Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2007, Article ID 20295, 17 pages
doi:10.1155/2007/20295

Research Article

A Wave Equation Associated with Mixed Nonhomogeneous Conditions: The Compactness and Connectivity of Weak Solution Set

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Received 17 April 2006; Revised 19 September 2006; Accepted 24 November 2006

Recommended by Jean Mawhin

The purpose of this paper is to show that the set of weak solutions of the initial-boundary value problem for the linear wave equation is nonempty, connected, and compact.

1. Introduction

In this paper, we consider the following problem: find a pair (u,P) of functions satisfying

$$\begin{gathered}u_{tt}-u_{xx}+Ku+\lambda u_t=F(x,t),\quad 0<x<1,\ 0<t<T,\\ u_x(0,t)=P(t),\qquad u(1,t)=0,\\ u(x,0)=u_0(x),\qquad u_t(x,0)=u_1(x),\end{gathered}\tag{1.1}$$

where the constants K, λ, the functions u_0, u_1, F are given before satisfying conditions specified later, and the unknown function $u(x,t)$ and the unknown boundary value $P(t)$ satisfy the following integral equation:

$$P(t)=g(t)+K_1\left|u(0,t)\right|^{\alpha-2}u(0,t)+\lambda_1\left|u_t(0,t)\right|^{\beta-2}u_t(0,t)-\int_0^t k(t-s)u(0,s)\,ds,\tag{1.2}$$

in which the constants K_1, λ_1, α, β and the functions g, k are also given before.

This paper is a continuation of authors' series of papers dealing with mixed problems for wave equations; see for instance the papers [1–5] among many others. In some special cases, the problem (1.1)-(1.2) is the mathematical model describing a shock problem involving a linear viscoelastic bar [1, 2, 4].

In [5], under conditions $(u_0,u_1) \in H^1 \times L^2$, $F \in L^2(Q_T)$, $g,k \in H^1(0,T)$, $K \in \mathbb{R}$, $\lambda, K_1 \geq 0$, $\lambda_1 > 0$, α, $\beta \geq 2$, Long and Giai have proved a theorem of global existence and uniqueness of a weak solution (u,P) of (1.1)-(1.2). The proof is based on a Galerkin method associated to a priori estimates, weak convergence, and compactness techniques. Furthermore, the asymptotic behavior of the solution (u,P) as $\lambda_1 \to 0_+$ and an asymptotic expansion of the solution (u,P) of (1.1)-(1.2) up to order $N+1/2$ have been established.

The purpose of this paper is to develop some aspects related to the article [5]. We show here that the set of solutions of (1.1)-(1.2) is nonempty and satisfies the classical Hukuhara-Kneser property, that is, the set of solutions is a compact connected set in an appropriate function space. Such structure of the solutions set for differential equations was studied in [6–10], we mention here [8–10] for Hukuhara-Kneser property and [6, 7] for R_δ-structure. It is well known that the "R_δ" results are really stronger than classical Kneser-type theorems, see [6, 7]. An important application of the above structures is that the existence of two different solutions implies the existence of a continuum of different solutions.

It is similar to the structure theorem in [8–10], a topological approach to proving connectivity of the solution set of operator equations was invented by Krasnosel'skii and Perov, one of the theorems in this paper is closely related to the following theorem.

Theorem 1.1 [11]. *Let $(E,|\cdot|)$ be a real Banach space, let D be a bounded open subset of E with boundary ∂D, closure $\overline{D}$, and let $T:\overline{D}\to E$ be a completely continuous operator. Assume that T satisfies the following conditions.*

(i) *T has no fixed points on ∂D and $\gamma(I-T,D)\neq 0$.*

(ii) *For each $\varepsilon>0$, there is a completely continuous operator T_ε such that $|T_\varepsilon(x)-T(x)| < \varepsilon$, for all $x\in\overline{D}$, and such that for each h with $|h|<\varepsilon$, the equation $x = T_\varepsilon(x)+h$ has at most one solution in $\overline{D}$.*

Then the set of fixed points of T is nonempty, compact, and connected.

The proof of the above theorem can be found in [11, Theorem 48.2]. We note that condition (i) is equivalent to the following condition.

($\tilde{\text{i}}$) T has no fixed points on ∂D and $\deg(I-T,D,0)\neq 0$.

Because of this, if a completely continuous operator T is defined on $\overline{D}$ and has no fixed points on ∂D, then the rotation $\gamma(I-T,D)$ coincides with the Leray-Schauder degree of $I-T$ on D with respect to the origin, see [11, Section 20.2].

Deimling also studied the compactness and connectivity of the set of all fixed points, these theorems are given in [12, page 212].

Applying Theorem 1.1, using the same methods as in [5, 13], we consider the structure of the solution set of (1.1)-(1.2). The paper consists of three sections and the main results, in the case $1<\alpha,\beta\leq 2$, will be presented in Sections 2, 3. We end the paper by a remark about the similar result to the one above in the cases $1<\alpha<2$, $\beta\geq 2$ or $1<\beta<2$, $\alpha\geq 2$. In our argument, the topological degree theory of compact vector fields and the following theorem will be used.

Theorem 1.2 [12, page 53]. *Let E, F be Banach spaces, let D be an open subset of E, and let $f:D\to F$ be continuous. Then for each $\varepsilon>0$, there is a mapping $f_\varepsilon:D\to F$ that is locally*

Lipschitz such that

$$\left| f(x) - f_\varepsilon(x) \right| < \varepsilon, \quad \forall x \in D, \tag{1.3}$$

and $f_\varepsilon(D)$ is a subset of the closed convex hull of $f(D)$.

2. Existence of weak solution

Put $\Omega = (0,1)$, $Q_T = \Omega \times (0,T)$, $T > 0$. We omit the definitions of usual function spaces: $C^m(\overline{\Omega})$, $L^p(\Omega)$, $W^{m,p}(\Omega)$. We denote $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$, $1 \leq p \leq \infty$, $m = 0,1,\dots$.

The norm in L^2 is denoted by $\|\cdot\|$. We also denote by $\langle\cdot,\cdot\rangle$ the scalar product in L^2 or pair of dual scalar products of a continuous linear functional with an element of a function space. We denote by $\|\cdot\|_X$ the norm of a Banach space X and by X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$, the Banach space of the real functions $u:(0,T) \to X$ measurable, such that

$$\begin{aligned} \|u\|_{L^p(0,T;X)} &= \left(\int_0^T \|u(t)\|_X^p dt \right)^{1/p} < \infty \quad \text{for } 1 \leq p < \infty, \\ \|u\|_{L^\infty(0,T;X)} &= \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X \quad \text{for } p = \infty. \end{aligned} \tag{2.1}$$

Let $u(t), u'(t) = u_t(t)$, $u''(t) = u_{tt}(t)$, $u_x(t)$, $u_{xx}(t)$ denote $u(x,t)$, $(\partial u/\partial t)(x,t)$, $(\partial^2 u/\partial t^2)(x,t)$, $(\partial u/\partial x)(x,t)$, $(\partial^2 u/\partial x^2)(x,t)$, respectively. We put

$$\begin{aligned} V &= \{v \in H^1(0,1) : v(1) = 0\}, \\ a(u,v) &= \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} dx. \end{aligned} \tag{2.2}$$

Then V is a closed subspace of H^1 and on V, $\|v\|_{H^1}$ and $\|v\|_V = \sqrt{a(v,v)} = \|v_x\|$ are two equivalent norms. We have the following lemma.

Lemma 2.1. *The imbedding $V \hookrightarrow C^0([0,1])$ is compact and*

$$\|v\|_{C^0([0,1])} \leq \|v\|_V \quad \forall v \in V. \tag{2.3}$$

The proof is straightforward and we omit the details.

We make the following assumptions:

(H_1) $u_0 \in V$ and $u_1 \in L^2$,
(H_2) $F \in L^2(Q_T)$,
(H_3) $g, k \in H^1(0,T)$,
(H_4) $K \in \mathbb{R}$, $\lambda, K_1 \geq 0$, $\lambda_1 > 0$,
(H_5) $1 < \alpha \leq 2$, $1 < \beta \leq 2$.

Theorem 2.2. *Let* (H_1)*–*(H_5) *hold. Let* $T>0$*. Then the problem (1.1)-(1.2) has at least one weak solution* (u,P) *such that*

$$\begin{gathered} u\in L^\infty(0,T;V), \qquad u_t\in L^\infty(0,T;L^2), \\ u(0,\cdot)\in W^{1,\beta}(0,T), \quad P\in L^{\beta'}(0,T), \quad \beta'=\frac{\beta}{\beta-1}. \end{gathered} \tag{2.4}$$

Proof of Theorem 2.2. The proof consists of Steps 1–3.

Step 1 (the Galerkin approximation). Let $\{w_j\}$ be a denumerable base of V. Find the approximate solution of problem (1.1)-(1.2) in the form

$$u_m(t)=\sum_{j=1}^m c_{mj}(t)w_j, \tag{2.5}$$

where the coefficient functions c_{mj} satisfy the system of ordinary differential equations

$$\begin{aligned} \langle u_m''(t),w_i\rangle+\langle u_{mx}(t),w_{ix}\rangle+P_m(t)w_i(0)+\langle Ku_m(t)+\lambda u_m'(t),w_i\rangle& \\ =\langle F(t),w_i\rangle, \quad 1\le i\le m,& \end{aligned} \tag{2.6}$$

$$P_m(t)=g(t)+K_1H_\alpha(u_m(0,t))+\lambda_1H_\beta(u_m'(0,t))-\int_0^t k(t-s)u_m(0,s)ds, \tag{2.7}$$

where $H_r(z)=|z|^{r-2}z$, $r\in\{\alpha,\beta\}$,

$$\begin{aligned} u_m(0)=u_{0m}&=\sum_{j=1}^m \alpha_{mj}w_j\longrightarrow u_0 \quad \text{strongly in } H^1, \\ u_m'(0)=u_{1m}&=\sum_{j=1}^m \beta_{mj}w_j\longrightarrow u_1 \quad \text{strongly in } L^2. \end{aligned} \tag{2.8}$$

Equation (2.6) is rewritten in the form

$$\begin{aligned} \sum_{j=1}^m\langle w_j,w_i\rangle\left(c_{mj}''(t)+\lambda c_{mj}'(t)\right)+\sum_{j=1}^m\left[\langle w_{jx},w_{ix}\rangle+K\langle w_j,w_i\rangle\right]c_{mj}(t)& \\ +P[c_m,c_m'](t)w_i(0)=\langle F(t),w_i\rangle, \quad 1\le i\le m,& \end{aligned} \tag{2.9}$$

where

$$\begin{gathered} P[c_m,c_m'](t)=g(t)-\int_0^t k(t-s)\sum_{j=1}^m c_j(s)w_j(0)ds+\widetilde{P}[c_m,c_m'](t), \\ \widetilde{P}:\mathbb{R}^{2m}\longrightarrow\mathbb{R}, \quad \widetilde{P}[y,z]=K_1H_\alpha\left(\sum_{j=1}^m y_jw_j(0)\right)+\lambda_1H_\beta\left(\sum_{j=1}^m z_jw_j(0)\right). \end{gathered} \tag{2.10}$$

Integrating (2.9) with respect to the time variable from 0 to t, we get

$$\sum_{j=1}^m b_{ij}\left(c_{mj}(t)e^{\lambda t}\right)'(t)+A_{mi}[c_m,c_m'](t)=\Psi_{mi}(t), \quad 1\le i\le m, \tag{2.11}$$

where

$$
\begin{gathered}
A_{mi}[c_m,c_m'](t)=e^{\lambda t}\left[\sum_{j=1}^{m}\mu_{ij}\int_0^t c_{mj}(s)ds+w_i(0)\int_0^t P[c_m,c_m'](s)ds\right],\\
\Psi_{mi}(t)=e^{\lambda t}\left[\sum_{j=1}^{m}b_{ij}(\beta_{mj}+\lambda\alpha_{mj})+\int_0^t\langle F(s),w_i\rangle(s)ds\right],\\
b_{ij}=\langle w_j,w_i\rangle,\quad \mu_{ij}=\langle w_{jx},w_{ix}\rangle+K\langle w_j,w_i\rangle.
\end{gathered}\tag{2.12}
$$

Integrating again (2.11), we obtain

$$
\sum_{j=1}^{m}b_{ij}c_{mj}(t)+e^{-\lambda t}\int_0^t A_{mi}[c_m,c_m'](r)dr=\widetilde{\Psi}_{mi}(t),\quad 1\le i\le m,\tag{2.13}
$$

where

$$
\widetilde{\Psi}_{mi}(t)=e^{-\lambda t}\left[\sum_{j=1}^{m}b_{ij}\alpha_{mj}+\int_0^t\Psi_{mi}(r)dr\right].\tag{2.14}
$$

Let $B=(b_{ij})$, with $b_{ij}=\langle w_j,w_i\rangle$. Then B is invertible. Hence, it follows from (2.13) that

$$
c_m(t)=-B^{-1}D_m[c_m,c_m'](t)+B^{-1}\widetilde{\Psi}_m(t)\equiv(Uc_m)_m(t),\tag{2.15}
$$

where

$$
\begin{gathered}
c_m(t)=(c_{m1}(t),\dots,c_{mm}(t)),\qquad (Uc_m)_m(t)=((Uc_m)_{m1}(t),\dots,(Uc_m)_{mm}(t)),\\
\widetilde{\Psi}_m(t)=(\widetilde{\Psi}_{m1}(t),\dots,\widetilde{\Psi}_{mm}(t)),\\
D_m[c_m,c_m'](t)=(D_{m1}[c_m,c_m'](t),\dots,D_{mm}[c_m,c_m'](t)),\\
D_{mi}[c_m,c_m'](t)=e^{-\lambda t}\int_0^t A_{mi}[c_m,c_m'](r)dr,\quad 1\le i\le m.
\end{gathered}\tag{2.16}
$$

Omitting the index m, the system (2.15)-(2.16) is rewritten in the form

$$
c=Uc,\tag{2.17}
$$

we note more that for $1\le i\le m$,

$$
\begin{gathered}
(Uc)_i(t)=-B^{-1}D_i[c,c'](t)+B^{-1}\widetilde{\Psi}_i(t),\\
D_i[c,c'](t)=e^{-\lambda t}\int_0^t A_i[c,c'](r)dr,\\
\widetilde{\Psi}_i(t)=e^{-\lambda t}\left[\sum_{j=1}^{m}b_{ij}\alpha_j+\int_0^t\Psi_i(r)dr\right],\\
\Psi_i(t)=e^{\lambda t}\left[\sum_{j=1}^{m}b_{ij}(\beta_j+\lambda\alpha_j)+\int_0^t\langle F(s),w_i\rangle(s)ds\right].
\end{gathered}\tag{2.18}
$$

For $T_m > 0$, $M > 0$ which are chosen later, we put

$$\overline{S} = \{c \in C^1([0,T_m];\mathbb{R}^m) : \|c\|_1 \le M\},$$
$$\|c\|_1 = \|c\|_0 + \|c'\|_0, \qquad \|c\|_0 = \sup_{0\le t\le T_m} |c(t)|_1, \qquad |c(t)|_1 = \sum_{i=1}^{m} |c_i(t)|. \tag{2.19}$$

Clearly, $\overline{S}$ is a closed convex and bounded subset of the Banach space $Y = C^1([0,T_m];\mathbb{R}^m)$. Consider the operator $U : \overline{S} \to Y$.

(a) At first, we choose $T_m > 0$, $M > 0$ such that U maps $\overline{S}$ into itself. Notice that since $\Psi(t) \in C^0([0,T_m];\mathbb{R}^m), \widetilde{\Psi}(t) \in Y$, so $A[c,c'](t), D[c,c'](t) \in Y$, for all $c \in Y$. Then, by (2.18), we have

$$\begin{gathered}
(Uc)_i'(t) = -B^{-1}D_i'[c,c'](t) + B^{-1}\widetilde{\Psi}_i'(t), \quad 1 \le i \le m, \\
D_i'[c,c'](t) = -\lambda e^{-\lambda t}\int_0^t A_i[c,c'](r)dr + e^{-\lambda t}A_i[c,c'](t), \\
\widetilde{\Psi}_i'(t) = -\lambda e^{-\lambda t}\left[\sum_{j=1}^{m} b_{ij}\alpha_j + \int_0^t \Psi_i(r)dr\right] + e^{-\lambda t}\Psi_i(t).
\end{gathered} \tag{2.20}$$

This implies that $U : Y \to Y$. Let $c \in \overline{S}$. We deduce from (2.18) and (2.20) that

$$\begin{gathered}
|(Uc)(t)|_1 \le \|B^{-1}\|[|D[c,c'](t)|_1 + |\widetilde{\Psi}(t)|_1], \\
|(Uc)'(t)|_1 \le \|B^{-1}\|[|D'[c,c'](t)|_1 + |\widetilde{\Psi}'(t)|_1], \\
|D[c,c'](t)|_1 \le \int_0^t |A[c,c'](r)|_1 dr \le T_m\|A[c,c']\|_0, \\
|D'[c,c'](t)|_1 \le (\lambda T_m + 1)\|A[c,c']\|_0.
\end{gathered} \tag{2.21}$$

On the other hand, by (2.12),

$$\sum_{i=1}^{m} |A_i[c,c'](t)| \le e^{\lambda t}\left[\widetilde{\mu}\int_0^t |c(s)|_1 ds + \widetilde{w}\int_0^t g(s)ds + T_m\sum_{i=1}^{m} N_{1i}(M) + N_{2i}(M)\right], \tag{2.22}$$

where

$$\begin{gathered}
\widetilde{\mu} = \max\left\{\sum_{j=1}^{m}\mu_{ij},\ 1 \le i \le m\right\}, \qquad \widetilde{w} = \sum_{i=1}^{m} w_i(0), \\
N_{1i}(M) = \|k\|_{L^1(0,T)}\sup\left\{\left|w_i(0)\sum_{j=1}^{m} y_j w_j(0)\right|, \|y\|_{\mathbb{R}^m} \le M\right\}, \quad 1 \le i \le m, \\
N_{2i}(M) = \sup\{|w_i(0)\widetilde{P}[y,z]|, \|y\|_{\mathbb{R}^m} \le M, \|z\|_{\mathbb{R}^m} \le M\}, \quad 1 \le i \le m.
\end{gathered} \tag{2.23}$$

This implies that

$$\begin{aligned} ||A[c,c']||_0 &\leq e^{\lambda T_m}\left[\widetilde{\mu}T_mM + T_m\widetilde{w}\|g\|_{L^\infty(0,T)} + T_m\sum_{i=1}^{m}N_{1i}(M)+N_{2i}(M)\right] \\ &\leq T_m e^{\lambda T_m}\left[\widetilde{\mu}M + \widetilde{w}\|g\|_{L^\infty(0,T)} + \sum_{i=1}^{m}N_{1i}(M)+N_{2i}(M)\right] \\ &\equiv T_m e^{\lambda T_m}\xi(M,T). \end{aligned} \tag{2.24}$$

Combining (2.21), (2.24), we get

$$\begin{aligned} \|Uc\|_0 &\leq ||B^{-1}||[T_m^2 e^{\lambda T_m}\xi(M,T) + \|\widetilde{\Psi}\|_{0*}], \\ ||(Uc)'||_0 &\leq ||B^{-1}||[(\lambda T_m+1)T_m e^{\lambda T_m}\xi(M,T) + \|\widetilde{\Psi}'\|_{0*}], \end{aligned} \tag{2.25}$$

so

$$\|Uc\|_1 \leq [(\lambda+1)T_m+1]T_m e^{\lambda T_m}\xi(M,T)||B^{-1}|| + \|\widetilde{\Psi}\|_{1*}||B^{-1}||, \tag{2.26}$$

where

$$\|\widetilde{\Psi}\|_{1*} = \|\widetilde{\Psi}\|_{0*} + \|\widetilde{\Psi}'\|_{0*} = \sup_{0\leq t\leq T}|\widetilde{\Psi}(t)|_1 + \sup_{0\leq t\leq T}|\widetilde{\Psi}'(t)|_1. \tag{2.27}$$

Choosing $M>0$ and $T_m>0$ such that

$$M > 2\|\widetilde{\Psi}\|_{1*}||B^{-1}||, \quad [(\lambda+1)T_m+1]T_m e^{\lambda T_m}\xi(M,T)||B^{-1}|| \leq \frac{1}{2}M. \tag{2.28}$$

Hence, $\|Uc\|_1 < M$ for all $c \in S$, this means that $U\overline{S} \subset \overline{S}$.

(b) Next, we show that $U : \overline{S} \to Y$ is continuous. Let $c, d \in \overline{S}$, we have

$$(Uc)_i(t) - (Ud)_i(t) = -B^{-1}[D_i[c,c'](t) - D_i[d,d'](t)], \quad 1 \leq i \leq m, \tag{2.29}$$

$$D_i[c,c'](t) - D_i[d,d'](t) = e^{-\lambda t}\int_0^t [A_i[c,c'](r) - A_i[d,d'](r)]dr, \quad 1 \leq i \leq m, \tag{2.30}$$

$$\begin{aligned} A_i[c,c'](t) - A_i[d,d'](t) &= e^{\lambda t}\sum_{j=1}^{m}\mu_{ij}\int_0^t [(c_j-d_j)(s)]ds \\ &\quad + w_i(0)e^{\lambda t}\int_0^t [P[c,c'](s) - P[d,d'](s)]ds, \end{aligned} \tag{2.31}$$

$$P[c,c'](t) - P[d,d'](t) = -\int_0^t k(t-s)\sum_{j=1}^{m}[c_j-d_j](s)w_j(0)ds + \widetilde{P}[c,c'](t) - \widetilde{P}[d,d'](t). \tag{2.32}$$

Using the inequality $\|x|^{p-1}x-|y|^{p-1}y|\le|x-y|^{p}$, for all $x,y\in\mathbb{R}$, and for all $p\in(0,1)$, we have

$$\widetilde{P}[c,c'](t)-\widetilde{P}[d,d'](t)\le K_1\left|\sum_{j=1}^{m}(c_j-d_j)(t)w_j(0)\right|^{\alpha-1}+\lambda_1\left|\sum_{j=1}^{m}(c'_j-d'_j)(t)w_j(0)\right|^{\beta-1}. \tag{2.33}$$

Thus, there exists a constant $\rho_1=\rho_1(k,\widetilde{w},K_1,\lambda_1)>0$ such that

$$\sup_{0\le t\le T_m}|P[c,c'](t)-P[d,d'](t)|\le\rho_1\left(\|c-d\|_0+\|c-d\|_0^{\alpha-1}+\|c'-d'\|_0^{\beta-1}\right), \tag{2.34}$$

so

$$\begin{aligned}&\|A[c,c']-A[d,d']\|_0\\&\quad\le e^{\lambda T_m}T_m\widetilde{\mu}\|c-d\|_0+e^{\lambda T_m}T_m\widetilde{w}\rho_1\left(\|c-d\|_0+\|c-d\|_0^{\alpha-1}+\|c'-d'\|_0^{\beta-1}\right),\end{aligned} \tag{2.35}$$

then

$$\begin{aligned}&\|Uc-Ud\|_0\\&\quad\le e^{\lambda T_m}T_m^2\widetilde{\mu}\|B^{-1}\|\|c-d\|_0+e^{\lambda T_m}T_m^2\widetilde{w}\rho_1\|B^{-1}\|\left(\|c-d\|_0+\|c-d\|_0^{\alpha-1}+\|c'-d'\|_0^{\beta-1}\right).\end{aligned} \tag{2.36}$$

Similarly, we also obtain from the equalities

$$\begin{aligned}(Uc)'_i(t)-(Ud)'_i(t)&=-B^{-1}[D'_i[c,c'](t)-D'_i[d,d'](t)],\\D'_i[c,c'](t)-D'_i[d,d'](t)&=-\lambda e^{-\lambda t}\int_0^t[A_i[c,c'](r)-A_i[d,d'](r)]dr\\&\quad+e^{-\lambda t}[A_i[c,c'](t)-A_i[d,d'](t)]\end{aligned} \tag{2.37}$$

that

$$\begin{aligned}&\|(Uc)'-(Ud)'\|_0\\&\quad\le\|B^{-1}\|(\lambda T_m+1)e^{\lambda T_m}T_m\widetilde{\mu}\|c-d\|_0\\&\qquad+\|B^{-1}\|(\lambda T_m+1)e^{\lambda T_m}T_m\widetilde{w}\rho_1\left(\|c-d\|_0+\|c-d\|_0^{\alpha-1}+\|c'-d'\|_0^{\beta-1}\right).\end{aligned} \tag{2.38}$$

Thus, the estimates (2.36) and (2.38) show that $U:\overline{S}\to Y$ is continuous.

(c) Now, we show that $U(\overline{S})$ is relatively compact in Y. Let $c \in \overline{S}$, $t,t' \in [0,T_m]$. From (2.18), we have

$$\begin{aligned}&(Uc)_i(t) - (Uc)_i(t') \\ &\quad = -B^{-1}\big[D_i[c,c'](t) - D_i[c,c'](t')\big] + B^{-1}\big[\widetilde{\Psi}_i(t) - \widetilde{\Psi}_i(t')\big], \quad 1 \le i \le m,\end{aligned} \tag{2.39}$$

$$\begin{aligned}&D_i[c,c'](t) - D_i[c,c'](t') \\ &\quad = \big(e^{-\lambda t} - e^{-\lambda t'}\big)\int_0^t A_i[c,c'](r)dr - e^{-\lambda t'}\int_t^{t'} A_i[c,c'](r)dr \\ &\quad = \big(-\lambda e^{-\lambda t_0}\big)(t-t')\int_0^t A_i[c,c'](r)dr - e^{-\lambda t'}\int_t^{t'} A_i[c,c'](r)dr, \quad 1 \le i \le m,\end{aligned} \tag{2.40}$$

$$\begin{aligned}\widetilde{\Psi}_i(t) - \widetilde{\Psi}_i(t') = {}& \big(-\lambda e^{-\lambda t_0}\big)(t-t')\sum_{j=1}^m b_{ij}\alpha_j \\ &+ \big(-\lambda e^{-\lambda t_0}\big)(t-t')\int_0^t \Psi_i(r)dr - e^{-\lambda t'}\int_t^{t'} \Psi_i(r)dr, \quad 1 \le i \le m,\end{aligned} \tag{2.41}$$

where $t_0 \in (t,t')$ or $t_0 \in (t',t)$. Then,

$$\begin{aligned}\big|D[c,c'](t) - D[c,c'](t')\big|_1 &\le (\lambda T_m + 1)\big\|A[c,c']\big\|_0|t-t'|, \\ \big|\widetilde{\Psi}(t) - \widetilde{\Psi}(t')\big|_1 &\le \sum_{i=1}^m\sum_{j=1}^m \big|b_{ij}\alpha_j\big|\lambda|t-t'| + (\lambda T_m + 1)\|\Psi\|_{0*}|t-t'|,\end{aligned} \tag{2.42}$$

where

$$\|\Psi\|_{0*} = \sup_{0\le t\le T}\big|\Psi(t)\big|_1. \tag{2.43}$$

From (2.39) and (2.42), we deduce that there exists a constant θ_1 independent of c, t, t' such that

$$\big|Uc(t) - Uc(t')\big|_1 \le \theta_1|t-t'|. \tag{2.44}$$

Similarly, by (2.20), there exists a constant θ_2 independent of c, t, t' such that

$$\big|(Uc)'(t) - (Uc)'(t')\big|_1 \le \theta_2|t-t'|. \tag{2.45}$$

It follows from $U\overline{S} \subset \overline{S}$ and the estimates (2.44), (2.45) that the set $U\overline{S}$ is uniformly bounded and equicontinuous in the Banach space Y with respect to the norm $\|\cdot\|_1$. Applying Arezela-Ascoli theorem, $U\overline{S}$ is relatively compact in Y.

By the Schauder fixed point theorem, U has a fixed point $c = (c_1,\dots,c_m) \in \overline{S}$. This implies that the system (2.6)–(2.8) has a solution $(u_m(t),P_m(t))$ with $u_m(t) = \sum_{j=1}^m c_{mj}(t)w_j$ on $[0,T_m]$. By the same technique as in the proof of [5, Theorem 1], we can prove the following steps, let us omit here.

Step 2 (a priori estimates). These estimates allow one to take $T_m = T$ for all m.

Step 3 (limiting process). There exists a subsequence of sequences $\{(u_m,P_m)\}$, still denoted by $\{(u_m,P_m)\}$, such that $\{(u_m,P_m)\}$ converges to (u,P) which is the weak solution of the problem. Theorem 2.2 is proved. □

3. Structure of weak solution set

In this section, we only consider the set of weak solutions which are obtained by the Galerkin method as above. Notations and norms will be the same as in Section 2.

THEOREM 3.1. *Let (H*$_1$*)–(H*$_5$*) hold. Let* $T>0$*. Then the set of weak solutions* (u,P) *to the problem (1.1)-(1.2) such that*

$$u\in L^{\infty}(0,T;V),\qquad u_t\in L^{\infty}(0,T;L^2),$$
$$u(0,\cdot)\in W^{1,\beta}(0,T),\quad P\in L^{\beta'}(0,T),\quad \beta'=\frac{\beta}{\beta-1},\tag{3.1}$$

is compact and connected.

Proof of the Theorem 3.1. The proof consists of Steps 4–6.

Step 4. The set of fixed points c of the operator $U:\overline{S}\to Y$ is nonempty, compact, and connected, where $\overline{S}=\{c\in C^1([0,T_m];\mathbb{R}^m):\|c\|_1\le M\}$ defined as in Theorem 2.2. Clearly, $\overline{S}$ is the closure of the open convex, and the bounded subset

$$S=\{c\in C^1([0,T_m];\mathbb{R}^m):\|c\|_1<M\},\tag{3.2}$$

with $M>0$, $T_m>0$ will be chosen later. □

Proof of Step 4. (i) At first, we have

$$\widetilde{P}:\mathbb{R}^{2m}\longrightarrow\mathbb{R},$$
$$\widetilde{P}[y,z]=K_1H_{\alpha}\left(\sum_{j=1}^{m}y_jw_j(0)\right)+\lambda_1H_{\beta}\left(\sum_{j=1}^{m}z_jw_j(0)\right)\tag{3.3}$$

is continuous, so for all $\varepsilon>0$, by Theorem 1.2, there exists a mapping $\widetilde{P}_{\varepsilon}:\mathbb{R}^{2m}\to\mathbb{R}$ that is locally a Lipschitz approximation of $\widetilde{P}$ such that

$$\left|\widetilde{P}[y,z]-\widetilde{P}_{\varepsilon}[y,z]\right|<\frac{\varepsilon}{\eta},\quad\forall y,z\in\mathbb{R}^m,\tag{3.4}$$

where $\eta>0$ may be chosen such that ε/η is as small as we wish. Clearly, $\widetilde{P}_{\varepsilon}$ is continuous.

(ii) Next, we define the following operators.

Let $U_\varepsilon : \overline{S} \to Y$ be defined as follows: for every $c \in \overline{S}$, $1 \le i \le m$,

$$\begin{aligned}
&(U_\varepsilon c)_i(t) = -B^{-1}D_{\varepsilon i}[c,c'](t) + B^{-1}\widetilde{\Psi}_i(t),\\
&D_{\varepsilon i}[c,c'](t) = e^{-\lambda t}\int_0^t A_{\varepsilon i}[c,c'](r)dr,\\
&A_{\varepsilon i}[c,c'](t) = e^{\lambda t}\left[\sum_{j=1}^m \mu_{ij}\int_0^t c_j(s)ds + w_i(0)\int_0^t P_\varepsilon[c,c'](s)ds\right],\\
&P_\varepsilon[c,c'](t) = g(t) - \int_0^t k(t-s)\sum_{j=1}^m c_j(s)w_j(0)ds + \widetilde{P}_\varepsilon[c,c'](t).
\end{aligned} \tag{3.5}$$

For every $0 \le \zeta \le 1$, let

$$V_\zeta : \overline{S} \longrightarrow Y \qquad c \longmapsto V_\zeta(c) \tag{3.6}$$

be defined by

$$(V_\zeta c)_i(t) = -\zeta B^{-1}D_i[c,c'](t) + B^{-1}\widetilde{\Psi}_i(t), \quad 1 \le i \le m. \tag{3.7}$$

We put

$$\begin{aligned}
N_{2i}^\varepsilon(M) &= \sup\left\{\left|w_i(0)\widetilde{P}_\varepsilon[y,z]\right|,\ \|y\|_{\mathbb{R}^m} \le M,\ \|z\|_{\mathbb{R}^m} \le M\right\}, \quad 1 \le i \le m,\\
\xi^\varepsilon(M,T) &= \widetilde{\mu}M + \widetilde{w}\|g\|_{L^\infty(0,T)} + \sum_{i=1}^m N_{1i}(M) + N_{2i}^\varepsilon(M), \quad 1 \le i \le m,
\end{aligned} \tag{3.8}$$

and choose $M > 0$, $T_m > 0$ such that

$$\begin{gathered}
M > 2\|\widetilde{\Psi}\|_{1*}\|B^{-1}\|, \quad T_m[(\lambda+1)T_m+1]e^{\lambda T_m}\xi(M,T)\|B^{-1}\| \le \frac{1}{2}M,\\
T_m[(\lambda+1)T_m+1]e^{\lambda T_m}\xi^\varepsilon(M,T)\|B^{-1}\| \le \frac{1}{2}M.
\end{gathered} \tag{3.9}$$

Analysis similar to that in the proof of Theorem 2.2 (Step 1) shows that U, $V_\zeta : \overline{S} \to Y$ are completely continuous. Furthermore, since

$$\|Uc\|_1 < M, \quad \|V_\zeta c\|_1 < M, \quad \forall c \in \overline{S},\ \forall \zeta \in [0,1], \tag{3.10}$$

the operators U, V_ζ ($V_\zeta = U$ when $\zeta = 1$) have no fixed points on ∂S. This means that

$$0 \notin (I - V_\zeta)\partial S. \tag{3.11}$$

We can show that $U_\varepsilon : \overline{S} \to Y$ is completely continuous. Let us only give the main ideas of the proof as follows.

(a) Replacing $\widetilde{P}$ by $\widetilde{P}_\varepsilon$ in the definition of operator $U : \overline{S} \to Y$, we obtain the operator $U_\varepsilon : \overline{S} \to Y$. The mapping $\widetilde{P}_\varepsilon$ is continuous, so we can define $\xi^\varepsilon(M,T)$ as in (3.8). By that and by M, T_m are chosen as above, the operator U_ε maps $\overline{S}$ into itself.

(b) For each $c \in \overline{S}$, for any $\widetilde{\varepsilon} > 0$, there exists $\delta > 0$ such that for all d in $\overline{S}$,

$$\|c-d\|_1 < \delta \Longrightarrow \left\|U_\varepsilon c - U_\varepsilon d\right\|_1 < \widetilde{\varepsilon}. \tag{3.12}$$

Indeed, since $\widetilde{P}_\varepsilon : \mathbb{R}^{2m} \to \mathbb{R}$ is continuous, $\widetilde{P}_\varepsilon$ is continuous uniformly in the set $[-2M, 2M]^{2m}$. It follows that, with $\widetilde{\varepsilon} > 0$ as above, there exists $\delta_1 > 0$ such that for all (y_1, z_1), $(y_2, z_2) \in [-2M, 2M]^{2m}$,

$$\begin{matrix} \left\|y_1 - y_2\right\|_{\mathbb{R}^m} < \delta_1 \\ \left\|z_1 - z_2\right\|_{\mathbb{R}^m} < \delta_1 \end{matrix} \Longrightarrow \left|\widetilde{P}_\varepsilon[y_1, z_1] - \widetilde{P}_\varepsilon[y_2, z_2]\right| < \frac{\widetilde{\varepsilon}}{3\widetilde{\eta}}, \tag{3.13}$$

where $\widetilde{\eta} > 0$ is large enough. Therefore, if $\delta > 0$ is small enough (only depending on δ_1), then

$$\left|P_\varepsilon[c, c'](t) - P_\varepsilon[d, d'](t)\right| \le \rho_2 \|c-d\|_0 + \frac{\widetilde{\varepsilon}}{6\widetilde{\eta}}, \tag{3.14}$$

for all $t \in [0, T_m]$, where $\rho_2 = \rho_2(k, \widetilde{w})$ is a positive constant. Thus,

$$\left\|A_\varepsilon[c, c'] - A_\varepsilon[d, d']\right\|_0 \le e^{\lambda T_m} T_m \widetilde{\mu} \|c-d\|_0 + e^{\lambda T_m} T_m \widetilde{w}\left(\rho_2 \|c-d\|_0 + \frac{\widetilde{\varepsilon}}{6\widetilde{\eta}}\right), \tag{3.15}$$

so

$$\left\|U_\varepsilon c - U_\varepsilon d\right\|_0 \le e^{\lambda T_m} T_m^2 \widetilde{\mu} \left\|B^{-1}\right\| \|c-d\|_0 + e^{\lambda T_m} T_m^2 \widetilde{w} \left\|B^{-1}\right\| \left(\rho_2 \|c-d\|_0 + \frac{\widetilde{\varepsilon}}{6\widetilde{\eta}}\right). \tag{3.16}$$

Similarly, we get

$$\begin{aligned} \left\|(U_\varepsilon c)' - (U_\varepsilon d)'\right\|_0 \le{} & \left\|B^{-1}\right\| (\lambda T_m + 1) e^{\lambda T_m} T_m \widetilde{\mu} \|c-d\|_0 \\ & + \left\|B^{-1}\right\| (\lambda T_m + 1) e^{\lambda T_m} T_m \widetilde{w} \left(\rho_2 \|c-d\|_0 + \frac{\widetilde{\varepsilon}}{6\widetilde{\eta}}\right). \end{aligned} \tag{3.17}$$

By $\widetilde{\eta}$ is large enough, the estimates (3.16) and (3.17) show that (3.12) holds.

Hence, $U_\varepsilon : \overline{S} \to Y$ is continuous.

(c) A similar argument to the one above for $U(\overline{S})$ can be used to show that $U_\varepsilon(\overline{S})$ is bounded and equicontinuous with respect to the norm $\|\cdot\|_1$. Then $U_\varepsilon : \overline{S} \to Y$ is completely continuous. Furthermore, by (2.29)–(2.31), (3.5), for all $c \in \overline{S}$, $1 \le i \le m$, we have

$$\begin{aligned} (Uc)_i(t) - (U_\varepsilon c)_i(t) &= -B^{-1}\left[D_i[c, c'](t) - D_{\varepsilon i}[c, c'](t)\right], \\ D_i[c, c'](t) - D_{\varepsilon i}[c, c'](t) &= e^{-\lambda t} \int_0^t \left[A_i[c, c'](r) - A_{\varepsilon i}[c, c'](r)\right] dr, \\ A_i[c, c'](t) - A_{\varepsilon i}[c, c'](t) &= w_i(0) e^{\lambda t} \int_0^t \left[P[c, c'](s) - P_\varepsilon[c, c'](s)\right] ds, \\ P[c, c'](t) - P_\varepsilon[c, c'](t) &= \widetilde{P}[c, c'](t) - \widetilde{P}_\varepsilon[c, c'](t), \end{aligned} \tag{3.18}$$

consequently

$$\left| Uc(t) - U_\varepsilon c(t) \right|_1 \leq \|B^{-1}\| T_m^2 \widetilde{w} e^{\lambda T_m} \frac{\varepsilon}{\eta}, \tag{3.19}$$

this implies that

$$\|Uc - U_\varepsilon c\|_0 < \frac{\varepsilon}{2}, \tag{3.20}$$

when η is large enough.

Similarly, we deduce from (2.37) and (3.5) that

$$\|(Uc)' - (U_\varepsilon c)'\|_0 < \frac{\varepsilon}{2}, \quad \forall c \in \overline{S}. \tag{3.21}$$

From (3.20), (3.21), we have

$$\|Uc - U_\varepsilon c\|_1 < \varepsilon, \quad \forall c \in \overline{S}. \tag{3.22}$$

(iii) We now prove that for each h with $\|h\|_1 < \varepsilon$, the equation

$$c = U_\varepsilon c + h \tag{3.23}$$

has at most one solution on $\overline{S}$. The proof is as follows.

Let $c = (c_1, \ldots, c_m)$, $d = (d_1, \ldots, d_m)$ be two solutions of (3.23). We need to prove that

$$c(t) = d(t) \quad \forall t \in [0, T_m]. \tag{3.24}$$

It is easy to see that

$$c(0) = d(0) = B^{-1}\widetilde{\Psi}_i(0) + h(0). \tag{3.25}$$

Put

$$\gamma = \sup\{\sigma \in [0, T_m] : c(t) = d(t)\ \forall t \in [0, \sigma]\}. \tag{3.26}$$

Clearly, by (3.25), $\gamma \geq 0$. Then $0 \leq \gamma \leq T_m$. In order to get $\gamma = T_m$, we suppose by contradiction that $\gamma < T_m$. For all $i = \overline{1, m}$, for all $t \in [0, T_m]$, we have

$$\begin{aligned}
c_i(t) - d_i(t) &= (U_\varepsilon c)_i(t) - (U_\varepsilon d)_i(t) = -B^{-1}[D_{\varepsilon i}[c, c'](t) - D_{\varepsilon i}[d, d'](t)], \\
D_{\varepsilon i}[c, c'](t) - D_{\varepsilon i}[d, d'](t) &= e^{-\lambda t} \int_0^t [A_{\varepsilon i}[c, c'](r) - A_{\varepsilon i}[d, d'](r)]\,dr, \\
c_i'(t) - d_i'(t) &= (U_\varepsilon c)_i'(t) - (U_\varepsilon d)_i'(t) = -B^{-1}[D_{\varepsilon i}'[c, c'](t) - D_{\varepsilon i}'[d, d'](t)], \\
D_{\varepsilon i}'[c, c'](t) - D_{\varepsilon i}'[d, d'](t) &= -\lambda e^{-\lambda t} \int_0^t [A_{\varepsilon i}[c, c'](r) - A_{\varepsilon i}[d, d'](r)]\,dr \\
&\quad + e^{-\lambda t}[A_{\varepsilon i}[c, c'](t) - A_{\varepsilon i}[d, d'](t)].
\end{aligned} \tag{3.27}$$

Hence, for all $t \in [0, T_m]$,

$$|c(t) - d(t)|_1 + |c'(t) - d'(t)|_1 \leq \|B^{-1}\|(1+\lambda)\int_0^t |A_\varepsilon[c,c'](r) - A_\varepsilon[d,d'](r)|_1 dr \\ + |A_\varepsilon[c,c'](t) - A_\varepsilon[d,d'](t)|_1, \tag{3.28}$$

in which

$$\begin{aligned} &|A_\varepsilon[c,c'](t) - A_\varepsilon[d,d'](t)|_1 \\ &\qquad \leq e^{\lambda t}\widetilde{\mu}\int_0^t |c(s) - d(s)|_1 ds + e^{\lambda T_m}\widetilde{w}\int_0^t |P_\varepsilon[c,c'](s) - P_\varepsilon[d,d'](s)| ds, \\ &|P_\varepsilon[c,c'](t) - P_\varepsilon[d,d'](t)| \\ &\qquad \leq \int_0^t \left| k(t-s)\sum_{j=1}^m [c_j(s) - d_j(s)]w_j(0)\right| ds + |\widetilde{P}_\varepsilon[c,c'](t) - \widetilde{P}_\varepsilon[d,d'](t)| \\ &\qquad \leq \|k\|_{L^\infty(0,T)}\widetilde{w}\int_0^t |c(s) - d(s)|_1 ds + |\widetilde{P}_\varepsilon[c,c'](t) - \widetilde{P}_\varepsilon[d,d'](t)|. \end{aligned} \tag{3.29}$$

Since $\widetilde{P}_\varepsilon : \mathbb{R}^{2m} \to \mathbb{R}$ is locally Lipschitz, with $c(\gamma) = d(\gamma) \in \mathbb{R}^m$ and $c'(\gamma) = d'(\gamma) \in \mathbb{R}^m$, there exist a ball $\widetilde{B}$ in $\mathbb{R}^{2m}$ of radius $r > 0$ centered at $(c(\gamma), c'(\gamma))$ and $L > 0$ such that

$$|\widetilde{P}_\varepsilon[y_1,z_1] - \widetilde{P}_\varepsilon[y_2,z_2]| \leq L(|y_1 - y_2|_1 + |z_1 - z_2|_1), \\ \forall (y_1,z_1),(y_2,z_2) \in \widetilde{B}. \tag{3.30}$$

On the other hand, the functions $c(t)$, $d(t)$, $c'(t)$, $d'(t)$ are continuous on $[0, T_m]$, and so are they at γ. Then, with $r > 0$ as above, there exists $\delta' > 0$ such that $(c(t), c'(t))$ and $(d(t), d'(t))$ belong to $\widetilde{B}$, for all $t \in [\gamma, \gamma + \delta'] \subset [0, T_m]$. This means that

$$|\widetilde{P}_\varepsilon[c,c'](t) - \widetilde{P}_\varepsilon[d,d'](t)| \leq L\phi(t), \quad \forall t \in [\gamma, \gamma + \delta'] \subset [0, T_m], \tag{3.31}$$

where

$$\phi(t) = |c(t) - d(t)|_1 + |c'(t) - d'(t)|_1. \tag{3.32}$$

Combining (3.29)–(3.31), note that $c(t) = d(t)$ for all $t \in [0, \gamma]$, we deduce that there exists a positive constant $K_1(T_m)$ (only depending on T_m) such that

$$\begin{aligned}
&\left|A_\varepsilon[c,c'](t)-A_\varepsilon[d,d'](t)\right|_1\\
&\quad\le e^{\lambda T_m}\widetilde{\mu}\int_0^t \left|c(s)-d(s)\right|_1 ds\\
&\quad\quad+e^{\lambda T_m}\widetilde{w}\int_0^t \left|\widetilde{P}_\varepsilon[c,c'](s)-\widetilde{P}_\varepsilon[d,d'](s)\right| ds\\
&\quad\quad+e^{\lambda T_m}\widetilde{w}^2\|k\|_{L^\infty(0,T)}\int_0^t d\tau\int_0^\tau \left|c(s)-d(s)\right|_1 ds\le e^{\lambda T_m}\widetilde{\mu}\int_0^t \phi(s)ds\\
&\quad\quad+e^{\lambda T_m}\widetilde{w}L\int_0^t\phi(s)ds+e^{\lambda T_m}\widetilde{w}^2\|k\|_{L^\infty(0,T)}T_m\int_0^t\phi(s)ds\\
&\quad\le K_1\left(T_m\right)\int_0^t\phi(s)ds,\quad \forall t\in[0,\gamma+\delta'].
\end{aligned}\tag{3.33}$$

By (3.28), (3.31), for all $t\in[0,\gamma+\delta']$, we have

$$\begin{aligned}
\phi(t)&\le\|B^{-1}\|(1+\lambda)K_1\left(T_m\right)\int_0^t d\tau\int_0^\tau\phi(s)ds+K_1\left(T_m\right)\int_0^t\phi(s)ds\\
&\le\|B^{-1}\|(1+\lambda)T_mK_1\left(T_m\right)\int_0^t\phi(s)ds+K_1\left(T_m\right)\int_0^t\phi(s)ds\\
&\le K_2\left(T_m\right)\int_0^t\phi(s)ds,
\end{aligned}\tag{3.34}$$

where $K_2(T_m)=\|B^{-1}\|(1+\lambda)T_mK_1(T_m)+K_1(T_m)$. Applying the Gronwall lemma, we have $\phi(t)=0$ for all $t\in[0,\gamma+\delta']$. Thus

$$c(t)=d(t),\quad\forall t\in[0,\gamma+\delta'].\tag{3.35}$$

This gives a contradiction with choosing γ as (3.26). So (3.24) follows.

(iv) Finally, it remains to show that

$$\deg\left(I-U,S,0\right)\neq 0.\tag{3.36}$$

The family of the compact operators V_ζ satisfies condition (3.11), applying the homotopy invariance property of the degree, we obtain that $\deg(I-V_\zeta,S,0)$ does not depend on λ. This implies that

$$\deg\left(I-V_1,S,0\right)=\deg\left(I-V_0,S,0\right),\tag{3.37}$$

where $I-V_1=I-U$ and V_0 is the constant mapping, since

$$\left(V_0c\right)_i(t)=B^{-1}\widetilde{\Psi}_i(t),\quad\forall c\in\overline{S},\ \forall t\in\left[0,T_m\right],\ \forall i=\overline{1,m}.\tag{3.38}$$

Thus $\deg(I-U,S,0)=\deg(I-V_0,S,0)=1$. This shows that (3.36) holds.

Combining (3.11), (3.22), (3.23), (3.36), and applying Theorem 1.1, Step 4 is proved.

Step 5. The set of the solutions (u_m, P_m) is nonempty, compact, and connected.

We obtain this result since the mapping in which for each $c_m = (c_{m1}, \dots, c_{mm})$ is corresponding with u_m such that $u_m(t) = \sum_{j=1}^{m} c_{mj}(t) w_j$ is continuous.

Step 6. The set of the weak solutions (u, P) which exist by passing the solutions (u_m, P_m) to the limit is nonempty, compact, and connected.

We also have this result since the mapping in which for each (u_m, P_m) is corresponding with the weak solution (u, P) is continuous.

Theorem 3.1 is proved completely. □

Remark 3.2. By the definition of the operator

$$\begin{gathered} \widetilde{P} : \mathbb{R}^{2m} \longrightarrow \mathbb{R}, \\ \widetilde{P}[y, z] = K_1 H_\alpha\left(\sum_{j=1}^{m} y_j w_j(0) \right) + \lambda_1 H_\beta\left(\sum_{j=1}^{m} z_j w_j(0) \right), \end{gathered} \tag{3.39}$$

and the inequalities

$$\begin{gathered} \left| H_\beta(x) - H_\beta(y) \right| \le |x - y|^{\beta - 1}, \quad \forall x,\ y \in \mathbb{R},\ \forall \beta \in (1, 2), \\ \left| H_\alpha(x) - H_\alpha(y) \right| \le (\alpha - 1) R^{\alpha - 2} |x - y|, \quad \forall x, y \in [-R, R],\ \forall R > 0,\ \forall \alpha \ge 2, \end{gathered} \tag{3.40}$$

in the same manner, it may be concluded that Theorem 3.1 also holds in the cases $1 < \alpha < 2$, $\beta \ge 2$ or $1 < \beta < 2$, $\alpha \ge 2$.

Acknowledgments

The authors wish to express their sincere thanks to Professor Jean Mawhin and the referees for their helpful suggestions and comments.

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