)^{1/q} \\ &\preceq \|a\|_p\alpha^{2m-n/p-\lambda^+},\end{aligned} \tag{2.40}$$

which converges to zero as $\alpha \to 0$.

Secondly, we claim that φ satisfies (1.6). To show the claim we use the Hölder inequality. Let $M > 1$, then we have

$$\begin{aligned}\int_{|y|\geq M} \frac{|\varphi(y)|}{|x-y|^{n-2m}}dy &\leq \|a\|_p\left(\int_{|y|\geq M} \frac{dy}{(1+|y|)^{(\mu-\lambda)q}|y|^{\lambda q}|x-y|^{(n-2m)q}}\right)^{1/q} \\ &\sim \|a\|_p\left(\int_{|y|\geq M} \frac{dy}{|y|^{\mu q}|x-y|^{(n-2m)q}}\right)^{1/q} \\ &= \|a\|_p(A(x))^{1/q}.\end{aligned} \tag{2.41}$$

Furthermore

$$
\begin{aligned}
A(x) \preceq & \sup_{|x|\leq M/2} \int_{|y|\geq M} \frac{dy}{|y|^{(n-2m+\mu)q}} \\
& + \sup_{|x|\geq M/2} \frac{1}{|x|^{\mu q}} \int_{(|y|\geq M)\cap(|x-y|\leq |x|/2)} \frac{dy}{|x-y|^{(n-2m)q}} \\
& + \sup_{|x|\geq M/2} \frac{1}{|x|^{(n-2m)q}} \int_{(|y|\geq M)\cap(|x|/2\leq|x-y|\leq 2|x|)} \frac{dy}{|y|^{\mu q}} \\
& + \sup_{|x|\geq M/2} \int_{(|y|\geq M)\cap(|x-y|\geq 2|x|)} \frac{dy}{|x-y|^{(n-2m+\mu)q}} \\
\preceq & \frac{1}{M^{(n-2m+\mu)q-n}} + \sup_{|z|\geq M/2} \frac{\operatorname{Log}(3|z|/M)}{|z|^{(n-2m)q}},
\end{aligned}
\tag{2.42}
$$

which converges to zero as $M \to \infty$. This ends the proof. □

Remark 2.14. It is obvious to see that for each $\varphi \in \mathcal{B}^+(\mathbb{R}^n)$, we have

$$
\frac{k_{m,n}}{(|x|+1)^{n-2m}} \int_{\mathbb{R}^n} \frac{\varphi(y)}{(|y|+1)^{n-2m}} dy \leq V\varphi(x). \tag{2.43}
$$

We precise in the following, some upper estimates on the m-potential of functions in the class $L^p(\mathbb{R}^n)/(1+|\cdot|)^{\mu-\lambda}|\cdot|^\lambda$. Indeed, put for a nonnegative function $a \in L^p(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$

$$
Wa(x) := V\left(\frac{a}{(1+|\cdot|)^{\mu-\lambda}|\cdot|^\lambda}\right)(x) = \int_{\mathbb{R}^n} G_{m,n}(x,y) \frac{a(y)}{(1+|y|)^{\mu-\lambda}|y|^\lambda} dy. \tag{2.44}
$$

Then we have the following.

Proposition 2.15. *Let $p > n/2m$ and $\lambda < 2m - n/p < \mu$. Then there exists $c > 0$ such that for each nonnegative function $a \in L^p(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we have the following estimates*

$$
Wa(x) \leq c\|a\|_p \begin{cases} \dfrac{1}{(1+|x|)^{n-2m}} \operatorname{Log}(|x|+1)^{p/(p-1)}, & \text{if } \mu + \dfrac{n}{p} = n \\ \dfrac{1}{(1+|x|)^{(n-2m)\wedge(\mu+n/p-2m)}}, & \text{if } \mu + \dfrac{n}{p} \neq n. \end{cases} \tag{2.45}
$$

Proof. Let $p > n/2m$ and $q \geq 1$ such that $1/p + 1/q = 1$. Let a be a nonnegative function in $L^p(\mathbb{R}^n)$ and $\lambda < 2m - n/p < \mu$. Put $\varphi(x) = a(x)/(1+|x|)^{\mu-\lambda}|x|^{\lambda}$, then by the Hölder inequality, we have for each $x \in \mathbb{R}^n$

$$\begin{aligned} V\varphi(x) &\leq \|a\|_p \left(\int_{\mathbb{R}^n} \frac{dy}{|x-y|^{(n-2m)q}(1+|y|)^{(\mu-\lambda)q}|y|^{\lambda q}} \right)^{1/q} \\ &= \|a\|_p (I(x))^{1/q}. \end{aligned} \tag{2.46}$$

Furthermore,

(i) if $|x| \leq 1$, we have by Lemma 2.12, that

$$\begin{aligned} I(x) &\preceq \int_{B(x,2)} \frac{dy}{|x-y|^{(n-2m)q}|y|^{q\lambda^+}} + \int_{B^c(x,2)} \frac{dy}{|x-y|^{(n-2m)q}|y|^{\mu q}} \\ &\preceq \int_{B(x,2)} \frac{dy}{|x-y|^{(n-2m)q}|y|^{q\lambda^+}} + \int_{B^c(0,2)} \frac{dy}{|x-y|^{(n-2m+\mu)q}} \\ &\preceq 1, \end{aligned} \tag{2.47}$$

(ii) if $|x| \geq 1$, we have

$$\begin{aligned} I(x) &\preceq \int_{(|y|\leq 1/2)} \frac{dy}{|x-y|^{(n-2m)q}|y|^{\lambda q}} + \int_{(|y|\geq 1/2)\cap(|x-y|\leq |x|/2)} \frac{dy}{|x-y|^{(n-2m)q}|y|^{\mu q}} \\ &\quad + \int_{(|y|\geq 1/2)\cap(|x|/2\leq |x-y|\leq 2|x|)} \frac{dy}{|x-y|^{(n-2m)q}|y|^{\mu q}} \\ &\quad + \int_{(|y|\geq 1/2)\cap(|x-y|\geq 2|x|)} \frac{dy}{|x-y|^{(n-2m)q}|y|^{\mu q}} \\ &\preceq \frac{1}{|x|^{(n-2m)q}} \int_{(|y|\leq 1/2)} \frac{dy}{|y|^{\lambda q}} + \frac{1}{|x|^{\mu q}} \int_{(|x-y|\leq |x|/2)} \frac{dy}{|x-y|^{(n-2m)q}} \\ &\quad + \frac{1}{|x|^{(n-2m)q}} \int_{(1/2\leq |y|\leq 3|x|)} \frac{dy}{|y|^{\mu q}} + \int_{(|x-y|>2|x|)} \frac{dy}{|x-y|^{(n-2m+\mu)q}} \\ &\preceq \frac{1}{|x|^{(n-2m)q}} \begin{cases} \operatorname{Log}(|x|+1), & \text{if } \mu + \dfrac{n}{p} = n \\ |x|^{n-\mu q}, & \text{if } \mu + \dfrac{n}{p} < n \\ 1, & \text{if } \mu + \dfrac{n}{p} > n. \end{cases} \end{aligned} \tag{2.48}$$

By combining the above inequalities, we get the result. □

Corollary 2.16. *The class of functions $L^\infty(\mathbb{R}^n)/(1+|\cdot|)^{\mu-\lambda}|\cdot|^{\lambda}$ is included in $K^\infty_{m,n}(\mathbb{R}^n)$ if and only if $\lambda < 2m < \mu$.*

Proof. "$\Leftarrow$" follows from Proposition 2.13.

"$\Rightarrow$" Suppose that the function φ defined on $\mathbb{R}^n$ by $\varphi(x) = 1/(1+|x|)^{\mu-\lambda}|x|^{\lambda}$ is in $K^{\infty}_{m,n}(\mathbb{R}^n)$. Then by Proposition 2.10, we have $\int_0^{\infty} r^{2m-1}\varphi(r)dr < \infty$. This implies that $\lambda < 2m < \mu$. □

Remark 2.17. Let $\lambda < 2m < \mu$ and $\varphi(x) = 1/(1+|x|)^{\mu-\lambda}|x|^{\lambda}$, for $x \in \mathbb{R}^n$, then by simple calculus, we obtain the following behaviour on the m-potential

$$V\varphi(x) \sim \begin{cases} \dfrac{1}{(1+|x|)^{n-2m}} \operatorname{Log}(|x|+1), & \text{if } \mu = n \\ \dfrac{1}{(1+|x|)^{(n-2m)\wedge(\mu-2m)}}, & \text{if } \mu \neq n. \end{cases} \tag{2.49}$$

3. First existence result

In this section, we aim at proving Theorem 1.4. The following lemmas are useful.

Lemma 3.1. *Let φ be a nonnegative function in $K^{\infty}_{m,n}(\mathbb{R}^n)$. Then we have*

$$\|V\varphi\|_{\infty} \leq \alpha_{\varphi} \leq 2^{n-2m}\|V\varphi\|_{\infty}. \tag{3.1}$$

Proof. By (1.3) we obtain easily that $\alpha_{\varphi} \leq 2^{n-2m}\|V\varphi\|_{\infty}$. On the other hand, by letting $|y| \to \infty$ in (1.24), we deduce from Fatou Lemma that $\|V\varphi\|_{\infty} \leq \alpha_{\varphi}$. □

Lemma 3.2. *Let φ be a nonnegative function in $K^{\infty}_{m,n}(\mathbb{R}^n)$. Then for each $x \in \mathbb{R}^n$, we have*

$$V(\varphi G_{m,n}(\cdot,y))(x) \leq \alpha_{\varphi} G_{m,n}(x,y). \tag{3.2}$$

Proof. The result holds by (1.24). □

In the sequel, let q be a nonnegative function in $K^{\infty}_{m,n}(\mathbb{R}^n)$ such that $\alpha_q \leq 1/2$. For $f \in \mathcal{B}^+(\mathbb{R}^n)$, we will define the potential kernel $V_q f := V_{m,n,q} f$ as a solution for the perturbed polyharmonic equation (1.9).

We put for $x, y \in \mathbb{R}^n$,

$$\mathcal{G}_{m,n}(x,y) = \begin{cases} \sum_{k\geq 0} (-1)^k (V(q\cdot))^k (G_{m,n}(\cdot,y))(x), & \text{if } x \neq y \\ \infty, & \text{if } x = y. \end{cases} \tag{3.3}$$

Then we have the following comparison result.

Lemma 3.3. *Let q be a nonnegative function in $K^{\infty}_{m,n}(\mathbb{R}^n)$ such that $\alpha_q \leq 1/2$. Then for $x, y \in \mathbb{R}^n$, we have*

$$(1-\alpha_q) G_{m,n}(x,y) \leq \mathcal{G}_{m,n}(x,y) \leq G_{m,n}(x,y). \tag{3.4}$$

Proof. Since $\alpha_q \leq 1/2$, we deduce from (3.2), that

$$\begin{aligned} |\mathcal{G}_{m,n}(x,y)| &\leq \sum_{k\geq 0} (\alpha_q)^k G_{m,n}(x,y) \\ &= \frac{1}{1-\alpha_q} G_{m,n}(x,y). \end{aligned} \tag{3.5}$$

Furthermore, we have for $x \neq y$ in $\mathbb{R}^n$

$$\mathcal{G}_{m,n}(x,y) = G_{m,n}(x,y) - V\left(q\mathcal{G}_{m,n}(\cdot,y)\right)(x), \tag{3.6}$$

which together with (3.2), imply that

$$\begin{aligned} \mathcal{G}_{m,n}(x,y) &\geq G_{m,n}(x,y) - \frac{\alpha_q}{1-\alpha_q} G_{m,n}(x,y) \\ &= \frac{1-2\alpha_q}{1-\alpha_q} G_{m,n}(x,y) \\ &\geq 0. \end{aligned} \tag{3.7}$$

Hence the result follows from (3.6) and (3.2). □

Let us define the operator V_q on $\mathcal{B}^+(\mathbb{R}^n)$ by

$$V_q f(x) = \int_B \mathcal{G}_{m,n}(x,y) f(y) dy, \quad x \in \mathbb{R}^n. \tag{3.8}$$

Then we obtain the following.

Lemma 3.4. *Let $f \in \mathcal{B}^+(\mathbb{R}^n)$. Then $V_q f$ satisfies the following resolvent equation*

$$Vf = V_q f + V_q(qVf) = V_q f + V\left(qV_q f\right). \tag{3.9}$$

Proof. From the expression of $\mathcal{G}_{m,n}$, we deduce that for $f \in \mathcal{B}^+(\mathbb{R}^n)$ such that $Vf < \infty$,

$$V_q f = \sum_{k\geq 0} (-1)^k \left(V(q\cdot)\right)^k Vf. \tag{3.10}$$

So we obtain that

$$\begin{aligned} V_q(qVf) &= \sum_{k\geq 0} (-1)^k \left(V(q\cdot)\right)^k [V(qVf)] \\ &= -\sum_{k\geq 1} (-1)^k \left(V(q\cdot)\right)^k Vf \\ &= Vf - V_q f. \end{aligned} \tag{3.11}$$

The second equality holds by integrating (3.6). □

Proposition 3.5. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ such that $Vf \in L^1_{loc}(\mathbb{R}^n)$. Then $V_q f$ is a solution (in the sense of distributions) of the perturbed polyharmonic equation (1.9).*

Proof. Using the resolvent equation (3.9), we have

$$V_q f = Vf - V(qV_q f). \tag{3.12}$$

Applying the operator $(-\Delta)^m$ on both sides of the above equality, we obtain that

$$(-\Delta)^m(V_q f) = f - qV_q f \quad \text{(in the sense of distributions).} \tag{3.13}$$

This completes the proof. □

Now, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $c > 0$. Then *by* (H_2), there exists a nonnegative function $q := q_c \in K^{\infty}_{m,n}(\mathbb{R}^n)$, such that $\alpha_q \leq 1/2$ and for each $x \in \mathbb{R}^n$, the map

$$t \longrightarrow t(q(x) - \varphi(x,t)) \text{ is continuous and nondecreasing on } [0,c], \tag{3.14}$$

which implies in particular that for each $x \in \mathbb{R}^n$ and $t \in [0,c]$,

$$0 \leq \varphi(x,t) \leq q(x), \tag{3.15}$$

Let

$$\Lambda := \{u \in \mathcal{B}^+(\mathbb{R}^n) : (1-\alpha_q)c \leq u \leq c\}. \tag{3.16}$$

We define the operator T on Λ by

$$Tu(x) := c(1 - V_q(q)(x)) + V_q[(q - \varphi(\cdot,u))u](x). \tag{3.17}$$

First, we prove that Λ is invariant under T. Indeed, for each $u \in \Lambda$, we have

$$Tu \leq c(1 - V_q(q)(x)) + cV_q(q)(x) \leq c. \tag{3.18}$$

Moreover, from (3.15), (3.4) and Lemma 3.1 we deduce that for each $u \in \Lambda$, we have

$$Tu \geq c(1 - V_q(q)(x)) \geq c(1 - V(q)(x)) \geq c(1-\alpha_q). \tag{3.19}$$

Next, we prove that the operator T is nondecreasing on Λ. Indeed, let $u, v \in \Lambda$ such that $u \leq v$, then from (3.14) we obtain that

$$Tv - Tu = V_q([(q - \varphi(\cdot,v))v] - [(q - \varphi(\cdot,u))u]) \geq 0. \tag{3.20}$$

Now, consider the sequence (u_k) defined by $u_0 = (1-\alpha_q)c$ and $u_{k+1} = Tu_k$, for $k \in \mathbb{N}$. Then since Λ is invariant under T, we obtain obviously that $u_1 = Tu_0 \geq u_0$ and so from the monotonicity of T, we have

$$u_0 \leq u_1 \leq \cdots \leq u_k \leq c. \tag{3.21}$$

So from (3.14) and the dominated convergence theorem we deduce that the sequence (u_k) converges to a function $u \in \Lambda$ which satisfies

$$u = c(1 - V_q(q)(x)) + V_q[(q - \varphi(\cdot, u))u](x). \tag{3.22}$$

That is

$$u - V_q(qu) = c(1 - V_q(q)(x)) - V_q(u\varphi(\cdot, u)). \tag{3.23}$$

Applying the operator $(I + V(q\cdot))$ on both sides of the above equality and using (3.9) we deduce that u satisfies

$$u = c - V(u\varphi(\cdot, u)). \tag{3.24}$$

Finally, we claim that u is a positive continuous solution for the Problem (1.6). To prove the claim, we use Lemma 2.4. Indeed, since $u \sim c$ on $\mathbb{R}^n$ and

$$0 \leq u\varphi(\cdot, u) \leq cq, \tag{3.25}$$

we deduce that either u and $u\varphi(\cdot, u)$ are in $L^1_{\text{loc}}(\mathbb{R}^n)$.

Now, from (3.24) we can easily see that $V(u\varphi(\cdot, u)) \in L^1_{\text{loc}}(\mathbb{R}^n)$. Hence u satisfies (in the sense of distributions) the elliptic differential equation

$$(-\Delta)^m u + u\varphi(\cdot, u) = f \quad \text{in } \mathbb{R}^n. \tag{3.26}$$

On the other hand, it follows from (3.25) that $u\varphi(\cdot, u) \in M_q$ and so by Proposition 2.8, we obtain that $V(u\varphi(\cdot, u))$ is in $C_0^+(\mathbb{R}^n)$.

This implies by (3.24) that $\lim_{|x| \to \infty} u(x) = c$, which completes the proof. □

Remark 3.6. Let $c > 0$ and u be a solution of (1.8). Then we have by Theorem 1.4 that for each $x \in \mathbb{R}^n$, $0 \leq u(x) \leq c$. Let q be the nonnegative function in $K^{\infty}_{m,n}(\mathbb{R}^n)$ given in the proof of Theorem 1.4. Then we deduce from (3.24) and (3.25), that

$$0 \leq c - u(x) = V(u\varphi(\cdot, u))(x) \leq cV(q)(x). \tag{3.27}$$

Example 3.7. Let $p > n/2m$ and a be a nonnegative function in $L^p(\mathbb{R}^n)$. Let $\lambda < 2m - n/p < \mu$ and α, β be two nonnegative constants.

Put $q(x) = a(x)/(1 + |x|)^{\mu - \lambda}|x|^{\lambda}$. Then, for each $c > 0$, the following polyharmonic problem

$$\begin{gathered}(-\triangle)^m u + \beta u^{\alpha+1} q = 0, \quad \text{in } \mathbb{R}^n \text{ (in the sense of distributions)} \\ \lim_{|x| \to \infty} u(x) = c,\end{gathered} \tag{3.28}$$

has a positive continuous solution satisfying $c/2 \leq u(x) \leq c$, provided that β is sufficiently small.

Moreover, by Remark 3.6 and Proposition 2.15, we have

$$0 \le c - u(x) \le c\|a\|_p \begin{cases} \dfrac{1}{(1+|x|)^{n-2m}} \operatorname{Log}(|x|+1)^{p/(p-1)}, & \text{if } \mu + \dfrac{n}{p} = n \\ \dfrac{1}{(1+|x|)^{(n-2m)\wedge(\mu+n/p-2m)}}, & \text{if } \mu + \dfrac{n}{p} \neq n. \end{cases} \tag{3.29}$$

Remark 3.8. It is interesting to compare the asymptotics (3.29) with the results of Trubek [10], for the case $m = 1$.

4. Second existence result

In this section, we aim at proving Theorem 1.5.

Proof of Theorem 1.5. Assuming (H_3) and (H_4), we will use the Schauder fixed point theorem. From (1.14), there exists $\eta > 0$ such that

$$h(t) \ge m_0 t, \quad \text{for each } t \in [0,\eta]. \tag{4.1}$$

On the other hand, let $\alpha \in (g^\infty, M_0)$, then by (1.15), there exists $\rho > 0$ such that for $t \ge \rho$, we have $g(t) \le \alpha t$. Put $\beta = \sup_{0\le t\le\rho} g(t)$. So we deduce that

$$0 \le g(t) \le \alpha t + \beta, \quad \text{for each } t \ge 0. \tag{4.2}$$

By Remark 2.14, we note that there exists a constant $\alpha_1 > 0$ such that

$$\frac{\alpha_1}{(1+|x|)^{n-2m}} \le Vq(x). \tag{4.3}$$

Let $a \in (0,\eta)$ and $b = \max\{a/\alpha_1, \beta/(1-\alpha\|Vq\|_\infty)\}$. So we consider the closed convex set

$$\Lambda = \left\{ u \in C_0(\mathbb{R}^n),\ \frac{a}{(1+|x|)^{n-2m}} \le u(x) \le bVq(x),\ \forall x \in \mathbb{R}^n \right\}. \tag{4.4}$$

Obviously by (4.3) we have that the set Λ is nonempty. Next we define the operator T on Λ by

$$Tu(x) = \int_{\mathbb{R}^n} G_{m,n}(x,y) f(y,u(y))\,dy. \tag{4.5}$$

Let us prove that $T\Lambda \subset \Lambda$. Let $u \in \Lambda$, then by (4.2) we have

$$\begin{aligned} Tu(x) &\le \int_{\mathbb{R}^n} G_{m,n}(x,y) q(y) g(u(y))\,dy \\ &\le \int_{\mathbb{R}^n} G_{m,n}(x,y) q(y) [\alpha u(y) + \beta]\,dy \\ &\le (\alpha b\|Vq\|_\infty + \beta) Vq(x) \\ &\le bVq(x). \end{aligned} \tag{4.6}$$

Moreover, since h is nondecreasing, we deduce by (4.1) and (1.14) that

$$\begin{aligned} Tu(x) &\geq \int_{\mathbb{R}^n} G_{m,n}(x,y)p(y)h(u(y))\,dy \\ &\geq \int_{\mathbb{R}^n} G_{m,n}(x,y)p(y)h\left(\frac{a}{(1+|y|)^{n-2m}}\right)dy \\ &\geq m_0 a\int_{\mathbb{R}^n} G_{m,n}(x,y)\frac{p(y)}{(1+|y|)^{n-2m}}\,dy \\ &\geq \frac{m_0 a k_{m,n}}{(1+|x|)^{n-2m}}\int_{\mathbb{R}^n}\frac{p(y)}{(1+|y|)^{2(n-2m)}}\,dy \\ &= \frac{a}{(1+|x|)^{n-2m}}. \end{aligned} \tag{4.7}$$

On the other hand, by (1.13), we have that for each $u \in \Lambda$

$$f(\cdot,u) \leq g(b\|Vq\|_\infty)q. \tag{4.8}$$

This implies by Proposition 2.8 that $Tu \in V(M_q) \subset C_0(\mathbb{R}^n)$. So $T\Lambda \subset \Lambda$.

Next, we prove the continuity of T in Λ. Let (u_k) be a sequence in Λ, which converges uniformly to a function $u \in \Lambda$. Then using (4.8) and (H_3), we deduce by Theorem 1.3 and the dominated convergence Theorem that for $x \in \mathbb{R}^n$,

$$Tu_k(x) \longrightarrow Tu(x) \quad \text{as } k \longrightarrow \infty. \tag{4.9}$$

Now, since $T\Lambda \subset V(M_q)$, we deduce by Proposition 2.8 that $T\Lambda$ is relatively compact in $C_0(\mathbb{R}^n)$, which implies that

$$\|Tu_k - Tu\|_\infty \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{4.10}$$

Hence T is a compact map from Λ to itself. So the Schauder fixed point theorem leads to the existence of $u \in \Lambda$ such that

$$u = V(f(\cdot,u)). \tag{4.11}$$

Finally by (4.8) and Lemma 2.4, we conclude that $y \to f(y,u(y))$ is in $L^1_{\text{loc}}(\mathbb{R}^n)$, which together with (4.11) imply that u satisfies (in the sense of distributions) the elliptic differential equation

$$(-\triangle)^m u = f(\cdot,u) \quad \text{in } \mathbb{R}^n. \tag{4.12}$$

This ends the proof. □

Example 4.1. Let p be a nonnegative function in $K^\infty_{m,n}(\mathbb{R}^n)$ and $0 \leq \alpha < 1$. Then the following problem

$$\begin{gathered} (-\triangle)^m u + p(x)u^\alpha = 0, \quad x \in \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = 0, \end{gathered} \tag{4.13}$$

has a positive solution $u \in C_0(\mathbb{R}^n)$ satisfying for each $x \in \mathbb{R}^n$

$$\frac{1}{(1+|x|)^{n-2m}} \preceq u(x) \preceq Vp(x). \tag{4.14}$$

5. Third existence result

In this section, we aim at proving Theorem 1.6.

Proof of Theorem 1.6. Let $c>0$ be the constant given by (H_7) and $c^* = c - \|V(q(\cdot,c))\|_\infty$. Let $\delta \in (0,c^*]$. We will use the Schauder fixed point theorem, so we consider the closed convex set

$$\Lambda = \{u \in C(\mathbb{R}^n \cup \{\infty\}) : \delta \le u(x) \le c,\ \forall x \in \mathbb{R}^n\} \tag{5.1}$$

and we define the integral operator T on Λ by

$$Tu(x) = \delta + V(f(\cdot,u))(x). \tag{5.2}$$

First, we prove that $T\Lambda \subset \Lambda$. Let $u \in \Lambda$, then since f is a nonnegative function, we have that $Tu(x) \ge \delta$, for each $x \in \mathbb{R}^n$. Moreover by (H_6), we have for $x \in \mathbb{R}^n$,

$$Tu(x) \le \delta + V(q(\cdot,u))(x) \le c^* + V(q(\cdot,c))(x) \le c. \tag{5.3}$$

Furthermore by (H_7), since for all $u \in \Lambda$, $f(\cdot,u) \in M_{q(\cdot,c)}$, then it follows from Proposition 2.8 that $V(f(\cdot,u)) \in C_0(\mathbb{R}^n)$ and more precisely $T\Lambda$ is relatively compact in $C(\mathbb{R}^n \cup \{\infty\})$. Therefore $T\Lambda \subset \Lambda$.

Next, let us prove the continuity of T in Λ. Let (u_k) be a sequence in Λ, which converges uniformly to a function $u \in \Lambda$. Since f is continuous with respect to the second variable, we deduce by the dominated convergence theorem that for each $x \in \mathbb{R}^n \cup \{\infty\}$,

$$Tu_k(x) \longrightarrow Tu(x) \quad \text{as } k \longrightarrow \infty. \tag{5.4}$$

Now, since $T\Lambda$ is relatively compact in $C(\mathbb{R}^n \cup \{\infty\})$, then

$$\|Tu_k - Tu\|_\infty \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{5.5}$$

Finally the Schauder fixed point theorem implies the existence of $u \in \Lambda$ such that

$$u(x) = \delta + V(f(\cdot,u))(x), \quad \forall x \in \mathbb{R}^n. \tag{5.6}$$

Using (H_6), (H_7) and Lemma 2.4, we deduce that the function $y \to f(y,u(y))$ is in $L^1_{\text{loc}}(\mathbb{R}^n)$. So u satisfies (in the sense of distributions) the elliptic differential equation

$$(-\triangle)^m u = f(\cdot,u) \quad \text{in } \mathbb{R}^n. \tag{5.7}$$

Moreover since $V(f(\cdot,u)) \in C_0(\mathbb{R}^n)$, then by (5.6) it follows that $\lim_{|x|\to\infty} u(x) = \delta$. This ends the proof. □

Corollary 5.1. *Assume that $q(x,t) = p(x)g(t)$, where g is a nonnegative nondecreasing measurable function and p is a nonnegative function in $K^{\infty}_{m,n}(\mathbb{R}^n)$. If the function g satisfies either $g(t) = o(t)$ as $t \to 0$ or $g(t) = o(t)$ as $t \to \infty$, then the problem (1.19) has a positive solution $u \in C(\mathbb{R}^n \cup \{\infty\})$.*

Example 5.2. Among the equations of form (1.1), we have the Emden-Fowler equation of order m

$$(-\triangle)^m u + p(x)u^{\alpha} = 0, \quad \alpha > 0,\ x \in \mathbb{R}^n,\ n > 2m, \tag{5.8}$$

where $p \in K^{\infty}_{m,n}(\mathbb{R}^n)$.

(i) For the sublinear ($0 < \alpha < 1$) or the superlinear ($\alpha > 1$) case, let $c > 0$ such that

$$\|Vp\|_{\infty} c^{\alpha-1} < 1. \tag{5.9}$$

Then applying Theorem 1.6, we deduce that for each $\delta \in (0, c(1 - c^{\alpha-1}\|Vp\|_{\infty}))$, (5.8) with $\alpha \neq 1$ has a continuous positive solution u in $\mathbb{R}^n$ with $\delta \le u(x) \le c$, for all $x \in \mathbb{R}^n$ and $\lim_{|x|\to\infty} u(x) = \delta$.

(ii) For the linear case ($\alpha = 1$). If $\|Vp\|_{\infty} < 1$, then applying Theorem 1.6, we deduce that for each $c > 0$ and $\delta \in (0, c(1 - \|Vp\|_{\infty}))$, (5.8) has a continuous positive solution u in $\mathbb{R}^n$ with $\delta \le u(x) \le c$, for all $x \in \mathbb{R}^n$ and $\lim_{|x|\to\infty} u(x) = \delta$.

Remark 5.3. We improve in this section the Yin's result in [11]. Indeed, Yin proved in particular the existence of bounded positive solutions for the Emden-Fowler equation

$$\triangle u + p(x)u^{\alpha} = 0, \quad 0 < \alpha \neq 1,\ x \in \mathbb{R}^n,\ n \ge 3, \tag{5.10}$$

provided that the function p satisfies

$$\int_0^{\infty} s \max_{|x|=s} \{p(x)\} ds < \infty. \tag{5.11}$$

However by taking $\lambda > (n-1)/2$ and

$$p(x) = p(x', x_n) = \frac{1}{(1 + x_n^2)\left(1 + \sum_{i=1}^{n-1} x_i^2\right)^{\lambda}}, \quad x \in \mathbb{R}^n, \tag{5.12}$$

then we have

$$\max_{|x|=s} p(x) \ge p(0,s) = \frac{1}{1+s^2} \tag{5.13}$$

which implies that (5.11) is not satisfied. On the other hand, we have that $p \in L^{\infty}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \subset K^{\infty}_{m,n}(\mathbb{R}^n)$. This implies by Corollary 5.1 that the Emden-Fowler equation (5.8) has a positive solution $u \in C(\mathbb{R}^n \cup \{\infty\})$, for each $m \ge 1$.

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