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Research Article

Existence Results for Polyharmonic Boundary Value Problems in the Unit Ball

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Here we study the polyharmonic nonlinear elliptic boundary value problem on the unit ball B in $\mathbb{R}^n$ $(n \geq 2)$ $(-\triangle)^m u + g(\cdot, u) = 0$, in B (in the sense of distributions) $\lim_{x \to \xi \in \partial B}(u(x)/(1-|x|^2)^{m-1}) = 0(\xi)$. Under appropriate conditions related to a Kato class on the nonlinearity $g(x,t)$, we give some existence results. Our approach is based on estimates for the polyharmonic Green function on B with zero Dirichlet boundary conditions, including a 3G-theorem, which leeds to some useful properties on functions belonging to the Kato class.

1. Introduction

In this paper, we deal with higher-order elliptic Dirichlet problems

$$
\begin{gathered}
(-\triangle)^m u + g(\cdot, u) = 0, \quad \text{in } B \text{ (in the sense of distributions)}, \\
\lim_{x \to \xi \in \partial B} \frac{u(x)}{(1-|x|^2)^{m-1}} = \theta(\xi),
\end{gathered}
\tag{1.1}
$$

where B is the unit ball in $\mathbb{R}^n$ $(n \geq 2)$, m is a positive integer, and θ is a nontrivial nonnegative continuous function on ∂B.

A basic result goes from Boggio in [1], where he gave an explicit formula for the Green function $G_{m,n}$ of $(-\triangle)^m$ on B with Dirichlet boundary conditions $(\partial/\partial \nu)^j u = 0$,

$0 \leq j \leq m-1$. Namely, Boggio showed that the Green function is positive and is given by

$$G_{m,n}(x,y) = k_{m,n}|x-y|^{2m-n} \int_1^{[x,y]/|x-y|} \frac{(v^2-1)^{m-1}}{v^{n-1}} dv, \tag{1.2}$$

with $k_{m,n}$ is a positive constant and $[x,y]^2 = |x-y|^2 + (1-|x|^2)(1-|y|^2)$, for x, y in B.

The positivity of $G_{m,n}$ does not hold for the Green function of the m-polyharmonic operator in an arbitrary bounded domain (see, e.g., [2]). Only for the case $m = 1$, we do not have this restriction.

In [3], Grunau and Sweers established two-sided estimates for $G_{m,n}$ and so they derived a 3G-theorem result. This was improved in [4], where the authors obtained from Boggio's formula more fine estimates on $G_{m,n}$. For instance, they gave a new form of the 3G-theorem: There exists $C_0 > 0$ such that for each $x, y, z \in B$,

$$\frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} \leq C_0\left[\left(\frac{\delta(z)}{\delta(x)}\right)^m G_{m,n}(x,z) + \left(\frac{\delta(z)}{\delta(y)}\right)^m G_{m,n}(y,z)\right], \tag{1.3}$$

where $\delta(x) = 1 - |x|$. In the case $m = 1$, the Green function G_Ω of an arbitrary bounded $C^{1,1}$ domain Ω satisfies (1.3). This has been proved by Kalton and Verbitsky [5] for $n \geq 3$ and by Selmi [6] for $n = 2$.

On the other hand, the classical 3G-theorem related to G_Ω (see [7, 8]) has been exploited to introduce the classical Kato class of functions $K_n(\Omega)$ (see [9, 7]), which was widely used in the study of some nonlinear differential equations (see [10, 11]). Similarly, in [4] the authors exploited the inequality (1.3) to introduce a new Kato class $K := K_{m,n}(B)$ (see Definition 1.1), such that $K_{1,n}(B)$ contains properly $K_n(B)$.

Definition 1.1. A Borel measurable function φ in B belongs to the class $K := K_{m,n}(B)$ if φ satisfies the following condition:

$$\lim_{\alpha \to 0}\left(\sup_{x \in B} \int_{B \cap B(x,\alpha)} \left(\frac{\delta(y)}{\delta(x)}\right)^m H_{m,n}(x,y)\,|\varphi(y)|\,dy\right) = 0. \tag{1.4}$$

In this paper, we will use properties of this class to investigate two existence results for problem (1.1). Our plan is as follows. In Section 2, we collect some properties of functions belonging to K. In particular, we derive from the 3G-theorem that for each $q \in K$, we have

$$\alpha_q := \sup_{x,y \in B} \int_B \frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)}\,|q(z)|\,dz < \infty. \tag{1.5}$$

In Section 3, we are interested in the following polyharmonic problem:

$$\begin{gathered}(-\triangle)^m u + u\varphi(\cdot,u) = f, \quad \text{in } B \text{ (in the sense of distributions)}, \\ \lim_{x \to \xi \in \partial B} \frac{u(x)}{(1-|x|^2)^{m-1}} = \theta(\xi),\end{gathered} \tag{1.6}$$

where θ is a nontrivial nonnegative continuous function on ∂B and the functions φ and f verify the following assumptions.

(H_1) φ is a nonnegative measurable function on $B \times (0,\infty)$.

(H_2) For each $\lambda > 0$, there exists $q_\lambda \in K^+$ with $\alpha_{q_\lambda} \leq 1/2$ and such that for each $x \in B$, the map $t \to t(q_\lambda(x) - \varphi(x,t))$ is continuous and nondecreasing on $[0,\lambda]$.

(H_3) f is a nonnegative measurable function on B such that the function $\gamma(x) := f(x)/(\delta(x))^{m-1}$ belongs to the class K.

Under these hypotheses, we give an existence result for problem (1.6). In fact, we will prove that (1.6) has a positive continuous solution u satisfying for $x \in B$ that

$$c\big(Vf(x) + \rho(x)\big) \leq u(x) \leq Vf(x) + \rho(x), \tag{1.7}$$

where c is a positive constant, $Vf(x) = \int_B G_{m,n}(x,y)f(y)dy$, and the function ρ is defined by $\rho(x) = (1-|x|^2)^{m-1}h(x)$ with h being the continuous solution of the Dirichlet problem $\triangle h = 0$, on B and $h_{|\partial B} = \theta$.

To establish this result, we will exploit the 3G-theorem to prove that if the coefficient $q \in K^+$ is sufficiently small and f is a positive function on B, then the equation

$$(-\triangle)^m u + qu = f \tag{1.8}$$

has a positive solution on B. In [12], Grunau and Sweers gave a similar result with operators perturbed by small lower-order terms:

$$(-\triangle)^m u + \sum_{|k|<2m} a_k(u) D^k u = f. \tag{1.9}$$

In the case $m = 1$, problem (1.6) has been studied by Mâagli and Masmoudi in [13], where they gave an existence and a uniqueness result in a bounded domain Ω.

In Section 4, we fix a positive harmonic function h_0 in B, continuous in $\overline{B}$ and we put $\rho_0(x) = (1-|x|^2)^{m-1}h_0(x)$. Then we aim at proving an existence result for problem (1.1) with g satisfying the following assumptions.

(A_1) g is a nonnegative Borel measurable function on $B \times (0,\infty)$, which is continuous with respect to the second variable.

(A_2) $g(x,t) \leq \psi(x,t)$, where ψ is a positive Borel measurable function in $B \times (0,\infty)$, such that the function $t \mapsto \psi(x,t)$ is nonincreasing on $(0,\infty)$.

(A_3) The function q defined on B by $q(x) = \psi(x,\rho_0(x))/\rho_0(x)$ belongs to the class K.

We will prove the following result. There exists a constant $c_1 > 0$ such that if $\theta \geq (1+c_1)h_0$ on ∂B, then problem (1.1) has a positive continuous solution u satisfying $\rho_0 \leq u \leq \rho$ on B.

This result is a followup to the one of Athreya [14], who considered the following problem:

$$\begin{aligned} \triangle u &= g(u) \quad \text{in } D, \\ u &= \varphi \quad \text{on } \partial D, \end{aligned} \tag{$*$}$$

where D is a simply connected bounded C^2-domain and $g(u) \leq \max(1, u^{-\alpha})$, for $0 < \alpha < 1$. Then he proved that there exists a constant $c > 0$ such that if $\varphi \geq (1+c)h_0$ on ∂D, then problem (*) has a positive continuous solution u such that $u \geq h_0$, where h_0 is a fixed positive harmonic function in D.

In order to simplify our statements, we define some convenient notations.

Notations 1.2. (i) $B := \{x \in \mathbb{R}^n;\ |x| < 1\}$ with $n \geq 2$.

(ii) On B^2 (i.e., $(x,y) \in B^2$),

$$[x,y]^2 = |x-y|^2 + (1-|x|^2)(1-|y|^2),$$
$$\delta(x) = 1-|x|. \tag{1.10}$$

Note that $[x,y]^2 \geq 1+|x|^2|y|^2-2|x||y| = (1-|x||y|)^2$. So we have

$$\max(\delta(x),\delta(y)) \leq [x,y]. \tag{1.11}$$

(iii) $\mathcal{B}(B)$ denotes the set of Borel measurable functions in B and $\mathcal{B}^+(B)$ denotes the set of nonnegative ones.

(iv) $C_0(B) := \{w$ continuous on B and $\lim_{|x|\to 1} w(x) = 0\}$.

(v) For $f \in \mathcal{B}^+(B)$, we put

$$Vf(x) := V_{m,n}f(x) = \int_B G_{m,n}(x,y)f(y)dy, \quad \text{for } x \in B. \tag{1.12}$$

The function Vf is called the m-potential of f and it is lower semicontinuous on B.

(vi) Let K^+ denote the set of nonnegative functions on the Kato class K.

(vii) For any $\varphi \in \mathcal{B}(B)$, we put

$$\|\varphi\|_B := \sup_{x\in B}\int_B \left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y)|\varphi(y)|dy. \tag{1.13}$$

(viii) Let f and g be two positive functions on a set S.

We call $f \sim g$ if there is $c > 0$ such that

$$\frac{1}{c}g(x) \leq f(x) \leq cg(x) \quad \forall x \in S. \tag{1.14}$$

We call $f \preceq g$ if there is $c > 0$ such that

$$f(x) \leq cg(x) \quad \forall x \in S. \tag{1.15}$$

2. Properties of the Kato class K

We collect in this section some properties of functions belonging to the Kato class K, which are useful at stating our existence results.

Proposition 2.1 (see [4]). *Let φ be a function in K. Then one has the following assertions.*

(i) $\|\varphi\|_B < \infty$.

(ii) *The function $x \to (\delta(x))^{2m-1}\varphi$ is in $L^1(B)$.*

Corollary 2.2. *Let $q \in K^+$. Then one has*

$$\alpha_q := \sup_{x,y\in B} \int_B \frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} q(z)dz < \infty \tag{2.1}$$

and for each $x \in B$,

$$V\left(qG_{m,n}(\cdot,y)\right)(x) \leq \alpha_q G_{m,n}(x,y). \tag{2.2}$$

Proof. The result holds by (1.3). □

For ζ in K^+, we denote

$$M_\zeta := \{\varphi \in K,\ |\varphi| \preceq \zeta\}. \tag{2.3}$$

From (1.3) and Proposition 2.1(i), we derive the following proposition.

Proposition 2.3 (see [15]). *For any function $\zeta \in K^+$, the family of functions*

$$\left\{ \frac{1}{(\delta(x))^{m-1}} \int_B (\delta(y))^{m-1} G_{m,n}(x,y)\varphi(y)dy;\ \varphi \in M_\zeta \right\} \tag{2.4}$$

is relatively compact in $C_0(B)$.

Proposition 2.4. *For each $q \in K^+$ and h a nonnegative harmonic function in B, one has for $x \in B$ that*

$$\int_B G_{m,n}(x,y)\left(1-|y|^2\right)^{m-1} h(y)q(y)dy \leq \alpha_q \left(1-|x|^2\right)^{m-1} h(x). \tag{2.5}$$

Proof. Let h be a nonnegative harmonic function in B. So by Herglotz representation theorem (see [16, page 29]), there exists a nonnegative measure μ on ∂B such that

$$h(y) = \int_{\partial B} P(y,\xi)\mu(d\xi), \tag{2.6}$$

where $P(y,\xi) = (1-|y|^2)/|y-\xi|^n$, for $y \in B$ and $\xi \in \partial B$. So we need only to verify the inequality for $h(y) = P(y,\xi)$ uniformly in $\xi \in \partial B$.

From expression (1.2) of $G_{m,n}$, it is clear that for each $x, y \in B$, we have

$$\lim_{z\to\xi} \frac{G_{m,n}(y,z)}{G_{m,n}(x,z)} = \frac{\left(1-|y|^2\right)^m}{\left(1-|x|^2\right)^m} \frac{|x-\xi|^n}{|y-\xi|^n} = \frac{\left(1-|y|^2\right)^{m-1}}{\left(1-|x|^2\right)^{m-1}} \frac{P(y,\xi)}{P(x,\xi)}. \tag{2.7}$$

Thus by Fatou lemma, we deduce that for each $x \in B$ and $\zeta \in \partial B$,

$$\begin{aligned}
&\int_B G_{m,n}(x,y)\frac{(1-|y|^2)^{m-1}}{(1-|x|^2)^{m-1}}\frac{P(y,\xi)}{P(x,\xi)}|q(y)|\,dy \\
&\quad\leq \liminf_{z\to\xi}\int_B G_{m,n}(x,y)\frac{G_{m,n}(y,z)}{G_{m,n}(x,z)}|q(y)|\,dy \leq \alpha_q,
\end{aligned}\tag{2.8}$$

which completes the proof. □

In the next results, we will give a class of functions included in K and we will precise estimates of the m-potential of some functions in this class.

Proposition 2.5. *Let $p > \max(n/2m, 1)$. Then for $\lambda < 2m - n/p$ if $n \geq 2m$, or $\lambda \leq 2m - n$ if $n < 2m$, one has*

$$\frac{L^p(B)}{(\delta(\cdot))^\lambda} \subset K. \tag{2.9}$$

To prove Proposition 2.5, we use the next two lemmas.

Lemma 2.6 (see [4]). *On B^2, one has the following.*

(i) *For $n > 2m$,*

$$G_{m,n}(x,y) \sim |x-y|^{2m-n}\min\left(1, \frac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}\right) \sim \frac{(\delta(x)\delta(y))^m}{|x-y|^{n-2m}[x,y]^{2m}}. \tag{2.10}$$

(ii) *For $n = 2m$,*

$$G_{m,n}(x,y) \sim \operatorname{Log}\left(1 + \frac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}\right) \sim \frac{(\delta(x)\delta(y))^m}{[x,y]^{2m}}\operatorname{Log}\left(1 + \frac{[x,y]^2}{|x-y|^2}\right). \tag{2.11}$$

(iii) *For $n < 2m$,*

$$G_{m,n}(x,y) \sim \frac{(\delta(x)\delta(y))^m}{[x,y]^n}. \tag{2.12}$$

Lemma 2.7. *Let $\lambda \in \mathbb{R}$. Then on B^2, one has*

$$\frac{1}{(\delta(y))^\lambda}\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y) \preceq \begin{cases} \dfrac{1}{|x-y|^{n-2m+\lambda^+}} & \text{if } n > 2m, \\[2ex] \dfrac{1}{|x-y|^{\lambda^+}}\operatorname{Log}\left(\dfrac{3}{|x-y|}\right) & \text{if } n = 2m, \\[2ex] (\delta(y))^{2m-n-\lambda^+} & \text{if } n < 2m, \end{cases} \tag{2.13}$$

where $\lambda^+ = \max(\lambda, 0)$.

Proof. Using Lemma 2.6, inequality (1.11), and the fact that $|x-y| \le [x,y]$, we deduce the following.

(i) If $n > 2m$, then for x, y in B, we have

$$\begin{aligned}\frac{1}{(\delta(y))^{\lambda}}\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y) &\preceq \frac{(\delta(y))^{2m-\lambda^+}}{|x-y|^{n-2m}[x,y]^{2m}}\\ &\preceq \frac{(\delta(y))^{2m-\lambda^+}}{|x-y|^{n-2m+\lambda^+}[x,y]^{2m-\lambda^+}}\\ &\preceq \frac{1}{|x-y|^{n-2m+\lambda^+}}.\end{aligned} \tag{2.14}$$

(ii) If $n = 2m$, then for x, y in B, we have

$$\begin{aligned}\frac{1}{(\delta(y))^{\lambda}}\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y) &\preceq \frac{(\delta(y))^{2m-\lambda^+}}{[x,y]^{2m}}\operatorname{Log}\left(1+\frac{[x,y]^2}{|x-y|^2}\right)\\ &\preceq \frac{(\delta(y))^{2m-\lambda^+}}{|x-y|^{\lambda^+}[x,y]^{2m-\lambda^+}}\operatorname{Log}\left(\frac{3}{|x-y|}\right)\\ &\preceq \frac{1}{|x-y|^{\lambda^+}}\operatorname{Log}\left(\frac{3}{|x-y|}\right).\end{aligned} \tag{2.15}$$

(iii) If $n < 2m$, then for x, y in B, we have

$$\frac{1}{(\delta(y))^{\lambda}}\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y) \preceq \frac{(\delta(y))^{2m-\lambda^+}}{[x,y]^n} \preceq (\delta(y))^{2m-n-\lambda^+}. \tag{2.16}$$

□

Proof of Proposition 2.5. Let $\alpha > 0$, $p > \max(n/2m, 1)$, and $q \ge 1$ such that $1/p + 1/q = 1$. To show the claim, we use the inequalities in Lemma 2.7 and the Hölder inequality. We distinguish three cases.

Case 1 ($n > 2m$). Let $p > n/2m$, $f \in L^p(B)$, and $\lambda < 2m - n/p$. Then, for $x \in B$, we have

$$\begin{aligned}\int_{B\cap B(x,\alpha)}\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y)\frac{|f(y)|}{(\delta(y))^{\lambda}}dy &\preceq \int_{B\cap B(x,\alpha)}\frac{|f(y)|}{|x-y|^{n-2m+\lambda^+}}dy\\ &\preceq \|f\|_p\left(\int_0^{\alpha} r^{n(1-q)+(2m-\lambda^+)q-1}dr\right)^{1/q}\\ &\preceq \|f\|_p\alpha^{2m-(n/p)-\lambda^+},\end{aligned} \tag{2.17}$$

which tends to zero as $\alpha \to 0$.

Case 2 ($n = 2m$). Let $p > 1$, $f \in L^p(B)$, and $\lambda < n/q$. Then, for $x \in B$, we have

$$\begin{aligned}\int_{B\cap B(x,\alpha)} \left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y)\frac{|f(y)|}{(\delta(y))^\lambda}dy &\preceq \int_{B\cap B(x,\alpha)} \frac{|f(y)|}{|x-y|^{\lambda^+}}\operatorname{Log}\left(\frac{3}{|x-y|}\right)dy \\ &\preceq \|f\|_p\left(\int_0^\alpha r^{n-1-\lambda^+q}\left(\operatorname{Log}\left(\frac{3}{r}\right)\right)^q dr\right)^{1/q},\end{aligned} \tag{2.18}$$

which tends to zero as $\alpha \to 0$.

Case 3 ($n < 2m$). Let $p > 1$, $f \in L^p(B)$, and $\lambda \le 2m - n$. Then, for $x \in B$, we have

$$\int_{B\cap B(x,\alpha)} \left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y)\frac{|f(y)|}{(\delta(y))^\lambda}dy \preceq \int_{B\cap B(x,\alpha)} (\delta(y))^{2m-n-\lambda^+}|f(y)|dy \preceq \|f\|_p\alpha^{n/q}, \tag{2.19}$$

which tends to zero as $\alpha \to 0$. This completes the proof. □

In the sequel, we put for $f \in \mathcal{B}(B)$ that

$$v(x) = \int_B G_{m,n}(x,y)\frac{|f(y)|}{(\delta(y))^\lambda}dy, \quad \text{for } x \in B. \tag{2.20}$$

Remark 2.8. From Lemma 2.6, we note that for each x, y in B, we have $(\delta(x)\delta(y))^m \preceq G_{m,n}(x,y)$. This implies that there exists a constant $c > 0$ such that for each $f \in \mathcal{B}(B)$, we have

$$c(\delta(x))^m\int_B \frac{|f(y)|}{(\delta(y))^{\lambda-m}}dy \le v(x). \tag{2.21}$$

In the next proposition, we will give upper estimates on the function v.

Proposition 2.9. *Let* $p > \max(n/2m, 1)$ *and* $\lambda < \min(m + 1 - (1/p), 2m - n/p)$*. Then there exists a constant* $c > 0$*, such that for each* $f \in L^p(B)$ *and* $x \in B$*, the following estimates hold:*

$$v(x) \le \begin{cases} c\|f\|_p(\delta(x))^{2m-\lambda-n/p} & \text{if } m - \dfrac{n}{p} < \lambda < \min\left(m+1-\dfrac{1}{p}, 2m-\dfrac{n}{p}\right), \\ c\|f\|_p(\delta(x))^m\left(\operatorname{Log}\left(\dfrac{2}{\delta(x)}\right)\right)^{1/q} & \text{if } \lambda = m - \dfrac{n}{p}, \\ c\|f\|_p(\delta(x))^m & \text{if } \lambda < m - \dfrac{n}{p}. \end{cases} \tag{2.22}$$

To prove Proposition 2.9, we need the following key lemma.

Lemma 2.10 (see [17]). *Let $x, y \in B$. Then one has the following properties.*
(1) *If $\delta(x)\delta(y) \leq |x-y|^2$, then $\max(\delta(x), \delta(y)) \leq ((\sqrt{5}+1)/2)|x-y|$.*
(2) *If $|x-y|^2 \leq \delta(x)\delta(y)$, then $((3-\sqrt{5})/2)\delta(x) \leq \delta(y) \leq ((3+\sqrt{5})/2)\delta(x)$.*

Proof of Proposition 2.9. Let $p > \max(n/2m, 1)$ and $f \in L^p(B)$. Then we have for $x \in B$ that

$$v(x) \leq \int_{B\cap D_1} G_{m,n}(x,y)\frac{|f(y)|}{(\delta(y))^{\lambda}}dy + \int_{B\cap D_2} G_{m,n}(x,y)\frac{|f(y)|}{(\delta(y))^{\lambda}}dy = I_1(x) + I_2(x), \tag{2.23}$$

where

$$D_1 = \{y \in B : |x-y|^2 \leq \delta(x)\delta(y)\}, \qquad D_2 = \{y \in B : |x-y|^2 \geq \delta(x)\delta(y)\}. \tag{2.24}$$

Let $q \geq 1$ such that $1/p + 1/q = 1$. We claim that for $x \in B$,

$$I_1(x) \preceq \|f\|_p (\delta(x))^{2m-\lambda-n/p}. \tag{2.25}$$

Indeed, for $x \in B$ and $y \in D_1$, we have by Lemma 2.10 that $|x-y| \leq ((\sqrt{5}+1)/2)\delta(x)$. Hence from the estimates on $G_{m,n}(x,y)$ in Lemma 2.6, we deduce that for $x \in B$ and $y \in D_1$,

$$\frac{1}{(\delta(y))^{\lambda}} G_{m,n}(x,y) \sim H_{m,n}(x,y) := \begin{cases} \dfrac{1}{|x-y|^{n-2m+\lambda}} & \text{if } n > 2m, \\ \operatorname{Log}\left(1 + \dfrac{(\delta(x))^{2m}}{|x-y|^{2m}}\right) & \text{if } n = 2m, \\ (\delta(x))^{2m-n} & \text{if } n < 2m. \end{cases} \tag{2.26}$$

Then, by elementary calculus, we deduce using the Hölder inequality that for $x \in B$,

$$I_1(x) \preceq \|f\|_p \left(\int_{(|x-y|\leq((\sqrt{5}+1)/2)\delta(x))\cap B} (H_{m,n}(x,y))^q dy\right)^{1/q} \preceq \|f\|_p (\delta(x))^{2m-\lambda-(n/p)}. \tag{2.27}$$

On the other hand, from the estimates on $G_{m,n}(x,y)$ in Lemma 2.6, we deduce by Lemma 2.10 and the fact that $t/(1+t) \leq \operatorname{Log}(1+t) \leq t$ that for $x \in B$ and $y \in D_2$, we have

$$G_{m,n}(x,y) \sim \frac{(\delta(x)\delta(y))^m}{|x-y|^n}. \tag{2.28}$$

So, we obtain by using the Hölder inequality that

$$I_2(x) \preceq \|f\|_p \left(\int_{D_2\cap B} \frac{(\delta(x))^{mq}(\delta(y))^{(m-\lambda)q}}{|x-y|^{nq}} dy\right)^{1/q}. \tag{2.29}$$

Now, we distinguish the following two cases.

Case 1. If $\lambda \leq m$, then we have

$$I_2(x) \preceq \|f\|_p (\delta(x))^m \left(\int_{(|x-y| \geq ((\sqrt{5}-1)/2)\delta(x)) \cap B} \frac{1}{|x-y|^{(n-m+\lambda)q}} dy \right)^{1/q}$$
$$\preceq \begin{cases} (\delta(x))^{2m-\lambda-n/p} & \text{if } m - \dfrac{n}{p} < \lambda \leq m, \\ (\delta(x))^m \left(\operatorname{Log}\left(\dfrac{2}{\delta(x)} \right) \right)^{1/q} & \text{if } \lambda = m - \dfrac{n}{p}, \\ (\delta(x))^m & \text{if } \lambda < m - \dfrac{n}{p}. \end{cases} \tag{2.30}$$

Case 2. If $\lambda > m$, then we have

$$I_2(x) \preceq \|f\|_p (\delta(x))^{2m-\lambda-n/p} \left(\int_{D_2 \cap B} \left(\frac{\delta(x)}{\delta(y)} \right)^{(\lambda-m)q} \frac{1}{|x-y|^n} dy \right)^{1/q}. \tag{2.31}$$

Then, since $0 < (\lambda - m)q < 1$, we have by [17, Corollary 2.8] that the function $x \to \int_{D_2 \cap B} (\delta(x)/\delta(y))^{(\lambda-m)q} (1/|x-y|^n) dy$ is bounded, and so

$$I_2(x) \preceq \|f\|_p (\delta(x))^{2m-\lambda-n/p}. \tag{2.32}$$

This completes the proof. □

Remark 2.11. By taking $p = +\infty$ (i.e., $q = 1$), in Propositions 2.5 and 2.9, we find again the results of Bachar et al. in [4, Example 3.9 and Proposition 3.10].

3. First existence result

In this section, we are interested in the existence of positive solutions for problem (1.6). To this end, we first introduce for $q \in K^+$, such that $\alpha_q \leq 1/2$, the potential kernel $V_q f := V_{m,n,q} f$ as a solution for the pertubed polyharmonic equation (1.8).

We put for $x, y \in B$ that

$$\mathcal{G}_{m,n}(x,y) = \sum_{k \geq 0} (-1)^k \left(V(q\cdot) \right)^k \left(G_{m,n}(\cdot, y) \right)(x). \tag{3.1}$$

Then we have the following comparison result.

Lemma 3.1. *Let $q \in K^+$ such that $\alpha_q \leq 1/2$. Then on B^2, one has*

$$(1 - \alpha_q) G_{m,n}(x,y) \leq \mathcal{G}_{m,n}(x,y) \leq G_{m,n}(x,y). \tag{3.2}$$

Proof. Since $\alpha_q \leq 1/2$, we deduce from (2.2) that

$$|\mathcal{G}_{m,n}(x,y)| \leq \sum_{k \geq 0} (\alpha_q)^k G_{m,n}(x,y) = \frac{1}{1-\alpha_q} G_{m,n}(x,y). \tag{3.3}$$

On the other hand, we note that for $x \neq y$ in B,

$$\mathcal{G}_{m,n}(x,y) = G_{m,n}(x,y) - V\left(q\mathcal{G}_{m,n}(\cdot,y)\right)(x). \tag{3.4}$$

Using these facts and (2.2), we obtain that

$$\mathcal{G}_{m,n}(x,y) \geq G_{m,n}(x,y) - \frac{\alpha_q}{1-\alpha_q}G_{m,n}(x,y) = \frac{1-2\alpha_q}{1-\alpha_q}G_{m,n}(x,y) \geq 0. \tag{3.5}$$

Hence the result follows from (3.4) and (2.2). □

In the sequel, for a given $q \in K^+$ such that $\alpha_q \leq 1/2$, we define the operator V_q on $\mathcal{B}^+(B)$ by

$$V_q f(x) = \int_B \mathcal{G}_{m,n}(x,y)f(y)dy, \quad x \in B. \tag{3.6}$$

Lemma 3.2. *Let $q \in K^+$ such that $\alpha_q \leq 1/2$ and $f \in \mathcal{B}^+(B)$. Then $V_q f$ satisfies the following resolvent equation:*

$$Vf = V_q f + V_q(qVf) = V_q f + V\left(qV_q f\right). \tag{3.7}$$

Proof. From the expression of $\mathcal{G}_{m,n}$, we deduce that for $f \in \mathcal{B}^+(B)$ such that $Vf < \infty$, we have

$$V_q f = \sum_{k\geq 0}(-1)^k\left(V(q\cdot)\right)^k Vf. \tag{3.8}$$

So we obtain that

$$V_q(qVf) = \sum_{k\geq 0}(-1)^k\left(V(q\cdot)\right)^k\left[V(qVf)\right] = -\sum_{k\geq 1}(-1)^k\left(V(q\cdot)\right)^k Vf = Vf - V_q f. \tag{3.9}$$

The second equality holds by integrating (3.4). □

Proposition 3.3. *Let $q \in K^+$ such that $\alpha_q \leq 1/2$. Let $f \in L^1_{\text{Loc}}(B)$ such that $Vf \in L^1_{\text{Loc}}(B)$. Then $V_q f$ is a solution of the perturbed polyharmonic equation (1.8).*

Proof. Using the resolvent equation (3.7), we have

$$V_q f = Vf - V\left(qV_q f\right). \tag{3.10}$$

Applying the operator $(-\Delta)^m$ on both sides of the above equality, we obtain that

$$(-\Delta)^m\left(V_q f\right) = f - qV_q f \quad \text{(in the sense of distributions)}. \tag{3.11}$$

This completes the proof. □

Now, we aim at proving an existence result for problem (1.6). We recall that the function ρ is defined on B by $\rho(x) = (1-|x|^2)^{m-1}h(x)$, where h is the continuous solution of the Dirichlet problem $\triangle h = 0$, on B and $h_{|\partial B} = \theta$.

The main result of this section is the following.

Theorem 3.4. *Assume (H_1)–(H_3) hold. Then problem (1.6) has a positive continuous solution u satisfying (1.7).*

Proof. Let $f \in \mathcal{B}^+(B)$ satisfy (H_3). Then $f(x) = (\delta(x))^{m-1}\gamma(x)$, where γ is in K^+. By Proposition 2.3, we have that $Vf/(\delta(\cdot))^{m-1}$ is in $C_0(B)$. So if we put $\beta = \|Vf + \rho\|_\infty$, we have obviously that $0 < \beta < \infty$.

Then by (H_2), there exists a function $q := q_\beta \in K^+$ such that $\alpha_q \le 1/2$ and for each $x \in B$, the map

$$t \longrightarrow t(q(x) - \varphi(x,t)) \quad \text{is continuous and nondecreasing on } [0,\beta], \tag{3.12}$$

which implies in particular that for each $x \in B$ and $t \in [0,\beta]$,

$$0 \le \varphi(x,t) \le q(x). \tag{3.13}$$

Let

$$\Lambda := \{u \in \mathcal{B}^+(B) : (1-\alpha_q)(Vf + \rho) \le u \le Vf + \rho\}. \tag{3.14}$$

We define the operator T on Λ by

$$Tu := V_q f + (\rho - V_q(q\rho)) + V_q[(q - \varphi(\cdot,u))u]. \tag{3.15}$$

We claim that Λ is invariant under T. Indeed, using (3.13) and (3.7), we have for each $u \in \Lambda$ that

$$\begin{aligned} Tu &\le V_q f + (\rho - V_q(q\rho)) + V_q((q - \varphi(\cdot,u))(Vf + \rho)) \\ &\le V_q f + (\rho - V_q(q\rho)) + V_q(q(Vf + \rho)) \le Vf + \rho. \end{aligned} \tag{3.16}$$

Moreover, from (3.13), (3.2), and Proposition 2.4, we deduce that for each $u \in \Lambda$, we have

$$Tu \ge V_q f + (\rho - V_q(q\rho)) \ge V_q f + (\rho - V(q\rho)) \ge (1-\alpha_q)(Vf + \rho). \tag{3.17}$$

Next, we will prove that the operator T is nondecreasing on Λ. Indeed, let $u, v \in \Lambda$ such that $u \le v$, then from (3.12), we obtain that

$$Tv - Tu = V_q([(q - \varphi(\cdot,v))v] - [(q - \varphi(\cdot,u))u]) \ge 0. \tag{3.18}$$

Now, we consider the sequence (u_k) defined by $u_0 = (1-\alpha_q)(Vf + \rho)$ and $u_{k+1} = Tu_k$ for $k \in \mathbb{N}$. Since Λ is invariant under T, then $u_1 = Tu_0 \ge u_0$, and so from the monotonicity of T, we deduce that

$$u_0 \le u_1 \le \cdots \le u_k \le u_{k+1} \le Vf + \rho. \tag{3.19}$$

Hence from (3.12) and the dominated convergence theorem, we deduce that the sequence (u_k) converges to a function u which satisfies

$$u = V_q f + (\rho - V_q(q\rho)) + V_q[(q - \varphi(\cdot,u))u]. \tag{3.20}$$

That is,

$$u - V_q(qu) = V_q f + (\rho - V_q(q\rho)) - V_q(u\varphi(\cdot,u)). \tag{3.21}$$

Applying the operator $(I + V(q\cdot))$ on both sides of the above equality and using (3.7), we deduce that u satisfies

$$u = Vf + \rho - V(u\varphi(\cdot,u)). \tag{3.22}$$

Finally, we need to verify that u is a positive continuous solution for problem (1.6). Since for each $x \in B$, $f(x) = (\delta(x))^{m-1}\gamma(x)$, where γ is in K^+, we deduce by Proposition 2.1 that $f \in L^1_{\mathrm{Loc}}(B)$ and by Proposition 2.3 we have that $Vf \in L^1_{\mathrm{Loc}}(B)$. Then, since $u \sim Vf + \rho$ and $u\varphi(\cdot,u) \le uq$, we deduce that either u and $u\varphi(\cdot,u)$ are in $L^1_{\mathrm{Loc}}(B)$. Now, from (3.22) we can easily see that $V(u\varphi(\cdot,u)) \in L^1_{\mathrm{Loc}}(B)$. Hence u satisfies (in the sense of distributions) the elliptic differential equation

$$(-\Delta)^m u + u\varphi(\cdot,u) = f \quad \text{in } B. \tag{3.23}$$

To prove continuity of u, we recall that $Vf/(\delta(\cdot))^{m-1} \in C_0(B)$. Then, there exists a function $\zeta \in C(\overline{B})$ such that $Vf(x) + \rho(x) = (\delta(x))^{m-1}\zeta(x)$, and so we have by Proposition 2.3 that $V((\delta(\cdot))^{m-1}\zeta q)$ is in $(\delta(\cdot))^{m-1}C_0(B)$. Now, since $V(u\varphi(\cdot,u))$ is lower semicontinuous and for $x \in B$, we have

$$V(u\varphi(\cdot,u))(x) \le V(uq) \le V\big(\big(\delta(\cdot)\big)^{m-1}\zeta q\big)(x), \tag{3.24}$$

then we deduce that $V(u\varphi(\cdot,u))$ is in $(\delta(\cdot))^{m-1}C_0(B)$. This implies by (3.22) that u is continuous on B and satisfies obviously $\lim_{x\to\zeta\in\partial B}(u(x)/(1-|x|^2)^{m-1}) = \theta(\zeta)$, which completes the proof. □

Example 3.5. Let α, β be two positive constants and q, γ are two functions in K^+. Then, for each $\theta \in C^+(\partial B)$, the following polyharmonic problem

$$\begin{gathered}(-\triangle)^m u + \beta u^{\alpha+1} q = \big(1-|x|^2\big)^{m-1}\gamma, \quad \text{in } B \text{ (in the sense of distributions)},\\ \lim_{x\to\xi\in\partial B} \frac{u(x)}{\big(1-|x|^2\big)^{m-1}} = \theta(\xi),\end{gathered} \tag{3.25}$$

has a positive continuous solution satisfying (1.7), provided that β is sufficiently small.

4. Second existence result

In this section, assuming that (A_1)–(A_3) hold, we aim at proving an existence result for problem (1.1). We recall that h_0 is a fixed positive harmonic function in B, continuous in $\overline{B}$. We put $\rho_0(x) = (1-|x|^2)^{m-1}h_0(x)$ and $\rho(x) = (1-|x|^2)^{m-1}h(x)$, where h is the continuous solution of the Dirichlet problem $\triangle h = 0$, on B and $h_{|\partial B=\theta}$.

The main result of this section is the following.

Theorem 4.1. *Assume (A_1)–(A_3) hold. Then there exists a constant $c_1 > 0$ such that if $\theta \ge (1+c_1)h_0$ on ∂B, then problem (1.1) has a positive continuous solution u satisfying for each*

$x \in B$ that

$$\rho_0(x) \le u(x) \le \rho(x). \tag{4.1}$$

To prove Theorem 4.1, we need the next lemma.

Lemma 4.2. *Assume that (A_2)-(A_3) hold. Then one has*

$$c_1 := \sup_{x\in\mathbb{R}_+^n} \frac{1}{\rho_0(x)} \int_{\mathbb{R}_+^n} G_{m,n}(x,y)\psi(y,\rho_0(y))\,dy < \infty. \tag{4.2}$$

Proof. By (A_3), the function q defined on B by $q(x) = \psi(x,\rho_0(x))/\rho_0(x)$ is in K^+. Then we deduce by Proposition 2.4 that

$$\frac{1}{\rho_0(x)} \int_{\mathbb{R}_+^n} G_{m,n}(x,y)\psi(y,\rho_0(y))\,dy = \int_{\mathbb{R}_+^n} G_{m,n}(x,y)\frac{(1-|y|^2)^{m-1}}{(1-|x|^2)^{m-1}}\frac{h_0(y)}{h_0(x)}q(y)\,dy \le \alpha_q. \tag{4.3}$$

The result holds from Corollary 2.2. □

In the sequel, we suppose
(A_4)

$$\theta(x) \ge (1+c_1)h_0(x), \quad \forall x \in \partial B. \tag{4.4}$$

Proof of Theorem 4.1. We will use a fixed point argument. Let

$$\Lambda = \{v \in C(\overline{B}) : \rho_0 \le u \le \rho\}. \tag{4.5}$$

Since $h = \theta$ on ∂B and h_0 is continuous in $\overline{B}$, then we obtain by (A_4) that $h \ge (1+c_1)h_0$ on $\overline{B}$. So Λ is a well-defined nonempty closed bounded and convex set in $C(\overline{B})$.

We define the operator T on Λ by

$$Tu(x) = \rho(x) - \int_B G_{m,n}(x,y)g(y,u(y))\,dy. \tag{4.6}$$

Since for $u \in \Lambda$ and $y \in B$, we have by (A_2) that

$$g(y,u(y)) \le \psi(y,\rho_0(y)) = \rho_0(y)q(y) \preceq q(y), \tag{4.7}$$

then the function $y \mapsto g(y,u(y))$ is in M_q. Hence by (A_3), Proposition 2.3, and the fact that $\rho \in C(\overline{B})$, we deduce that $T\Lambda$ is relatively compact in $C(\overline{B})$.

Moreover by hypothesis (A_2), we have for $x \in B$ that

$$\int_{\mathbb{R}_+^n} G_{m,n}(x,y)g(y,u(y))\,dy \le \int_{\mathbb{R}_+^n} G_{m,n}(x,y)\psi(y,\rho_0(y))\,dy \le c_1\rho_0(x). \tag{4.8}$$

Then since by (A_4), $\rho \ge (1+c_1)\rho_0$, we obtain that $T\Lambda \subset \Lambda$.

Next, let us prove the continuity of T in the uniform norm. Let $(u_k)_k$ be a sequence in Λ which converges uniformly to $u \in \Lambda$, then since g is continuous with respect to the second variable, we deduce by the dominated convergence theorem that

$$\forall x \in \overline{B}, \quad Tu_k(x) \longrightarrow Tu(x) \quad \text{as } k \longrightarrow \infty. \tag{4.9}$$

Now since $T\Lambda$ is relatively compact in $C(\overline{B})$, then

$$||Tu_k - Tu||_\infty \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{4.10}$$

Finally the Schauder fixed point theorem implies the existence of $u \in \Lambda$ such that $Tu = u$. That is, for $x \in B$, we have

$$u(x) = \rho(x) - \int_B G_{m,n}(x,y)g(y,u(y))\,dy. \tag{4.11}$$

Using (A_2), (A_3), and Proposition 2.1, we deduce that $y \to g(y,u(y))$ is in $L^1_{\mathrm{Loc}}(\mathbb{R}^n_+)$. So u satisfies (in the sense of distributions) the elliptic differential equation

$$(-\Delta)^m u = g(\cdot,u) \quad \text{in } B. \tag{4.12}$$

Furthermore, since $h_{|\partial B=\theta}$, then we deduce by Proposition 2.3 that $\lim_{x\to\xi\in\partial B}(u(x)/(1-|x|^2)^{m-1}) = \theta(\xi)$. This completes the proof. □

Example 4.3. Let $p > n/2m$, $\lambda < 2m - n/p$, and $\alpha > 0$. Let $f \in L^p(B)$ and let h_0 be a positive harmonic function in B, which is bounded and continuous in $\overline{B}$. Then we have for $x \in B$ that $\delta(x) \preceq h_0(x)$.

Now, let g be a nonnegative Borel measurable function on $B \times (0,\infty)$, which is continuous with respect to the second variable and satisfies

$$g(x,t) \preceq \frac{t^{-\alpha} f(x)}{(\delta(x))^{\lambda - m\alpha}}, \tag{4.13}$$

then there exists a constant $c_1 > 0$ such that if $\theta \geq (1+c_1)h_0$ on ∂B, problem (1.1) has a continuous positive solution satisfying (4.1).

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